Chapter 28

SOME OCEANOGRAPHIC AND ENGINEERING CONSIDERATIONS IN MARINE PIPE LINE CONSTRUCTION*

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ABSTRACT

The proper design of a pipe line for the transport of gas or oil from sea to land requires the solution of a number of engineering problems either not encountered in pipe line engineering on land or found to be of a different nature in the marine environment than in the terrestrial environment. These include: (1) consideration of the vertical stability of the pipe, (2) consideration of the lateral stability of the pipe and its vertical risers in the presence of wave-induced forces, and (3) consideration of the longitudinal stability of the pipe in the presence of thermally induced tensile and compressive forces. The first of these considerations is treated in the present paper.

In those areas where the bearing capacity of the upper sediments is small, as is the case for certain regions of the Gulf shelf, downward sag of a pipe line can occur and entrenchment of the line to considerable depths may be necessary in order that excessive stresses within the pipe be avoided. Because both the flexural and longitudinal tensile stresses, occurring simultaneously, can be important in a sagging pipe line, both must be evaluated. Appropriate formulas and graphs are presented for this purpose. From these and a knowledge of the sediment characteristics along the proposed pipe line route, it is possible to determine whether or not regions of critical sag might develop in a pipe of given specifications.

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* Contribution from the Department of Oceanography of the Agricultural and Mechanical College of Texas, No. 14. This paper is based in part upon research sponsored by the United Gas Pipe Line Company through the Texas A & M Research Foundation.

INTRODUCTION

The present and potential source of offshore oil and gas in tideland regions of the Gulf of Mexico demands an economical mode of transportation from sea to land. Pipe lines can meet this demand, if designed not only to endure the processes of deterioration in the sea, but also to withstand the internal stresses induced by lack of adequate support, by severe wave loads, or by thermal changes.

An introduction to the scope of problems encountered in the design and installation of pipe lines to be laid upon or beneath long stretches of the sediments such as the continental shelf of the Gulf of Mexico has been given in another paper (Reid, 1951). The purpose here is to expand upon some of the physical problems which are encountered and to present, in summary, the results of theory and techniques which may be useful in the design of a marine pipe line.

Some of the specific questions which arise in connection with the laying of a pipe line offshore are:

(1) What route should be followed in reaching a certain offshore destination?

(2) Can the pipe be laid upon the bottom or must it be buried within the sediments?

(3) If the burial of the pipe line is indicated, what should be the depth of burial?

(4) Will the pipe sink into the sediments; if so, how much sag will be experienced and what will be the stresses induced thereby?

(5) Will support of the pipe in regions of weak sediment be required in order to insure vertical stability of the pipe, either from the standpoint of downward sag due to excessive net weight of pipe or from the standpoint of buckling associated with thermal expansion?

The engineers of the United Gas Pipe Line Company were confronted with problems of this nature in planning the 15 miles of 20.5 inch pipe and 10 miles of 14 inch pipe which has recently been laid within the sediments of the Atchafalaya Bay, Louisiana, and the adjacent Gulf. (A discussion of the preliminary investigation appears in the Petroleum Engineer, March 1951, and the installation of this line is discussed by Paul Reed, 1951.)

Such questions can be answered or at least partially answered by considering the vertical stability of the pipe in the light of the general stratigraphy and strength distribution of the sediments along the path of the pipe line.

VERTICAL STABILITY OF THE PIPE

GENERAL DISCUSSION

Adequate support of a pipe line resting upon or passing through the sediments of a marine environment, such as encountered on the Gulf Shelf, cannot be taken for granted. The conditions of sediment strength and degree of consolidation are considerably different from those which are encountered in the case of ordinary soils. According to the soil mechanics classification given by Terzaghi and Peck (1948), those soils having an unconfined compression strength of greater than 8,000 pounds per square foot are considered extremely stiff and those soils having a strength of less than 500 pounds per square foot are considered very soft. The different degrees of stiffness which make up the classification are contained between these extremes. In comparison, the mean unconfined compressive strength of silty clays and clayey silts encountered in the upper strata of the sediments of the Atchafalaya Bay and adjoining Gulf region, for example, has been found to be approximately 80 pounds per square foot.* Values range from less than 10 to about 250 pounds per square foot. All of these values fall in the very soft category. From the standpoint of pipe line engineering, it appears necessary to refine the classification since the relatively stronger portion of the very soft sediments can adequately support certain pipe lines. As an arbitrary limit those sediments having an unconfined compressive strength of less than 100 pounds per square foot (or a shear strength of less than 50 pounds per square foot) will be referred to hereafter as extremely soft. The extremely soft and very soft silty clays are of recent origin and increase in thickness (from a few feet to about 15 feet) with distance from shore in the Atchafalaya Bay area, forming a wedge of weak deposits resting on top of relatively stronger, and more consolidated, marsh deposits of considerable thickness. Even the latter deposits are soft in terms of the above classification.

This is a greatly oversimplified picture of the stratigraphy. Superimposed on this structure are "pockets" of nearly fluid sediment which apparently extend to depths as great as 10 or 15 feet. These pockets lie principally between regions of hard reef, and consequently represent a situation to be considered with caution because of the possibility of differential sag of the pipe. A route which passes through such zones may demand entrenchment of the pipe to considerable depth in order to avoid the possibility of overstressing in the pipe walls due to sag. Whether or not such sag could be critical depends upon such factors as the net weight and length of the section subject to deformation, the initial tension in the pipe, the strength of the sediments adjacent to the weak zone, and the depth of the weak zone.

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* Leipper, et. al. (1951). Oceanographic Analysis of Marine Pipe Line Problems. Final Report to the United Gas Pipe Line Company.

The sag of a pipe section, having a length of the order of 200 feet or more, introduces a complex problem from the standpoint of computation of induced stresses. One is dealing here with a beam which is so long that when vertical deformation occurs it is accompanied by a significant elongation. The firmer sediment adjacent to the weak zone will tend to restrain the movement of the pipe at the ends of the sagging portion of the pipe so that practically all of the elongation will occur in the sagging section. This can induce a net axial tension of considerable magnitude. The tensile stress thereby induced in the material is in addition to the tensile and compressive flexural stresses induced by the bending of the pipe.

In the case of a very long pipe the bending effect can become so small that the sagging pipe can be considered essentially as a flexible cable. In this case the pipe will assume the shape of a catenary under the action of a uniform load per unit length, with the tensile force carrying the full load. If the pipe section is very short or if the deflection is very small, then the theory of simple bending may apply. In this case the net tension would be negligible and the load is carried entirely by shear forces. The situation regarding pipe sag in the sediments in general, involves both bending stresses and net tension, and the load is carried partially by shear and partially by tension. In order to insure a safe design where sag is likely to occur, it is therefore necessary to compute both flexural and pure tensile stresses induced by the sag.

CRITERION FOR SINKING OF THE PIPE

An offshore pipe line which is resting upon the bottom will exert a downward load on the underlying sediment which is simply the submerged weight of the pipe in water, or absolute weight minus the weight of water displaced by the pipe. In order that static equilibrium exist, the sediment must develop an equal and opposite reaction. There is a maximum reaction which the sediment can exert. This may be referred to as the ultimate load bearing capacity of the sediment. In general the bearing capacity depends not only upon the nature of the sediment but is a function of the applied load distribution as well. Thus a pipe will have an effect on the sediments which differs from that which a flat plate of the same weight would induce. The criterion for sinking of the pipe is that the net downward gravitational load exerted by the pipe is greater than the ultimate load bearing capacity of the sediment.

If the pipe is entrenched within the sediment, the problem of evaluating the net downward gravitational load of the pipe becomes somewhat complex. Evidently the load exerted by the entrenched pipe depends upon the structural nature of the sediment itself. In contrast to a suspension (which represents a dispersion of discrete particles in a fluid), the sediment consists of a continuous network of solid materials which includes water within the interstices of the structure. The solid phase presumably supports its own weight and does not add to the hydrostatic pressure of the water phase as in the case of a suspension. In a fluid sediment, neglecting capillary forces, there will be a buoyant force exerted upon the pipe which will be equal to the weight of water displaced by the pipe. However, since only part of the total volume of sediment is water, the buoyancy will be less than that experienced by a pipe submerged in water alone. The water content of a sediment is generally evaluated in terms of the per cent of the dry mass of sediment. This will be denoted by the symbol Q. The buoyancy per unit volume of the pipe in the sediment, however, is equal to the mass of water per unit volume of the sediment. If B represents the buoyant force per unit volume of the pipe, then

(1) $$B = \frac{\rho_s g}{1 + 100/Q},$$

where ρ_s represents the density of the sediment (i.e., the wet density). As an example, consider a sediment having a specific gravity of 1.4 and a moisture content of 100 per cent of the dry weight. In this case $\rho_s g =$ 87.4 pounds per cubic foot, which leads to a value of B of 43.7 pounds per cubic foot. This represents a buoyant force which is about 70 per cent of that which would be experienced by a pipe submerged in water alone.

If equilibrium is to exist, the sediment must support a greater percentage of the actual weight of the pipe than in the case of a pipe resting upon the bottom. The maximum reaction which the sediment can develop with respect to the pipe will in general depend upon the adhesive property of the sediment with respect to the pipe, the pressure existing at the depth of entrenchment, the shear strength of the sediment, the cohesive property of the sediment, and the size of the pipe. The combined effect of bearing reaction at the bottom of the pipe and adhesion along the sides and top of the pipe, under conditions of maximum restraint, represents the ultimate load bearing capacity in this case.

For silty clay sediments, the ultimate load bearing capacity is evidently independent of the pressure within the sediment, and depends only upon the shear strength and load distribution. If P_b represents the ultimate load bearing capacity per unit length of pipe, D the overall diameter of the protected pipe, and τ_u the ultimate shear strength * of the sediment, then presumably

(2) $$P_b = k_b D \tau_u,$$

* This can be measured directly for a sample of sediment or can be taken as one-half of the ultimate unconfined compressive strength for clayey sediment.

for a silty clay. For the case of a flat strip load of width D acting on a flat surface of clay soil, Terzaghi (1943) gives 5.14 for the factor of proportionality k_b. In the case of a pipe, k_b is probably much smaller than this, judging from the limited information available on sinking of pipes. The circular shape of the pipe evidently leads to a stress concentration in the sediments beneath the pipe which is greater than that experienced in the case of a flat plate. For concrete coated pipe entrenched in silty clay sediment, an approximate value of $k_b = 2$ is indicated from the experience gained in the installation of United's pipe line.

It must be emphasized, however, that further information is needed for establishing the empirical validity of equation (2) as well as evaluating the proportionality factor. At the present time it is not possible to state the exact threshold of equilibrium existing for a pipe loaded sediment. It can be stated, however, that there is a significant probability that sinking will occur if the net load per unit length, exerted by the pipe, exceeds $2D\tau_u$, and little chance that it will not sink if the load exceeds $5D\tau_u$. If sinking is to be avoided the net load should be less than $2D\tau_u$.

SYMMETRICAL SAG OF A PIPE LINE IN PLASTICALLY DEFORMED SEDIMENT

The problem of determining the combined flexural and tensile stresses induced in the case of differential sinking of the pipe line is examined in this section. Two simple end conditions are considered for the sagging section of pipe. In the section which follows, an analysis of the end conditions for a relatively firm (but non rigid) supporting material is made by taking into account the elastic deformation of this material.

If the vertical restraint per unit length, P_m, offered by the plastically deformed sediment is uniformly weak in the zone of pipe sag, and if the conditions of support at the ends of the sagging section are similar, then the vertical deformation of the pipe will be symmetrical with respect to the center of sag.

The net vertical load on the pipe per unit length, w, is simply the net weight of the pipe in the sediment minus the reaction P_m. If w_p represents the weight of pipe per unit length in air (including the weight of transported fluid-gas or petroleum), then

$$(3) \qquad w = w_p - \frac{\pi}{4} D^2 B - P_m ,$$

where B is the buoyancy as already defined. The reaction, P_m, of the distorted sediment is not necessarily the same as the bearing capacity, P_b, of the undisturbed sediment.

The criterion for sinking, however, is that

$$\left(w_p - \frac{\pi}{4} D^2 B\right) > P_b .$$

Thus, if the net weight is great enough, then the sediment deformation exceeds the elastic state and its structure is broken down, reducing the possible reaction which it can develop.

The Theoretical Model

A model is envisaged in which a pipe line spans a pocket of excessively weak sediment of horizontal distance l_o along the pipe, resulting in a net downward force w on the pipe over the length l_o . The weak material is homogeneous from the standpoint of maximum restraint, and is of sufficient depth that the point of maximum sag of the pipe does not reach a layer of strong sediment below. Furthermore, conditions of elastic flexural deformation and elongation are presumed, such that the amount of sag is very small compared to l_o and hence the slope of the sagging section is very much less than unity. The assumptions regarding the loading of the pipe and the elastic theory are summarized below:

(1) The net downward force per unit length, w , acting on the pipe is uniform along the pipe and essentially normal to the pipe.
(2) The downward force is independent of the vertical deformation of the pipe.
(3) The end conditions are the same at each end of the sagging section, such that the sag is symmetrical with respect to the point of maximum sag.
(4) The tension due to the axial elongation of the pipe is uniform throughout the entire length of the sagging section of the pipe.
(5) Plane transverse sections of the pipe remain plane after combined bending and extension of the pipe.
(6) The modulus of elasticity in tension is the same as that in compression for the pipe material.
(7) The proportional elastic limit of the pipe material is not exceeded.
(8) The axis of the pipe is initially straight.
(9) The slope of the sagging pipe is so small that the rate of change of the slope per unit length of pipe represents the curvature of the pipe.
(10) The pipe is of uniform cross section.

The assumption of negligible tension cannot be made for a sagging pipe line, as is done in the case of a simple beam, because of the magnitude of the deflection involved. If the maximum deflection in a fixed end beam is of the order of magnitude or greater than the width of the beam, then the restoring moment associated with the induced tension becomes appreciable. * In order to keep the complexity of the problem at a minimum, the assumption (4) is made. This appears reasonable provided that the tension is large compared with the limited amount of longitudinal restraint provided by the sediments in the weak zone.

Probably the most severe restrictions regarding the application of the theory are (1), (2) and (3), and to a less extent (4). In applying the theory to a real situation one must keep these assumptions in mind. Examples illustrating the use of the theory are given at the end of Part I, and certain modifications in the application of the theory are discussed.

The Basic Equations of Combined Flexure and Elongation in a Pipe

A schematic diagram of the sagging section of pipe is shown in Figure 1B, and the equilibrium of forces and moments is represented graphically by Figure 1C. The origin of the coordinate system is taken at the point of maximum sag; x represents the horizontal distance measured positively to the right of this point, and y is the vertical distance measured positively upward from this point. This allows a convenient form for the equations governing the deflection, in view of the fact that symmetrical sag is considered. The bending moment at the origin is denoted by M_o, and the total axial tension after deformation of the pipe is represented by N. An initial tension may exist in the pipe line, due to thermal or pressure effects within the pipe. This is denoted by N_o, and is represented graphically in Figure 1A. The shear at the center of sag is zero for symmetrical sag. In view of assumptions (1) to (10), the equation representing the balance of moments about point A in Figure 1C is

$$M = EI \frac{d^2y}{dx^2} = M_o - \frac{wx^2}{2} + Ny, \tag{4}$$

where M represents the bending moment within the pipe at section A of distance x from the origin. The quantity E is the modulus of elasticity in tension (or compression) of the pipe material, and I

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* Usually the assumption (7) would be transcended in a short beam before the restriction involved here would govern; this however depends upon the flexibility of the beam.

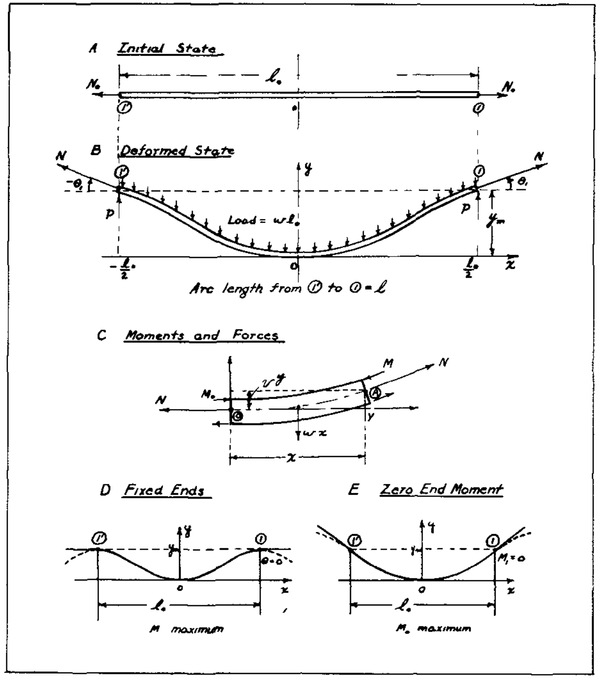

Fig. 1. Schematic diagrams of symmetrical, free sag in a pipe: (A) initial state, (B) deformed state, (C) balance of forces and moments in a section of the pipe, (D) illustration of fixed end condition, (E) illustration of condition of ends free to turn.

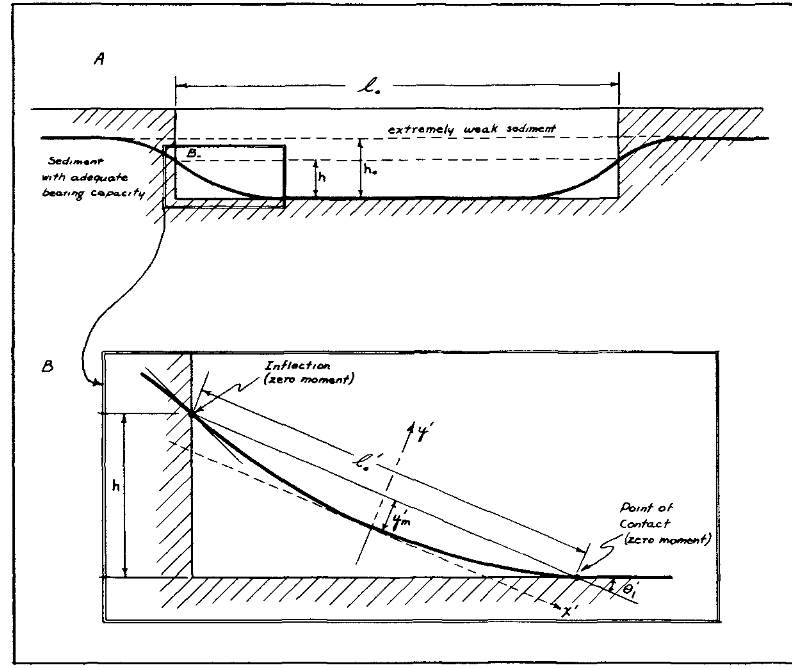

Fig. 2. Schematic diagram of restricted sag, where a portion of sagging section is supported by firmer sediment at depth h_0 below original level of pipe. Equivalent free sag for a portion of the pipe illustrated in diagram B.

is the area moment of inertia of the cross-section. The latter can be expressed as follows:

(5) $$I = A_s r^2,$$

where A_s is the cross-sectional area of the steel in the pipe, and r is the radius of gyration of the cross-section of steel, taken about the neutral axis.

The balance of vertical forces between O and A is given by

(6) $$V = \frac{dM}{dx} = -wx + N\frac{dy}{dx}$$

where V is the shear force at section A. Consequently the shear at the end of the sagging section is

(7) $$V_1 = -\frac{1}{2}w l_o + N\theta_1 ,$$

where θ_1 is the end slope at $x = l_o/2$ (the negative of that at $x = -l_o/2$). Because of the small values to which the quantity θ is restricted, it represents therefore the angle (in radians) between the pipe and the horizontal plane. It will be noted from this that the shear forces at each end carry only part of the total net weight of the sagging pipe, unless the ends are held rigid and the angle θ is zero. The total net weight wl_o, of course, must ultimately be sustained by the vertical reaction of the supporting material at each end of the sagging section.

The General Solution of the Equations for Symmetrical Sag

The solution of (4) for the vertical deflection of the pipe is

(8) $$y = \frac{1}{N}\left\{(M_o - w\lambda^2)\left[\left(\cosh\frac{x}{\lambda}\right) - 1\right] + \frac{1}{2}wx^2\right\},$$

where λ is a characteristic length defined by

(9) $$\lambda = \sqrt{\frac{EI}{N}} .$$

The conditions $y = 0$ and $dy/dx = 0$ at $x = 0$ are employed in arriving at equation (8).

The expressions for the slope, bending moment and shear at any point in the sagging pipe can be derived from equation (8) as follows:

$$\theta \equiv \frac{dy}{dx} = \frac{1}{N}\left\{\frac{1}{\lambda}\left(M_o - w\lambda^2\right)\left(\sinh x/\lambda\right) + wx\right\}, \tag{10}$$

$$M = \left(M_o - w\lambda^2\right)\left(\cosh x/\lambda\right) + w\lambda^2, \tag{11}$$

and

$$V = \frac{1}{\lambda}\left(M_o - w\lambda^2\right)\sinh x/\lambda . \tag{12}$$

It can be shown furthermore that in the limit (8) reduces to

$$y = \frac{1}{EI}\left(M_o \frac{x^2}{2} - w\frac{x^4}{24}\right), \tag{8a}$$

when N is extremely small. This equation, representing a special case of the more general relation (8), is that which the simple theory of flexure yields.

On the other hand, if N is very large then V is small and (8) reduces to

$$y = \frac{wx^2}{2N}, \tag{8b}$$

which is the approximate form of catenary sag associated with tension N. These two limiting cases serve as checks on the more general theory.

<u>Application of Hooke's Law for Evaluation of the Tension</u>

Since the tension is one of the sought variables of the problem, an additional equation involving N is necessary in order to make the solution unique. This can be established by applying Hooke's law to the over-all extension of the sagging pipe section. If ℓ represents the length of the pipe section between points (1') and (1) after vertical deformation has occurred, and ℓ_i is the initial length between the same points in the pipe prior to deformation, then the overall strain is given by

$$\frac{\ell - \ell_i}{\ell_i} = \frac{N - N_o}{A_s E}, \tag{13}$$

where $N - N_o$ is the increase in tension due to the longitudinal strain induced by the sag. The length of the deformed section can be found from the approximate expression:

$$\ell = \ell_o + \int_0^{\ell_o/2} \theta^2 dx, \tag{14}$$

which is quite valid as long as $|\theta| << 1$.

Longitudinal Slippage at the Ends of the Pipe.

The quantity ℓ_i is not necessarily the same as ℓ_o, because if longitudinal slippage of the pipe occurs at the ends, then the original length of the section between points (1') and (1) will be greater than ℓ_o and the resulting tension in the pipe will be lower than that for the case of no slippage. The amount of slippage will depend upon the longitudinal restraint offered by the stronger sediment adjacent to the zone in which sag occurs. If the sediment exerting this restraining force is perfectly rigid then no slippage will occur and ℓ_i will equal ℓ_o. If, on the other hand, the sediment at the ends of the pipe offers very little restraint, then ℓ_i will be nearly the same as ℓ and the resulting value of $(N - N_o)$ will be small.

The amount of slippage at each end of the sagging section is $(\ell_i - \ell_o)/2$. This slippage is proportional to the increase in tension at the ends of the sagging section. The slippage is also proportional to the effective length of pipe, adjacent to the sagging section, which undergoes elongation. This effective length is determined the final balance existing between the longitudinal restraint exerted by the surrounding sediments, and the increase in tension $(N - N_o)$ at the ends of the sagging section. From these considerations it can be shown that the following approximation is applicable

$$\ell_i - \ell_o = \frac{(N - N_o)^2}{f_r A_s E}, \tag{15}$$

where f_r represents the maximum longitudinal restraining force per unit length of pipe offered by the sediments adjacent to the weak zone.

If the pipe is buried in the sediments in the adjacent sections, then f_r can be expressed in terms of the ultimate shear strength, τ_u', of the relatively strong sediment as follows:

$$f_r = \pi D \tau_u', \tag{16}$$

where it is assumed that shear occurs at or near the surface of the pipe of overall diameter D.

For a pipe lying on the bottom f_r is equal to the coefficient of friction between the pipe and bottom multiplied by the submerged weight of pipe per unit length.

The Characteristic Dimensionless Parameters

In order to make the functional relationships existing between the basic variables of the problem as simple as possible it is convenient to introduce the dimensionless parameters given in the Table I.

Table I

Eq.	Name	Symbol		Definition
(17)	Bending moment factor	m	=	$\frac{M}{w l_o^2}$
(18)	End shear factor	p	=	$\frac{V_1}{w l_o}$
(19)	Tension factor	n^2	=	$\frac{N l_o^2}{EI}$
(20)	Flexibility parameter	q	=	$\frac{w l_o^4}{EIr}$ *

The definition (17) can be applied to the moments at the middle and at the ends. The quantity

$$m_o = \frac{M_o}{w l_o^2} \tag{21}$$

* The quantity r is the radius of gyration of the cross section as defined previously and equals $\sqrt{I/A_s}$.

is the bending moment factor at the point of maximum sag, and

$$(22) \qquad m_1 = \frac{M_1}{w \, l_0^2}$$

is the bending moment factor at the ends of the sagging section.

Likewise, as a special case of (19):

$$(23) \qquad n_0^2 = \frac{N_0 \, l_0^2}{EI},$$

which is the intial tension factor.

By making use of equations (7) to (11), the following relations can be established:

$$(24) \qquad \varphi = \frac{1}{2} - \frac{n^2}{q} \cdot \frac{l_0}{r} \theta_1 ,$$

$$(25) \qquad m_0 = \frac{-\varphi}{n \sinh n/2} + \frac{1}{n^2} ,$$

$$(26) \qquad m_1 = \frac{-\varphi}{n \tanh n/2} + \frac{1}{n^2} ,$$

and

$$(27) \qquad \frac{y_m}{r} = \frac{q}{n^2}\left[\frac{1}{8} + \varphi \, \frac{1 - \cosh n/2}{n \sinh n/2}\right],$$

where y_m is the maximum vertical deflection or simply the sag. Special forms of these expressions are given below for the two commonly visualized end conditions.

Case I: Rigid Ends with Zero Slope

This condition is illustrated schematically in Figure 1D and represents the situation for which $\theta_1 = 0$. In this case the equations (24) to (27) take the form:

$$(24a) \qquad \varphi = \frac{1}{2} ,$$

(25a) $$m_0 = -\frac{1}{2n \sinh n/2} + \frac{1}{n^2},$$

(26a) $$m_1 = -\frac{1}{2n \tanh n/2} + \frac{1}{n^2},$$

and

(27a) $$\frac{y_m}{r} = \frac{q}{n^2}\left[\frac{1}{8} + \frac{1-\cosh n/2}{2n \sinh n/2}\right].$$

The maximum bending moment factor (and hence the maximum bending moment) occurs at the ends for this condition and is given by m_1 .

Case II: Ends Free to Turn

In this case $m_1 = 0$, which implies a maximum slope (or inflection) at the ends of the sagging section. For this condition:

(24b) $$\frac{l_0}{r}\theta_1 = \frac{q}{n^2}\left(\frac{1}{2} - \frac{\tanh n/2}{n}\right) * ,$$

(25b) $$m_0 = \frac{1}{n^2}\left(1 - \frac{1}{\cosh n/2}\right),$$

(26b) $$p = \frac{1}{n}\tanh n/2,$$

and

(27b) $$\frac{y_m}{r} = \frac{q}{n^2}\left[\frac{1}{8} + \frac{1-\cosh n/2}{n^2 \cosh n/2}\right].$$

In this case the maximum bending moment occurs at the center of sag.

The Relation between the Tension Factor and Flexibility

It should be noted from the relations above that the bending moment factors m_0 and m_1 and also the shear factor p are fully

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* Note that l_0/r represents the slenderness ratio parameter, which is the critical variable in the stability theory of columns.

determined by the tension factor n^2. This factor must be determined from the flexibility parameter q, the end conditions, and the initial tension. Equations (10), (13), (14), (15) together with (25a) or (25b) yield the additional relations required:

(28) $$q = k_1 k_2 f(n),$$

where

(29) $$k_1 = \sqrt{1 - n_o^2/n^2},$$

(30) $$k_2 = \sqrt{1 + s\, q^{-3/4}(n^2 - n_o^2)},$$

and

(31) $$s = \frac{w}{f_r}\left(\frac{E A_s}{w r}\right)^{1/4} = \frac{w}{f_r}\left(\frac{E I}{w r^3}\right)^{1/4}.$$

The function $f(n)$ depends upon the end condition.* For case I ($\theta_1 = 0$):

(32) $$f(n) = f_1(n) = \sqrt{24}\, n^4 \left\{ n^2 - 9n + 24 + 3n\left[\frac{n + 3(1 - e^{-n})}{1 - \cosh n}\right]\right\}^{-1/2},$$

while for case II ($m_1 = 0$):

(33) $$f(n) = f_2(n) = \sqrt{24}\, n^4 \left\{ n^2 - 24 + \frac{12}{n}\left(\frac{n - 5\sinh n}{1 + \cosh n}\right)\right\}^{-1/2}.$$

The quantity s is a dimensionless parameter which may be referred to as the slippage coefficient. The slippage is large when the restraining force f_r is small compared with w. If the ends of the pipe are held so that no slippage occurs, then $k_2 = 1$.

If both s and n_o are zero then $k_1 = k_2 = 1$ and (28) reduces to

(28a) $$q = f_1(n) \quad \text{for} \quad \theta_1 = 0,$$

or

(28b) $$q = f_2(n) \quad \text{for} \quad m_1 = 0.$$

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* See Table VIII in the Appendix for tabulated values of $f(n)$ computed for different values of n ranging from .01 to 1,000.

The function $f_1(n)$ [i.e., q for $n_o^2 = 0$, $s = 0$, $\theta_1 = 0$] is shown graphically in Figure 3 by the dashed curve. The function $f_2(n)$ is shown graphically by the full curve labeled $n_o^2 = 0$ in the same graph. By means of this graph it is readily possible to determine n^2 corresponding to a given value of q for the case of no initial tension and no slippage. Curves of n^2 versus q corresponding to four different finite values of n_o^2 are also shown in Figure 3 for the end condition $m_1 = 0$.

The expressions for $f(n)$ in equations (32) and (33) reduce to simple forms for the limiting conditions of very small and very large values of n :

$$(35a,b) \quad \lim_{n\to 0} \frac{f_1(n)}{n} = \sqrt{\frac{9!}{6}} \doteq 246 ; \qquad \lim_{n\to 0} \frac{f_2(n)}{n} = \sqrt{\frac{8!}{17}} \doteq 48.7 ;$$

and

$$(36) \qquad \lim_{n\to\infty} \frac{f_1(n)}{n^3} = \lim_{n\to\infty} \frac{f_2(n)}{n^3} = \sqrt{24} \doteq 4.90 .$$

Thus for the case of no end slippage and no initial tension

$$(37) \qquad n^2 = \frac{q^{2/3}}{(24)^{1/3}} ,$$

for large values of q . Equation (37) is a good approximation if:

$$q > 10^6 , \qquad \text{for} \quad \theta_1 = 0 ;$$

or

$$q > 10^4 , \qquad \text{for} \quad m_1 = 0 .$$

Physically this means that for sufficiently large values of q, the pipe acts essentially as a flexible cable. Equation (37) can be put in the form

$$(37a) \qquad N = \left[\frac{E A_s}{24} \left(w l_o \right)^2 \right]^{1/3} ,$$

which is the expression for the tension in a flexible cable of length l_o , provided that the sag is very small relative to l_o.

If slippage or initial tension is appreciable, or if q is not sufficiently large, then the general expressions for $f(n)$ must be used and the factors k_1 and k_2 must be taken into consideration. A more complete graph could be constructed so that n^2 could be readily determined from q, n_0^2 and s. Lacking such a complete graph, the evaluation of n^2 can be carried out by successive approximation without too much difficulty (see examples illustrating this procedure at the end of part I).

It will be noted from Figure 3 that the scales for n^2 and q^2 are logarithmic, allowing for a wide range of values in both parameters. The general curves approach the limiting asymptotic relations for very large or very small values of q. The limiting asymptotes appear as straight lines on the graph, thus making it a simple matter to extrapolate the curves for values lying outside the range of the graph.

Determination of the Bending Moment Factor

For the end condition of case I m_1 is of prime concern, while for case II m_0 is the important factor for determination of the maximum bending moment in the pipe. As indicated above, these factors are functions of n^2 only and are represented graphically in Figure 4. The factor m_1 corresponding to $\theta_1 = 0$ is represented by the dashed curve, while m_0 for the case of $m_1 = 0$ is represented by the full curve.

The limiting expressions for these functions can be obtained from equations (26a) and (25b); see Table II below:

Table II
Limiting values of m_0 and m_1

Case I ($\theta_1 = 0$)		Case II ($m_1 = 0$)	
Value of m_1	Condition	Value of m_0	Condition
$-\frac{1}{12}$	$n^2 < 3$	$\frac{1}{8}$	$n^2 < 5$
$-\frac{1}{2n}$	$n^2 > 1000$	$\frac{1}{n^2}$	$n^2 > 500$

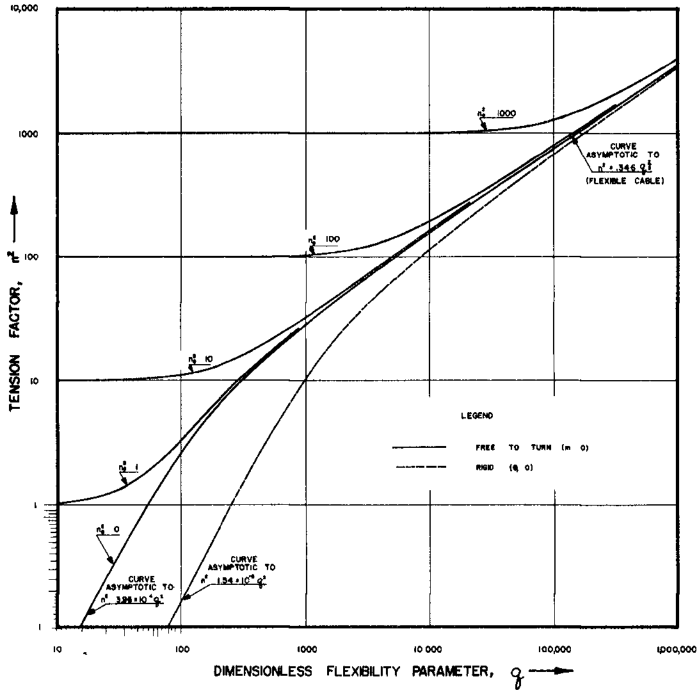


Fig. 3. Dimensionless tension factor n^2, as a function of the dimensionless flexibility parameter, for different values of initial tension as indicated, and for two different end conditions. All curves apply to the case of no end slippage ($\mathcal{S} = 0$).

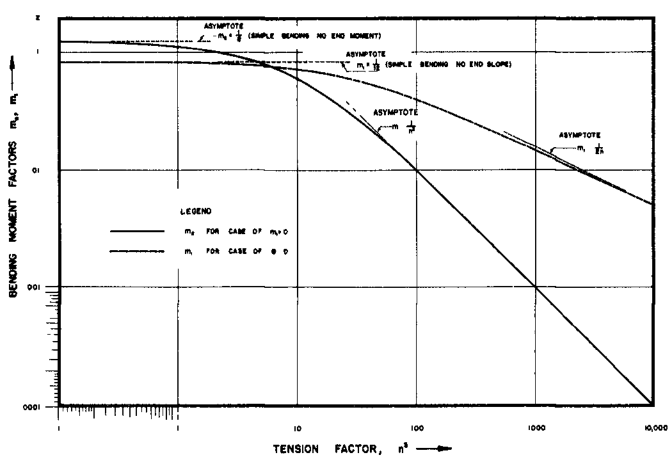


Fig. 4. Maximum bending moment factors, m_o and m_1 (m_o, for the case of no end moment, and m_1 for the case of zero end slope) as functions of the tension factor, n^2.

The limiting expressions for M_1 and M_o in the case of small tension are consequently:

(38) $$M_1 = -\frac{1}{12} w l_o^2 , \quad \text{for } \theta_1 = 0,$$

and

(39) $$M_o = \frac{1}{8} w l_o^2 , \quad \text{for } M_1 = 0,$$

which are the same expressions obtained from the theory of simple bending. For very large tension, on the other hand, the bending moments are a very small fraction of $w l_o^2$. In case II, the tension becomes the prime factor in the determination of the stress in the pipe, when n^2 is large.

Figure 5 shows the bending moment factors as a function of the flexibility for the case of no end slippage. The dashed curve represents m_1 versus q for $\theta_1 = 0$ and $n_o^2 = 0$. The full curves represent m_o versus q for the different values of n_o^2 indicated, for the condition $m_1 = 0$.

Practical Forms of the Equations for Slope and Relative Sag

The equations (27a), (24b) and (27b) can be simplified by use of (28). The formula for the slope in case II becomes

(40) $$\frac{l_o}{r} \theta_1 = k_1 k_2 k_3 n ,$$

where k_1 and k_2 are the same as defined in (29) and (30) and k_3 is a coefficient which depends upon n. It can be shown that for the entire range of n:

$$(\text{small } n) \sqrt{\frac{70}{17}} < k_3 < \sqrt{6} \quad (\text{large } n).$$

The average value of k_3 is about 2.24, and the following approximation will yield values of θ_1 which are never more than 10 per cent in error:

(40a) $$\theta_1 \doteq 2.24\, k_1 k_2 \frac{r}{l_o} n , \quad (m_1 = 0).$$

The formula for the relative sag can be simplified in a similar manner:

(41) $$\frac{y_m}{r} = k_1 k_2 k_4 n.$$

The coefficient k_4 is determined by n, however for the entire range of n;

$$(\text{large } n) \qquad \sqrt{3/8} < k_4 < \sqrt{105/256} \qquad (\text{small } n).$$

The average value of k_4 is about 0.626 for either of the end conditions examined here. This means that the following approximation will yield values of y_m which are accurate to within about 2 per cent:

(41a) $$y_m \doteq 0.626\, k_1 k_2\, r\, n, \quad \begin{Bmatrix} \theta_1 = 0 \\ \text{or} \\ m_1 = 0 \end{Bmatrix}.$$

It will be noted that

$$k_1 n = \sqrt{n^2 - n_o^2},$$

so that both θ_1 and y_m are proportional to the square root of the increase in tension in the pipe due to sag. Furthermore, if there is no end slippage the coefficient k_2 is unity.

For small values of q the equations (27a) and (27b) reduce to the forms:

(42a) $$y_m = \frac{1}{384}\,\frac{w l_o^4}{EI}, \quad \text{for } \theta_1 = 0;$$

and

(42b) $$y_m = \frac{5}{384}\,\frac{w l_o^4}{EI}, \quad \text{for } m_1 = 0,$$

provided that there is no initial tension. These equations are also obtained from the theory of simple bending.

Limit of Application of the Simple Theory of Bending

The simple theory of bending of a beam under the action of a uniform load per unit length is seen to be a limiting case of the theory

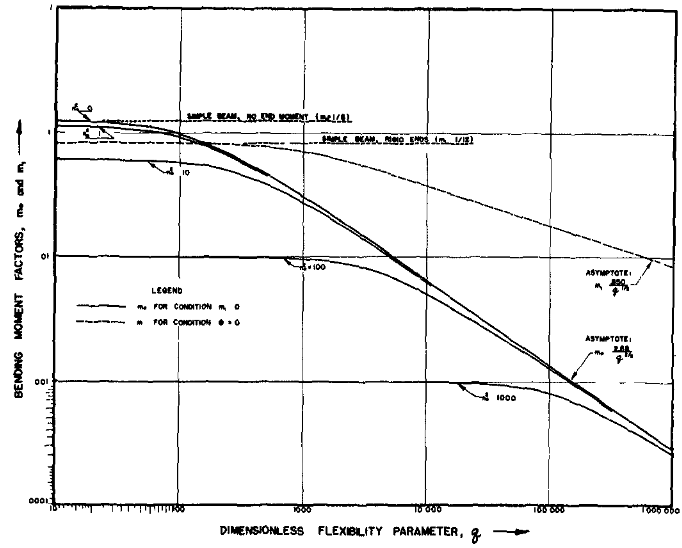


Fig. 5. Maximum bending moment factors (m_0 and m_1) as functions of the flexibility parameter q, for different values of initial tension factor, n_0^2, as indicated. All curves constructed for the case of no end slippage ($\Delta = 0$).

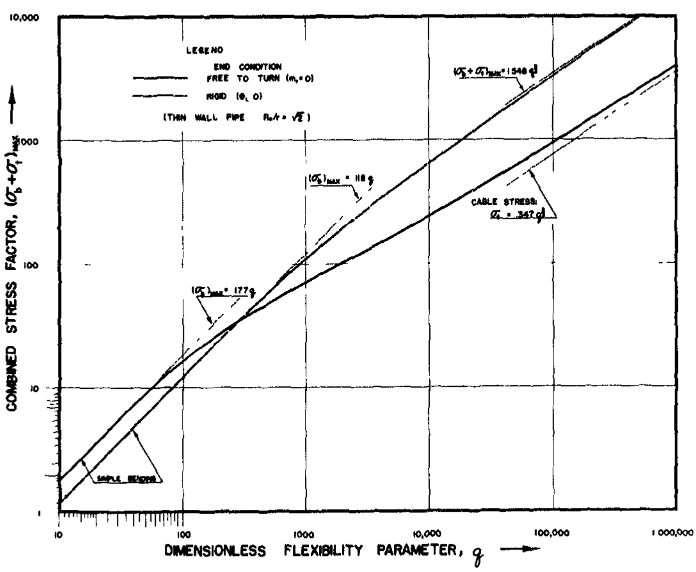


Fig. 6. Maximum combined stress factors, for the two indicated end conditions, as functions of the flexibility parameter q. Both curves apply to the case of no end slippage and no initial tension for a pipe with relatively thin walls compared to its diameter. Limiting relations for very small q and very large q indicated by light dashed lines.

of combined flexure and elongation given here. The question of whether or not the generalized theory is required for a solution to the problem depends upon the value of the dimensionless flexibility parameter and the initial tension factor.

It is evident from Figure 5 that when an initial tension corresponding to a value of n_o^2 greater than unity exists, then the simple theory of bending is not valid. Furthermore, Figure 5 indicates that, for $n_o^2 = 0$, there is also a practical limit of flexibility beyond which the simple theory of bending becomes invalid. For a 10 per cent tolerance of error in the simple theory, the upper limit of flexibility (q_c) in the application of the simple theory is:

(43a) $$q_c = 700, \text{ for } \theta_1 = 0 \quad \text{(rigid ends)},$$

or

(43b) $$q_c = 50, \text{ for } m_1 = 0 \quad \text{(ends free to turn)}.$$

From equations (42a) and (42b) and the critical limits of q given above, it is evident that the upper limits of the relative deflection for which the simple theory is valid are:

(44a) $$\left(\frac{y_m}{r}\right)_c \doteq 1.8, \text{ for } \theta_1 = 0,$$

and

(44b) $$\left(\frac{y_m}{r}\right)_c \doteq 0.65, \text{ for } m_1 = 0,$$

That is, if the simple theory is to apply, then the maximum deflection must be less than the order of magnitude of the radius of gyration of the cross section.

It is possible, for given pipe specifications, to interpret the above criteria, for the allowable use of simple bending theory, in terms of the length ($\ell_o)_c$ for different values of net loading w . Table III gives this information for two different pipe sizes, and for values of net load from 1 lb. per ft. to an extreme value of 1000 lbs. per ft. The load is governed by the weight of pipe (including ballast, if any), the weight of fluid in the pipe, and the characteristics of the sediment as discussed above, and is not necessarily governed by the pipe specifications alone. The stress S_b induced by simple bending under the conditions stated is also included in Table III to indicate that, in most of the situations represented, this is not the governing factor so far as the validity of the simple theory is concerned. In other words, taking the 20.5 inch pipe with $w = 10$ lbs. per ft. as an example, it is apparent that there is considerable latitude for increase in ℓ_o beyond the critical value of 210 feet given for case II, so far as the stress is concerned. However, the generalized theory must be used in order to determine the sag characteristics for ℓ_o beyond this value. It can be shown, in fact, that for sufficient length this pipe would fail essentially in elongation rather than in bending, under conditions of free sag with the load of 10 lbs. per ft.

Table III

Critical values of pipe length #, corresponding to different net loads per unit length, beyond which simple bending theory is invalid; and bending stress corresponding to these conditions

(a) 10 inch O.D. steel pipe: (1/2 inch walls) $I = 169$ in.^{4} $A_s = 14.9$ in.^{2} $r = 3.36$ in.				
	Case I: $\theta_1 = 0$		Case II: $m_1 = 0$	
w lb./ft.	$(\ell_o)_c$ ft.	S_b psi	$(\ell_o)_c$ ft.	S_b psi
1	265	2,080	149	985
10	149	6,580	84	3,130
100	84	20,900	47	9,800
1000	47	(60,400)*	26	30,000
(b) 20.5 inch O.D. steel pipe: (3/4 inch walls) $I = 2270$ in.^{4} $A_s = 46.5$ in.^{2} $r = 6.98$ in.				
	Case I: $\theta_1 = 0$		Case II: $m_1 = 0$	
w lb./ft.	$(\ell_o)_c$ ft.	S_b psi	$(\ell_o)_c$ ft.	S_b psi
1	664	2,000	372	938
10	372	6,250	210	2,980
100	210	20,000	118	9,440
1000	118	(63,100)*	66	29,500

\# Based upon the assumption that there is no initial tension; E taken as 30×10^6 psi.

* Values in parenthesis represent stresses beyond the endurance limit for ordinary steels.

Induced Stresses

The bending moment induces a non-uniform normal stress across the section of the pipe. The maximum value of this stress, S_b , occurs in the material farthest from the neutral surface of bending, and is given by

$$S_b = \frac{|M| R_o}{I} = \frac{|m| w l_o^2 R_o}{I}, \tag{45}$$

where R_o is the outside radius of the steel pipe. The bending stress is zero at the neutral surface of bending and varies from $-S_b$ at the concave side to $+S_b$ at the convex side of bending. The tension induced by axial elongation of the pipe gives rise to a uniform stress, S_t , given by

$$S_t = \frac{N}{A_s} = \frac{n^2 EI}{A_s l_o^2} . \tag{46}$$

Thermal stress associated with restraint of axial elongation or contraction of the pipe is included in this term, since in determining N thermal effects must be taken into account in the term N_o .

The vertical shearing force gives rise to a non-uniform shear stress at each section of the pipe. The mean value of this shear stress is

$$\overline{S_s} = \frac{V}{A_s} = \frac{p w l_o}{A_s} . \tag{47}$$

The transverse shear stress varies from zero in the steel farthest from the neutral surface to a maximum at the neutral surface. For a pipe of standard wall thickness relative to the diameter, the maximum shear stress is approximately $2\overline{S_s}$.

The fluid pressure within the pipe will give rise to still another stress due to the circumferential elongation of the pipe. This stress is called the hoop stress, S_h , and is a uniform normal stress which is perpendicular to the normal stresses induced by axial elongation and bending. If Δp represents the difference in pressure between the inside and outside of the pipe, then

$$S_h = \frac{D_i}{D_o - D_i} \Delta p , \tag{48}$$

where D_o and D_i are the outside and inside diameters of the pipe, respectively.

The formula, for the most severe combined stress at a given section of the pipe depends upon the value of the criterion parameter $4\,\overline{S}_s^{\,2}/S_b$. Table IV gives the appropriate expression for the governing stress (either $S_{t,m}$ or $S_{s,m}$) for three different conditions imposed upon the criterion parameter.

TABLE IV

The Governing Combined Stress
and the Criterion for its Choice

Condition	Governing Combined Stress
(A) $0 \leq \dfrac{4\overline{S}_s^{\,2}}{S_b} < S_1$	$S_{t,m} = S_b + S_t$
(B) $S_2 > \dfrac{4\overline{S}_s^{\,2}}{S_b} > S_1$	$S_{t,m} = \frac{1}{2}\left\{(S_t+S_h) + \sqrt{(S_t-S_h)^2 + 1\ldots}\right.$
(C) $S_2 < \dfrac{4\overline{S}_s^{\,2}}{S_b} > S_1$	$S_{s,m} = \frac{1}{2}\sqrt{(S_t-S_h)^2 + 16\overline{S}_s^{\,2}}$
where $S_1 = (S_b+S_t) - S_h$; $S_2 = \dfrac{S_t\,S_h}{S_b}$	

It will be noted from Table IV that condition (C) implies that the maximum shear stress $S_{s,m}$ governs. This presumes that the yield limit of stress in shear is just half that in tension for the pipe steel.

Under conditions (A) the pipe would fail in tension at a point farthest from the neutral surface, provided that $S_{t,m}$ were great enough. Under condition (B) the pipe would fail in tension at the neutral surface, along a plane which forms an angle of less than 90° with the neutral surface. Under condition (C) failure, if it occurred, would manifest itself by shear at the neutral surface, along a plane which forms an angle of less than 90° with the neutral surface.

The position along the pipe line at which the maximum stress occurs depends upon the end conditions. The stress S_b is a maximum at the ends of the sagging section for the case of zero end slope; while in the case of ends free to turn, the maximum value of S_b would occur at the center of sag. The stress $\overline{S_s}$, on the other hand, is a maximum at the ends in both cases, but its magnitude depends upon the end condition. The stresses S_t and S_h are presumed to be independent of position along the pipe.

For practical purposes, the condition (A) can be presumed for nearly all cases of pipe sag, and the governing stress therefore is

(49)
$$S_{t,m} = S_b + S_t ,$$

where S_b is the value occurring at the position of maximum flexure. The validity of this assumption, however, can be checked by computing $4(\overline{S}_s)^2/S_b$. This must be less than the value of $(S_b + S_t) - S_h$, otherwise $S_b + S_t$ is not the maximum combined stress.

The Dimensionless Stress Parameters

It is convenient to introduce the following dimensionless stress parameters:

(50)
$$\sigma_b = \left(\frac{l_o}{r}\right)^2 \frac{S_b}{E} ,$$

and

(51)
$$\sigma_t = \left(\frac{l_o}{r}\right)^2 \frac{S_t}{E} .$$

From equations (45) and (46) it can be shown that the stress parameters are related to the characteristic parameters m, n, and q as follows :

(52)
$$\sigma_b = \frac{R_o}{r} m q ,$$

and

(53)
$$\sigma_t = n^2 .$$

The maximum combined stress for condition (A) is therefore

$$(54) \qquad S_b + S_t = \left(\frac{r}{l_o}\right)^2 E\,(\sigma_b + \sigma_t) = \left(\frac{r}{l_o}\right)^2 E\left(\frac{R_o}{r}\, m q + n^2\right).$$

The quantity $(\sigma_b + \sigma_t)$ can be shown graphically as a function of q for given values of n_o^2, Δ, and R_o/r.

For all practical purposes, the value of R_o/r for most pipes may be taken as $\sqrt{2}$. This is theoretically correct for a circular pipe with thin walls. However, if greater accuracy is desired, the following relation can be used:

$$(55) \qquad \frac{R_o}{r} = \frac{2}{\sqrt{1 + (R_i/R_o)^2}},$$

where R_i is the inside radius.

Figure 6 has been constructed using $R_o/r = \sqrt{2}$, for the case of $n_o^2 = 0$ and $\Delta = 0$. The value of the combined stress factor $(\sigma_b + \sigma_t)$ given as a function q corresponds to the combined stress at the point of maximum bending in the sagging pipe. Curves for the two investigated end conditions, $\Theta_1 = 0$ and $m_1 = 0$, are shown. In the case of ends free to turn, the stress factor approaches that for simple bending ($\sigma_t = 0$) for very small values of q. For very large values of q, on the other hand, the combined stress factor approaches that corresponding to the stress in a flexible cable.

In the case of $\Theta_1 = 0$, the curve approaches that for simple bending, for low values of q. However, at high values of q the values of $(\sigma_b + \sigma_t)$ are considerably greater than the corresponding values for the case of $m_1 = 0$.

In the general problem for which $n_o^2 \neq 0$ and $\Delta \neq 0$, it is necessary to make use of the relations (52) and (53) in order to compute the combined stress factor. The value of m to be used in relation (52) is the maximum factor for the particular end condition.

Table IV gives the limiting expressions for σ_b, σ_t, and $(\sigma_b + \sigma_t)$ for the case of no initial tension and no end slippage.

Table V

Limiting expressions for the bending and the tensile stress factors for the case of $n_o^2 = 0$ and $s = 0$

End Condition	Stress factors		Stress factors and combined stress factor for a pipe with thin walls $R_o/r = \sqrt{2}$			Values of q for which limiting relation applies
	σ_b	σ_t	σ_b	σ_t	$\sigma_b + \sigma_t$	
Case I ($\theta_1 = 0$)	$\frac{1}{12}\frac{R_o}{r}q$	$\frac{q^2}{60,480}$	$.118q$	$1.65\times10^{-5} \times q^2$	$.118q$	$q < 1000$
	$\left(\frac{3}{8}\right)^{1/2}\frac{R_o}{r}q^{2/3}$	$\frac{q^{2/3}}{(24)^{1/3}}$	$1.201q^{2/3}$	$.347q^{2/3}$	$1.548q^{2/3}$	$q > 10^6$
Case II ($m_1 = 0$)	$\frac{1}{8}\frac{R_o}{r}q$	$\frac{17q^2}{40,320}$	$.177q$	$4.21\times10^{-4} \times q^2$	$.177q$	$q < 100$
	$\frac{(24)^{1/3}R_o}{r}q^{1/3}$	$\frac{q^{2/3}}{(24)^{1/3}}$	$4.08q^{1/3}$	$.347q^{2/3}$	$.347q^{2/3}$	$q > 10^7$

Resumé of Theory of Free Sag

The independent dimensionless variables in the problem of pipe sag are the flexibility parameter q, the initial tension factor n_o^2, and the end slippage coefficient s. Given these, the factor n^2 can be ascertained, and consequently m_o, m_1, p, θ, l_o/r, y_m/r, σ_t, σ_b, and $\sigma_t + \sigma_b$ can be evaluated for a given end condition. The basic quantities sought can then be found from the dimensionless parameters by applying the definitions of these parameters given by relations (17) to (23) together with (50) and (51). The basic physical quantities which must be known in order to determine the actual tension, bending moments, etc., are summarized in Table VI.

Table VI

Category	Physical Quantity	Units in the ft.-lb.-sec. system
loading factors	N_o	lbs.
	w	lbs. per ft.
	f_r	lbs. per ft.
span of weak zone	ℓ_o	ft.
pipe specification factors	E	lbs. per sq. ft.
	I	$ft.^4$
	A_s	$ft.^2$

The problem is thus formally solved, but only for the two specific end conditions chosen, and subject to the restrictions implied in the basic assumptions (principally numbers 1, 2, 9 and 10).

The theory of pipe sag presented above is not necessarily restricted to the case of sag into weak sediments, but may apply to a wide class of situations encountered in the installation of the pipe line. The situations encountered in nature may be classified as either simple or complex from the standpoint of the application of theory. The problem falls in the simple category if the conditions are such that the assumptions in the theory are fulfilled for all practical purposes. If this is not the case, then the problem is complex. However, it may be possible, by proper separation of the problem into various parts, to apply the theory in modified form. Such a technique must be used in analyzing the situation of restricted sag, where a portion of the sagging section of pipe is supported by firm sediments after a certain amount of sinking occurs.

Restricted Sag

The theory of free sag is subject to the condition

$$y_m < h,$$

where h is the vertical distance from the ends of the sagging pipe section to the bottom of the weak sediment zone. Under certain conditions the sag may be great enough that the central portion of the sagging section rests upon the firmer sediments at the base of the weak zone (see Figure 2A). In this case the portion of pipe subject to free sag is not ℓ_o but is a smaller length ℓ_o' . The equivalent free sag problem for the length l_o' is represented graphically in Figure 2B. It is considered that the slope of this section is small enough that the unit load, w, is nearly normal to the pipe as previously assumed. The vertical scale in the schematic diagrams of Figure 2 is greatly exaggerated.

For the case of zero moment at the ends of section l_o', the equivalent end slope, referred to the rotated coordinates in Figure 2B, is

(56) $$\theta_1' = \frac{h}{l_o'} .$$

The section l_o' is subject to an effective initial tension factor $(n_o')^2$ given by

(57) $$(n_o')^2 = n_o^2\left(\frac{l_o'}{l_o}\right)^2 + \frac{l_o'}{l_o}\left(\frac{h}{r}\right)^2,$$

where n_o^2 is the initial tension factor of the straight pipe.

By making use of equations (28), (29), and (40a); together with (56) and (57), for the special case of $n_o^2 = 0$ and no end slippage, the following important relations are obtained:

(58) $$q \doteq .279\,\frac{h}{r}\,\frac{f_2(n')}{n'}\left[\frac{5(n')^2}{(h/r)^2} - 1\right]^{-4},$$

and

(59) $$\frac{l_o'}{l_o} \doteq \frac{(n')^2}{(h/r)^2} - \frac{1}{5},$$

where

(60) $$(n')^2 = \frac{N(l_o')^2}{EI}.$$

The function $f_2(n')$ is that given by equation (33), where n is replaced by n'. The graph of this function, as already noted, is represented by the full curve labled $n_o^2 = 0$ in Figure 3; in this graph, enter with $(n')^2$ on the vertical scale and read $f(n')$ on the horizontal scale.

The quantities $(n')^2$ and ℓ_o are the key factors to determine, because once these are determined, the tension and bending moment can easily be evaluated. Equation (58) is not explicit in n' and therefore must be solved for n' by successive approximation and/or graphically. The maximum possible range of ℓ_o'/ℓ_o is 0 to 1/2, and therefore must lie in the range:

(61) $$0.20\left(\frac{h}{r}\right)^2 < (n')^2 < 0.70\left(\frac{h}{r}\right)^2 ,$$

according to equation (59). This serves as a guide in the selection of values for a solution to equation (58). The upper limit $0.70(h/r)^2$ corresponds to $\ell_o' = \ell_o/2$. This represents the condition for tangency of the pipe at the bottom, with zero moment at the point of contact. In the case where the pipe is tangent to the bottom but receives no support, $(n')^2$ is equal to $0.64(h/r)^2$ according to equation (41a); under this condition a bending moment does exist at the point of contact.

The maximum bending moment factor for the condition of no end moment is m_o'. This can be determined from equation (25b) by replacing n by n' or can be obtained from the solid curve given in Figure 4 by entering with $(n')^2$ on the horizontal scale and reading m_o' on the vertical scale. The bending moment can then be computed from the relation

(62a) $$M_o' = m_o' \, w(\ell_o')^2 .$$

For the case of rigid ends of the section ℓ_o, the problem becomes complex, for in this event the section ℓ_o' is no longer symmetrical with reference to the rotated coordinates of Figure 2B. One end is rigid and subject to a maximum bending moment while the other end is free of moment. However, as a first approximation the equations (24) and (26) can be used to determine the maximum bending moment factor. The angle θ_1' for this case will be $-h/\ell_o'$; therefore

(63) $$m_1' \doteq -\frac{\left[\frac{1}{2} + \frac{(n')^2}{q'}\frac{h}{r}\right]}{n' \tanh \frac{n'}{2}} + \frac{1}{(n')^2} ,$$

where

(64) $$q' = q\left(\frac{\ell_o'}{\ell_o}\right)^4 .$$

The values of n' and l_o', obtained from equations (58) and (59) can be used in (63) and (64) as a first approximation. Finally the end moment can be found from

$$M_1' = m_1' \, w \, (l_o')^2, \tag{62b}$$

and the tension can be found from equation (60).

DEFORMATION OF A PIPE LINE IN AN ELASTICALLY DEFORMED MATERIAL

In general the relatively strong material which supports the pipe at the ends of the sagging section will be deformed itself. This may be a quasi-elastic deformation if the material is sufficiently strong, or it may be predominantly a plastic deformation. The condition of rigid ends is a hypothetical situation representing the limiting case of elastic deformation of a material of infinite strength and infinite elastic modulus. On the other hand, the condition of zero moment at the ends of the section l_o is a special case of plastic deformation of the supporting material, where the net upward unit load on the pipe is the same in magnitude as the downward unit load in the weak sediment zone.

The situation of elastic deformation of a relatively strong supporting material is considered here. By examining the mutual distortion of pipe and supporting material under the action of a known total load on the pipe, it is possible to arrive at a "modified rigid end" condition, which makes the determination of stresses associated with pipe sag more realistic.

The Basic Equations

The equation for balance of vertical forces on a small section of pipe is given by

$$EI \frac{d^4 y'}{dx'^4} - N \frac{d^2 y'}{dx'^2} = f, \tag{65}$$

where f is the net force per unit length acting in the direction of y'. As before y' is taken vertically upwards. Equation (65) is subject to the assumptions (4) to (10) inclusive given in the preceding section, but the restrictions imposed by (1), (2), and (3) are dropped.

The assumption which is made in place of (1) and (2) is that

$$f = -E_e y' \,, \quad \text{or} \quad E_e = \left|\frac{f/D}{y'/D}\right| > 0 \,, \tag{66}$$

where E_e is the effective elastic modulus of the sediment, and y' is measured upward from the equilibrium position of the pipe.

The other principal assumption is that the supporting material, which is subject to elastic deformation, is of semi-infinite extent in the positive x' direction. The origin is shifted horizontally to the end of the sagging pipe section, so that $x' = 0$ is the dividing line between the weak sediment and the relatively stronger, supporting material adjacent to it. Furthermore, it is presumed that y' approaches zero for very large values of x', i.e. it reaches the equilibrium position at great distance from the end at which the sag load is applied.

Under these conditions, the solution of (65) is given by one of the following equations:

$$y' = \frac{y'_1}{\sin \nu c}\, e^{-\beta x'} \sin \nu (c - x') \,, \quad \text{for } \gamma < 1 \,; \tag{67a}$$

$$y' = \frac{y'_1}{c}\, e^{-\alpha x'} (c - x') \,, \quad \text{for } \gamma = 1 \,; \tag{67b}$$

$$y' = \frac{y'_1}{e^{n_1 c} - e^{n_2 c}} \left[e^{n_1 (c - x')} - e^{n_2 (c - x')} \right], \quad \text{for } \gamma > 1 \,; \tag{67c}$$

where

$$\gamma = \frac{N}{2\sqrt{EI\, E_e}} \,, \tag{68}$$

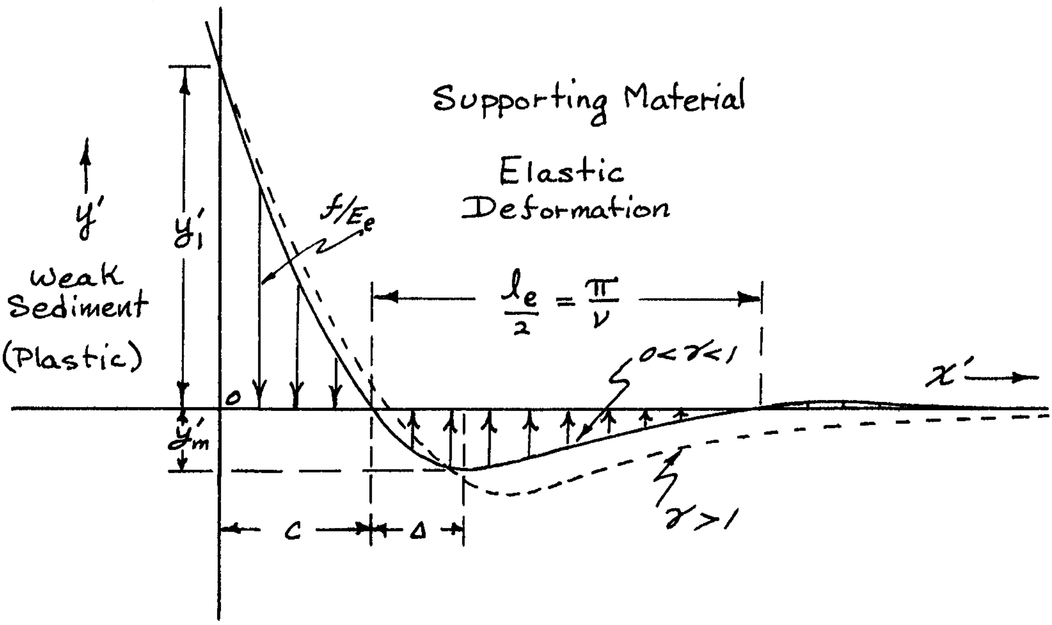
$$\alpha = \left(\frac{E_e}{EI} \right)^{1/4} \,, \tag{69}$$

$$\beta = \alpha \sqrt{\frac{1+\gamma}{2}} \,, \quad \nu = \alpha \sqrt{\frac{1-\gamma}{2}} \,, \quad \nu' = \alpha \sqrt{\frac{\gamma - 1}{2}} \,, \tag{70a,b,c}$$

(71a,b) $$n_1 = \beta + \nu', \quad n_2 = \beta - \nu'.$$

The constant y_1' represents the vertical displacement of the pipe at the end $x' = 0$, and C represents the position of $y' = 0$. The quantity γ is a dimensionless quantity which will be referred to as the tension parameter. The quantities α, β, ν, ν', n_1, and n_2 have the dimensions of wave number (reciprocal length).

In the case of $\gamma < 1$, the distortion of the pipe is in the form of a damped sine wave, of wave length $2\pi/\nu$. For $\gamma \geq 1$ the distortic is critically damped. A schematic diagram of the distortion of the pipe is shown below:



The distribution of reaction, f, is indicated by the vertical arrows.

The distortion y_m' indicated in the above diagram occurs at the distance Δ from the position of zero distortion. For the case of $\gamma < 1$:

(72) $$\tan \nu\Delta = \frac{\nu}{\beta} = \sqrt{\frac{1-\gamma}{1+\gamma}}.$$

In terms of the wave length l_e, the value of Δ lies in the range:

$$0 < \Delta < l_e/8,$$

the upper limit corresponding to $\gamma = 0$ (no tension). The position of zero slope (corresponding to $y' = y'_m$), zero moment, and zero shear occur at distances of Δ, 2Δ, and 3Δ from the position of $y' = 0$, respectively. A special case of (67a) for the condition $N = 0$ has been investigated previously by Timoshenko (1930)*. In this case, $\beta = \nu = \alpha/\sqrt{2}$.

For all values of the tension parameter γ, i.e., for either damped sine distortion or critically damped distortion, the distortion becomes negligible in a distance of about $2\pi/\alpha$ from the end. This distance will be small if the elastic modulus of the supporting material is large. For all practical purposes, if the supporting material is at least as long as $2\pi/\alpha$, then the relations (67a,b,c) are valid.

Maximum Total Reaction and Moment, for Elastic Deformation

The total net reaction of the supporting material is denoted by F. This is defined by the relation

$$F = \int_0^\infty f \, dx'. \tag{73}$$

In general, the value of F depends upon the arbitrary end conditions y_1' and C as well as upon N, and the characteristics of the pipe and its supporting material. The extreme value of reaction, F_{ult}, which can occur under the condition of elastic deformation of the supporting material is given by the approximate relation:

$$F_{ult} \doteq \left(0.80 + 1.32\sqrt{\gamma + 1.11}\right)\frac{f_c}{\alpha}, \quad \text{for } \gamma < 10, \tag{74}$$

where f_c is the critical limit of f beyond which the deformation of the supporting material becomes plastic. The conditions for maximum reaction are that

$$y_1 = C = 0 \quad \left(\frac{y_1}{\sin \nu C} \neq 0\right),$$

---

* This is an extension of the original investigation of a beam on an elastic foundation which was carried out by Winkler (1867).

and

$$|y'_m| = |y'_c| = \frac{|f_c|}{E_e} .$$

The bending moment in the pipe likewise depends upon y'_1 and c in general. The extreme value which can be obtained, M_{ult}, under elastic conditions, is given by the following approximate relation:

(75) $$M_{ult} \doteq -(3.86 + 4.20\gamma)\frac{f_c}{\alpha^2} , \quad \text{for } \gamma < 10 .$$

The condition for this extreme value is that

$$y'_1 = -y'_m ,$$

and

$$|y'_m| = |y'_c| .$$

It should be noted that the extreme values of F and M do <u>not</u> occur under the same conditions of c .

End Conditions and Total Reaction Relationships

In general the following useful relations hold for all values of γ :

(76) $$M_1 = \frac{1}{\alpha^2}\left[(1+2\gamma)E_e y'_1 + 2\beta F\right] ,$$

(77) $$\theta_1 = -\frac{\alpha^2}{E_e}F - 2\beta y'_1 ,$$

and

(78) $$F = N\theta_1 - V_1 = \frac{1}{2}w l_o .$$

For the special case of zero tension $(\gamma = 0)$:

(76a) $$M_1 = \frac{1}{\alpha^2}\left(E_e y'_1 + \sqrt{2}\,\alpha F\right) ,$$

(77a) $$\theta_1 = -\frac{\alpha^2}{E_e}F - \sqrt{2}\,\alpha\, y_1',$$

and

(78a) $$F = -V_1 = \frac{1}{2}w l_o .$$

Furthermore, if $y_1 = 0$, then

(79a) $$M_1 = \frac{\sqrt{2}}{\alpha}F ,$$

and

(79b) $$\theta_1 = -\frac{F}{\alpha^2 EI} = -\frac{\alpha^2}{E_e}F .$$

The end moment corresponding to the maximum total reaction of the supporting material (for non-plastic deformation) can be found from (79a) by substituting for F the value F_{ult} from equation (74).

The Modified Rigid End Condition for Free Sag

By eliminating the dimensionless shear factor φ between equations (24) and (26) and employing the definition of m_1 from equation (22), the general relationship between M_1 and θ_1 at the ends of that portion of the pipe in the weak sediment zone is obtained. Another relationship between M_1 and θ_1 which must be satisfied if the pipe is supported by an elastically deformed material is given by equation (76) and (77), where y_1' is eliminated between the latter equations. Consequently, if these relations are to be satisfied simultaneously, then it follows that:

(80) $$M_1 = \left\{\frac{\frac{1}{2}w l_o \frac{a}{b} - F^*}{2a\beta + F^*}\right\} b ,$$

(81) $$\theta_1 = \frac{M_1 + b}{a} ,$$

and

(82) $$y_1 = \frac{M_1 - \frac{2\beta}{\alpha^2} F}{F^*} ;$$

where

(83) $$a = \frac{N l_o}{n \tanh n/2} ,$$

(84) $$b = -w l_o^2 (m_1)_{\theta_1 = 0} = -(M_1)_{\theta_1 = 0} > 0 ,$$

and

(85) $$F^* = \frac{(1+2\gamma) E_e}{\alpha^2}$$

The maximum bending moment in the pipe occurs within the supporting material at the distance $l_m = C_m + 3\Delta$ from the position at which M_1 occurs. The value of the maximum moment M_{max} for the case of $\gamma < 1$ is given by

(85) $$M_{max} = M_1 \frac{\sin \nu C_m}{\sin[-\nu(C_m + 2\Delta)]} e^{-\beta(C_m + 3\Delta)} ,$$

where C_m is given by

(86) $$\operatorname{ctn} \nu C_m = \frac{1}{2\nu\beta} \left(\frac{M_1}{EI y_1'} - \alpha^2 \gamma \right) ,$$

and Δ is given by equation (72).

The equations (80) to (86) represent the generalized relationships for a "nominally rigid end condition". That is, the supporting material can be considered nominally rigid if the deformations experienced in it are below the critical limit, y_c' , and consequently are small compared to the maximum sag of the pipe, which occurs in the plastically deformed weak sediment zone. The maximum bending moment which can occur in the pipe for a given sag, depends essentially upon the modulus E_e of the supporting material. For extremely large values of E_e (approaching that of concrete or steel), the deflection y_1' and the slope θ_1 become negligible and the maximum moment is essentially $(M_1)_{\theta_1 = 0}$. That is, the condition of infinite elastic modulus of the supporting material represents the truly rigid end condition, and is actually never attained in nature.

In order to carry out any computations for the "nominally rigid end condition", it is necessary to determine the tension factor n^2 or N itself. The value of n^2 for the extreme end conditions $\theta_1 = 0$ and $m_1 = 0$ can be determined. In general the value of n^2 for the "nominally rigid end condition" will lie somewhere between the values based upon the above end conditions, since the angle θ_1 will lie between zero and the value of θ_1 corresponding to $m_1 = 0$. As a first approximation, the mean value of n^2 determined for the two extreme end conditions can be used. It will be noted from Figure 3 that this approximation cannot be more than 15% in error if q is greater than 10,000. For smaller values of q, the slope θ_1 can be determined from the first approximation of n^2; a second approximation for n^2 can then be obtained by interpolating between the values corresponding to $\theta_1 = 0$ and $m_1 = 0$, by comparing the above value of θ_1 with that corresponding to the condition $m_1 = 0$. In most cases of practical interest, q will be sufficiently large, such that a second approximation of n^2 is unnecessary.

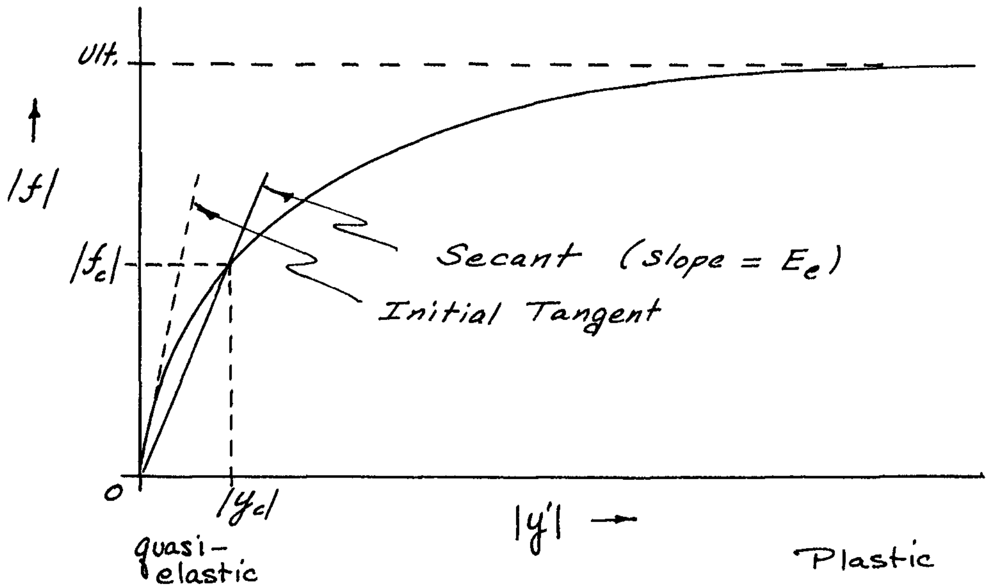
The Critical Limit of Deformation

If the deformation of the supporting material is increased to the point where significant plastic yielding of the material occurs, then the reaction becomes nearly independent of deformation. This means that the load, f, becomes essentially uniform over that portion of the pipe for which the deformation has exceeded the critical limit. As a result, the bending moment in the pipe is redistributed and the maximum value is reduced.

It was assumed in the development of the "nominally rigid end condition" that the supporting material is elastic in the sense that f is proportional in magnitude to the deformation, up to the critical limit, y_c'. In a material such as stiff clay, the load-deformation relationship* is not linear, and consequently the value of y_c' is difficult to define.

A schematic diagram of the typical load-deformation curve for clay is illustrated below:

* As found in the laboratory as well as the settlement load relationship observed in the field.



As a first approximation, the critical load, f_c, can be taken as one-half the maximum reaction of the material, and the critical deformation, y'_c, is that deformation corresponding to f_c on the load deformation curve. Consequently, the quantity $E_e = |f_c/y'_c|$ represents a secant modulus of the material; its value is roughly about one-half the value of the initial tangent modulus. The rate of increase of f with y' for deformations greater than y'_c becomes so small that for all practical purposes, the material is plastically deformed beyond this limit.

The application of equations (66) to (86) for a supporting material such as clay is admittedly an approximation. Nevertheless, the resulting bending moment is a much more reasonable estimate than that which would be obtained by considering the supporting material as rigid. Furthermore, it can be shown that if the estimate of E_e is in error by as much as 50 per cent, the resulting error in the estimated value of M_1 and M_{max} is less than 12 per cent for values of E_e of the order of 10,000 lbs./ft.^{2}.

The value of M_{ult}, on the other hand, is not as accurate. The value of M_{ult} is proportional to

$$\sqrt{\frac{|f_c|}{D}\,\frac{|y'_c|}{D}}$$

The quantity inside the radical represents twice the apparent modulus of resilience of the supporting material. (Actually a material such as clay suffers a permanent deformation even for $y < y_c$, so that the true resilience is less than this.) An error of 50 per cent in the apparent modulus of resilience will lead to an error of about 25 per cent in M_{ult} .

APPLICATION: SAG OF A LIQUID FILLED TEN-INCH PIPE

An Example of Free Sag

Suppose that the proposed route of a ten-inch pipe line traverses a band of extremely soft sediment, which is about 500 feet wide and about 12 feet deep at the anticipated crossing, and is several miles in length transverse to the pipe. The weak sediment is sandwiched between two extensive reef bodies, consisting of uncemented oyster shells in a clay matrix.

Tests carried out in the field disclose that the weak sediment has a bearing capacity of only 70 lbs. per ft. for a coated section of the pipe*. The base pipe is 10 inches O.D. with 1/2 inch steel walls, 5/8 inch coating of Somastic, and 1 inch coating of concrete for protection. Its weight per unit length is estimated to be 95 lbs. per ft. when loaded with liquid petroleum. This takes into account the uplift due to the moisture content of the sediment, which is 43.7 lbs. per cu. ft. The reef material has a bearing capacity significantly greater than the submerged weight of the loaded pipe. Consequently differential sag of a 500 foot section of the pipe would occur under these conditions and may lead to severe stresses in the pipe.

To avoid this situation, two alternatives present themselves: (1) reroute the pipe line so as to by-pass the weak sediment zone, or (2) entrench the pipe to a depth of about 12 feet within the reef zones so as to reduce or eliminate the anticipated vertical distortion of the pipe. Either of these alternatives might be costly, and therefore it is worthwhile to determine quantitatively if the pipe could be allowed to sag without danger of overstressing.

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* This bearing capacity corresponds to a maximum shearing resistance of the sediment of roughly 30 lbs. per ft.^{2}.

The specifications of the pipe are summarized below:

$$\text{I.D. Steel} = 9 \text{ in.}$$

$$\text{O.D. Steel} = 10 \text{ in.}$$

$$\text{Overall } D \text{ (protected)} = 13 \text{ in.}$$

$$A_s = 14.9 \text{ in.}^2$$

$$I = 169 \text{ in.}^4$$

$$r = 3.36 \text{ in.}$$

$$EI = 35.2 \times 10^6 \text{lb.ft.}^2$$

$$EIr = 9.85 \times 10^6 \text{lb.ft.}^3$$

$$R_o/r = 1.49$$

It will be assumed at first that there is no tension in the pipe prior to sag, and furthermore that there is no longitudinal slippage of the pipe in the supporting reef material. Thus the loading and length factors can be summarized as follows:

$$w = 95 - 70 = 25 \text{ lbs./ft.}$$

$$\ell_o = 500 \text{ ft.}$$

$$f_r = 0 \; (s = 0)$$

$$N_o = 0 \; (n_o^2 = 0)$$

Therefore, using equation (20), the flexibility parameter can be computed:

$$q = \frac{25\ (500)^4}{9.85 \times 10^6} = 159{,}000.$$

The total net load of the sagging section of pipe is 12,500 lbs. The reef material therefore must carry 6,250 lbs. at each end of the sagging section in addition to the load exerted by the pipe line passing over the reef. If the reef material is extremely stiff then possibly this load could be carried without significant distortion of the reef material. If so the rigid end condition, $\theta_1 = 0$, might apply. However, if the reef material has a bearing capacity of only 120 lbs. per ft., then the material would be plastically deformed. The upward net load on the pipe would be about

25 lbs. per ft. in the reef material, and the moment would be released altogether at the ends of the 500 foot section. The moment distribution and deflection of the supported pipe would be similar but opposite in sign to that occurring in the unsupported pipe section. This condition represents the end condition for which $m_1 = 0$.

Calculations made on the basis of the two extreme end conditions place upper and lower limits on the maximum stress that could be expected for the situation being considered. Such calculations are presented below:

Method of Determination	Case I: $\theta_1 = 0$	Case II: $m_1 = 0$
Fig. 3 using $q = 159{,}000$	$n^2 = 920 = \sigma_t$	$n^2 = 1{,}000 = \sigma_t$
	$n = 30.4$	$n = 31.7$
Equa. 19	$N = 130{,}000$ lbs.	$N = 141{,}000$ lbs.
Equa. 41a ($k_1 = k_2 = 1$)	$y_m = 5.33$ ft.	$y_m = 5.53$ ft.
Equa. 40a		$\theta_1 = .0398$ rad $= 2.3°$
Fig. 4	$m_1 = -.0153$	
$m_o = 1/n^2$ for $n^2 > 500$	$m_o = .00108$	$m_o = .000995$
Equa. 22	$M_1 = -95{,}500$ lb.-ft.	
Equa. 21	$M_o = 6{,}750$ lb.-ft.	$M_o = 6{,}220$ lb.-ft.
Equa. 52	$(\sigma_b)_{max} = 3{,}620$ (using m_1)	$(\sigma_b)_{max} = 236$ (using m_o)
	$(\sigma_b + \sigma_t) = 4{,}540$	$(\sigma_b + \sigma_t) = 1240$
Equa. (54) using $(\sigma_b + \sigma_t)_{max}$	$(S_b + S_t)_{max} = 42{,}650$ psi	$(S_b + S_t)_{max} = 11{,}670$ psi
	occurs at ends	occurs at center

If the elastic limit of the material in tension is in the neighborhood of 30,000 psi, then the value of maximum stress for the condition $\theta_1 = 0$ indicates that the assumption of elasticity has been transcended, and the value of stress is quantitatively in error but qualitatively is indicative of an unsafe situation. Actually, the maximum stress will lie somewhere between the limits 11,670 psi and 42,650 psi, depending upon the strength characteristics of the supporting reef material.

To test whether or not $S_b + S_t$ is the maximum combined stress, the value of shear stress must be computed. For the case rigid ends, $\bar{S}_s$ is a maximum, since the entire net vertical load of the pipe is carried by shear and none in tension at the ends. This end shear is 6,250 lbs, so that $\bar{S}_s$ = 420 psi (from equation 47). The value of S_{b_2} is 34,000 psi (from the value of σ_b given above), consequently $4(\bar{S}_s)^2/S_b$ is only about 20 psi. From condition (A) of Table IV, this means that the combined stress $S_b + S_t$ does govern in this case as presumed, since it is very unlikely that the hoop stress is greater than 42,630 psi (this would require an internal pressure of about 4,740 psi in excess of the environmental pressure outside the pipe).

The value of sag for the case of plastic yielding of the supporting material will be greatest. For the end condition $m_1 = 0$ the center of sag will be $2\,y_m$ below the equilibrium position of the pipe or about 11 feet for the situation above. This is less than the depth of the weak sediment zone in this case and therefore the situation is truly one of free sag.

Effect of Initial Tension for the Rigid End Condition

A reduction in temperature of a very long pipe line below that at which the pipe was initially installed can induce an axial tensile stress in the pipe if the longitudinal restraint provided by sediments and protecting coating inhibit the contraction of the pipe.* For a 50°F change in temperature, the maximum thermal stress which can be induced in the steel is about 9,750 psi. In the 10 inch pipe examined above, this is equivalent to an axial tensile force of N_o = 145,000 lbs.

What affect would an initial tension of this magnitude have on the sagging section of pipe with rigid end conditions?

For the case of no end slippage as before ($s = 0$, $k_2 = 1$), the value n^2 must be evaluated from the equations (28) and (29):

$$n^2 = \frac{n_o^2}{1 - \left[\frac{q}{f_1(n)}\right]^2} .$$

From equation (23), using the value of N_o above:

$$n_o^2 = 1,000 .$$

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* For the case of initial compression see Summary and Conclusions and also the Appendix.

For the case of $m_1 = 0$, $n^2 = 1{,}500$ from Figure 3; however for the case of $\theta_1 = 0$, the value is probably somewhat less than this. Using $n^2 = 1{,}500$ as a first approximation, the value of $f_1(n)$ can be found from the dashed curve in Figure 3 by reading the value on the q scale corresponding to the value n^2 above. This yields $f_1(n) = 320{,}000$. Using this estimate of $f_1(n)$, together with the value of q already computed, a second approximation of n^2 can be found as follows:

$$n^2 = \frac{1000}{\left\{1 - \left[\frac{159{,}000}{320{,}000}\right]^2\right\}} = 1330 .$$

A third approximation following the same procedure, but using the mean value of the first and second approximations of n^2 to find $f_1(n)$, yields:

$$n^2 = 1400,$$

which is sufficiently accurate. Thus from equation (19) the final tension N is 197,000 lbs. This is an increase of only 52,000 lbs. above the initial value, and the sag corresponding to this, from equation (41a), is only 3.5 ft.

The value of maximum stress is found to be 41,100 psi, which is only slightly less than that found for the case of zero initial tension. The reason for the small difference in stress is that the increase in pure tensile stress is offset by the reduction in bending moment for the higher value of tension parameter.

Effect of End Slippage

In the previous computations it was assumed that the slippage was zero ($\Delta = 0$, or $k_2 = 1$). Strictly speaking, this requires an indefinitely large value of longitudinal restraint f_r; however, in the problem being considered, if f_r is about 50 w or 1250 lbs. per ft., then k_2 would differ from unity by only 10 per cent. For the 10 inch nominal pipe entrenched in the supporting material, this would require a value of τ_u' (equation 16) of approximately 400 lbs. per sq. ft., which is not unreasonably large.

Suppose now that f_r were only 25 lbs. per ft. ($f_r/w = 1$). This situation could obtain for a 10 inch pipe resting upon the supporting material (but not entrenched), provided that the coefficient of friction between the pipe and the underlying material were about 3/10.

The slippage coefficient is given by equation (41):

$$s = \frac{25}{25}\left[\frac{30 \times 10^6 \times 14.9}{25 \times 0.28}\right]^{1/4} = 89.4 \ .$$

For the case of no initial tension and zero end slope, equations (28), (29) and (30) reduce to

$$f_1(n) = \frac{159{,}000}{\sqrt{1 + .0112\, n^2}} \ ,$$

where q is taken as 159,000 as before. The value of n^2 will be lower than in the case of no slippage; however, using $n^2 = 920$ as found previously, the approximate value of $f_1(n)$ is 47,000 as computed from the last equation above. This corresponds to a value of n^2 of 380 (Figure 3), which is the first approximation of n^2. Using this value, a second approximation, $n^2 = 510$, can be found by the same procedure. The true value of n^2, which represents a root of the equation,

$$F(n) \equiv f_1(n)\sqrt{1 + .0112\, n^2} - 159{,}000 = 0,$$

must lie between the first and second approximation.

It is found that

$$F(n) = -51{,}400, \text{ for } n^2 = 380 \ ,$$

and

$$F(n) = 22{,}300, \text{ for } n^2 = 510 \ .$$

Hence the root as found by linear interpolation is

$$n^2 = \frac{51{,}400}{73{,}700}(510 - 380) = 470 \ .$$

If this is carried one step further a value of 480 is found to satisfy $F(n) = 0$ more closely. This is of sufficient accuracy.

Using $n^2 = 480$ (or $n = 21.9$) the value of k_2 is found to be 2.52, and the maximum sag (equation 41a) is

$$y_m = 0.626 \times 2.52 \times 21.9 \times 0.28 = 9.7 \text{ ft.}$$

From Figure 4, a value of m_1 of -.021 for the case of $\theta_1 = 0$, is obtained. The tension and bending moments evaluated from the value of n^2 and m_1 above are 67,600 lbs. and -131,000 lb.-ft. respectively.

Finally the stresses can be computed (equations 45 and 46):

$$S_b = 46,800 \text{ psi},$$

$$S_t = 4,500 \text{ psi},$$

and

$$S_b + S_t = 51,300 \text{ psi}.$$

The value of maximum total stress is about 20 per cent greater than in the case of no end slippage. In many cases the end slippage will have considerably less affect - the case examined here probably represents an extreme condition of slippage.

If both initial tension and end slippage exist, the problem of determining n^2 can be solved by successive approximation in a similar manner.

Modified Rigid Ends

It was stated that the maximum stress in the 10 inch pipe will lie between the limits 11,670 psi and 42,650 psi (for the case of $\delta = 0$, $n_o^2 = 0$) depending upon the strength characteristics of the supporting material. Two different conditions of elastic deformation of the supporting material are examined here in order to give a more realistic idea of the stress.

For the case of $\delta = 0$, $n_o^2 = 0$, the value of N was found to be 130,000 lbs. for the rigid end condition and 141,000 lbs. for released end moment. The mean value 135,000 lbs. will be presumed for the case of elastic deformation of the supporting sediment investigated below.

Suppose that the supporting material has an effective modulus (E_e) of 12,000 lbs. per ft.^{2}; this is representative of the somewhat stiffer soft clays of the Atchafalaya region. If the value of $\mathcal{S}_c$ for this material is not exceeded then equations of elastic deformation will apply.

The following characteristic parameters are evaluated from equations (68) through (70b):

$$\gamma = 0.10,$$

$$\alpha = .136 \text{ ft.}^{-1}, \quad \alpha^2 = .0185 \text{ ft.}^{-2},$$

$$\beta = .101 \text{ ft.}^{-1},$$

$$\nu = .0913 \text{ ft.}^{-1},$$

and

$$\ell_e = \frac{2\pi}{\nu} = 69 \text{ ft.}$$

Thus the pipe deflection in the elastically deformed supporting material is in the form of a non-critically damped sine wave (since $\gamma < 1$) with a wave length of 69 ft. If the supporting reef material is at least 69 ft. wide then the elastic theory as given here is applicable.

From equations (74) and (75) it is found that

$$F_{ult} = 16.5 \; f_c \,,$$

and

$$M_{ult} = 232 \; f_c \,.$$

It is seen immediately from these relations that a supporting material with an f_c of only 100 lbs. per ft. could not possibly support the load or moment indicated by the condition of rigid ends. It will be found that the value of f_c must be about 20 times greater than this if the supporting material is to be free of plastic deformation.

The next step is to compute a , b , and F^* using relations (83), (84), and (85):

$$a = 2.18 \times 10^6,$$

$$b = 95{,}500,$$

$$F^* = 779{,}000 \text{ lbs.}$$

Inserting these values in equations (80), (81), and (82) yields:

$$M_1 = -49{,}100 \text{ lb. ft.},$$

$$\theta_1 = .0207 \text{ radians},$$

and

$$y_1' = 0.15 \text{ ft.}$$

The deflection y_1' = 0.15 ft. represents the extreme value of elastic distortion of the supporting material in this case, so that the lower limit of f_c is

$$f_c = 12{,}000 \times .15 = 1800 \text{ lbs./ft.}$$

This would require a shear strength (τ_u) of at least 830 lbs./ft.^{2} (if k_b is taken as 2.0, equation 2).

From equation (86), the condition of maximum moment requires that

$$\nu c_m = -1.95 \text{ radians},$$

and from equation (72):

$$\nu \Delta = 0.735 \text{ radians.}$$

Thus the position of maximum moment from the end of the weak zone is

$$\ell_m = c_m + 3\Delta = 2.8 \text{ ft.}$$

Using equation (85) the maximum moment can now be computed:

$$M_{max} = -54{,}000 \text{ lb. ft.}$$

Finally the maximum total stress can be computed

$$S_b + S_t = 28{,}250 \text{ psi.}$$

A similar analysis for the case of $E_e = 2{,}000$ lbs./ft.^{2} leads to a somewhat smaller total stress. The results of these computations are summarized in Table VII. In this case the pipe deflection in the supported zone is still a non-critically damped sine wave, but the wave length is about 120 ft., indicating that the distortion of the supporting material is spread out over a greater length along the pipe, leading to a redistribution and reduction of the maximum bending moment. The strength f_c required in this case is at least 740 lbs./ft. or greater, in order that the supporting material is truly in an elastic state of deformation.

All of the computations for the example of free sag in a 10 inch pipe line of 500 foot sag length are summarized in Table VII for convenience of comparison. As mentioned previously, the condition $m_1 = 0$, or complete release of the moment at the ends of the weak zone, represents the special condition of plastic deformation of the supporting material, for which $f_c = w = 25$ lbs./ft. The problem of partially elastic and partially plastic deformation of the supporting material has not been investigated.

An Example of Restricted Sag

In the previous example the amount of total sag was found, for each of the end conditions investigated, to be less than the depth of the weak sediment zone (as measured from the equilibrium position of the pipe, which occurs at moderate horizontal distance from the weak sediment zone). The example therefore was truly one of free sag. Suppose now that the width of the weak zone is ten times that of the above example, i.e. ℓ_o = 5,000 feet or about 1 mile. A ten inch nominal pipe will again be considered but in this case it will be presumed that the net downward unit force on the pipe, w, is only 10 lbs. per ft. Thus if the pipe were to sag freely in the weak zone of sediment, then the supporting material at the ends would have to carry a load of 25,000 lbs. at each end of the sagging section, which is only about four times that considered in the previous example.

TABLE VII

SUMMARY OF COMPUTATIONS OF FREE SAG FOR A TEN INCH STEEL PIPE:

Net downward load in weak sediment 25 lb/ft (w)
Width of weak sediment zone 500 ft (ℓ_o)
Pipe data: $EI = 35.2 \times 10^6$ lb ft^2
$A_s = 14.9$ in^2
$D = 13.0$ in (protected)

Initial Tension End Slippage	$n_o^2 = 0$ $\delta = 0$ ($f_r > 50\,w$)				$n_o^2 = 1000$ $\delta = 0$	$n_o^2 = 0$ $f_r/w = 1$
State of Adjacent Sediment	Plastic Deformation	Elastic Deformation		Rigid	Rigid	Rigid
End Cond.	$M_1 = 0$	---	---	$\theta_1 = 0$	$\theta_1 = 0$	$\theta_1 = 0$
E_e (lbs/ft^2)	0****	2,000	12,000	∞	∞	∞
N_o (lbs)	0	0	0	0	145,000	0
N (lbs)	141,000	135,000	135,000	130,000	197,000	67,600
M_o (lb ft)	6,220	6,500	6,500	6,750	4,500	13,100
M_1 (lb ft)	0	-34,400	-49,100	-95,500	-78,700	-131,000
M_{max} (lb ft)*	±6,220	-40,700	-54,000	-95,500	-78,700	-131,000
S_t (psi)	9,450	9,050	9,050	8,650	13,200	4,500
$(S_b)_{max}$ (psi)	2,220	14,450	19,200	34,000	27,900	46,800
y_1' (ft)	5.5	0.37	0.15	0	0	0
y_m (ft)	5.5	5.4	5.4	5.3	3.5	9.7
θ_1 (rad)	.0398	.0275	.0207	0	0	0
f_c (lb/ft)	25	$\geq$ 740	$\geq$ 1800	∞	∞	∞
ℓ_1 (ft)***	250	28.0	12.9	0	0	0
ℓ_m (ft)*	±250	7.3'	2.8'	0	0	0
$y_1' + y_m$ (ft)	11.0	5.8	5.6	5.3	3.5	9.7
$(S_t + S_b)_m$ (psi)	11,670	23,500	28,250	42,650**	41,100**	51,300**

* Maximum moment occurs at distance ℓ_m from end of weak zone (or $\ell_o/2 + \ell_m$ from center of sag)

** Stress which would exist if the elastic limit of the material were great enough

*** Equilibrium level of pipe reached at distance ℓ_1 from end of weak zone

**** Limiting case of $/f/ = /w/$

However the flexibility parameter, q, (using the same values of E, I, and r as before) is 6.34×10^8 or 4,000 times greater than before. For such an extreme flexibility, the pipe acts as a flexible cable and the asymptotic equation (36) can be used to compute the tension parameter. In fact for $s = 0$, $n_o^2 = 0$:

$$n = \left(\frac{q}{4.9}\right)^{1/3} = 505;$$

and according to equation (41a) the maximum free sag, y_m, would be about 89 feet! The tensile stress due to elongation alone, according to equation (37a) and/or (46), would be 24,200 psi, and if the end conditions are considered rigid then a maximum total stress of over 100,000 psi could be developed (see Table V).

It is inconceivable that an extremely weak sediment zone of a depth comparable to the computed free sag above could exist. Consequently the pipe will actually sag until it rests upon the firmer sediments at the base of the weak zone, and the full elongation of the pipe, expected in the case of free sag, will not be realized.

Suppose that the average depth of the weak zone h_o (Figure 2A) is 10 feet. If the adjacent supporting material is of sufficient strength, then the difference $h_o - h$ will be small, and hence h can be approximated by h_o. Consequently, under this condition the relative depth h/r will be 35.7 and according to equation (61) the tension parameter for the length of pipe ℓ_o' (Figure 2B) will lie in the range:

$$255 < (n')^2 < 893,$$

provided that no initial tension existed preceding the sag of the pipe into the weak zone.

For the condition $q = 6.34 \times 10^8$ ($w = 10$ lbs./ft. and $\ell_o =$ 5,000 ft.), equation (58) reduces to

$$\text{(58a)} \qquad \left[\frac{(n')^2}{255} - 1\right]^4 - 1.57 \times 10^{-5}\,\frac{f_2(n')}{n'} = 0$$

Using the range of $(n')^2$ stated above as a guide, the appropriate root of this equation can be ascertained by successive approximation, or by graphical procedure. Since in this case the unrealized free sag is much greater than the depth of the weak sediment zone, it is expected that the ratio ℓ_o'/ℓ_o will be quite small. Consequently, the value of $(n')^2$ will probably lie closer to the lower limit 255 than to the upper limit. Using this as a starting point, the value of $f_2(n')$ as determined from Figure 3 is 20,500 (full curve labled $n_o^2 = 0$; $f_2(n')$ read from the q scale).

By evaluating the approximate value of the ratio $f_2(n')/n'$ (using n' = 16), a first approximation of $(n')^2$ can then be determined explicitly from (58a). This yields $(n')^2$ = 350.

A second approximation evaluated in the same way but using $(n')^2$ = 350 to obtain $f_2(n')/n'$ yields $(n')^2$ = 360. The convergence to the final value is quite rapid in this case (a third approximation yields the same result to within one per cent accuracy).

Using $(n')^2$ = 360, the length ratio l_o'/l_o according to equation (59) is .082, or l_o' = 410 feet. The tension accordingly is 75,500 lbs. (equation 60), which corresponds to a tensile stress, S_t, of about 5,100 psi. The above tension actually applies to the case where the moments at both ends of the section l_o' are zero. However it will be noted that for the value of q' corresponding to $(n')^2$ = 360, the values of $(n')^2$ for the two extreme end conditions differ by only 20 per cent (Figure 3). Consequently it is presumed that the tension found above is approximately valid for different end conditions which might be considered.

If it is supposed now that the supporting material adjacent to the region of sag is quasi-rigid so that the pipe is essentially horizontal, then the moment factor at this end is given approximately by equation (63). Using $(n')^2$ = 360 and q' as found from equation (64) (l_o'/l_o = .082), the calculated moment factor is about -.047, and the corresponding bending moment is about -79,000 lb.-ft. (equation 62b). This corresponds to a maximum bending stress of 28,000 psi and consequently the maximum combined stress in the pipe is approximately

$$(S_b + S_t) = 33{,}000 \text{ psi.}$$

It will be noted that this applies for the case of $n_o^2 = 0$, $s = 0$ and $\theta_1 = 0$.

The effect of end slippage and/or elastic deformation of the supporting material could be carried out in a manner similar to that already presented in the problem of free sag. However, due to the complexity of the restricted sag problem and the nature of the approximations already made in this application, it would appear that such refinements are not justifiable, unless the asymmetry of the section l_o' is likewise taken into account. This would require re-examination of the basic equation (4) and its solution (8); for in this case a finite shear V_o would exist at the center of the sagging pipe section.

On examining the question further, a better approximation might be had by considering the section l_o' as one-half of a symmetrical, freely sagging section formed by deleting the section of pipe which is fully supported at the base of the weak sediment zone. The difficulty in applying the free sag theory in this case is that there exists a finite shear and zero moment at the point of tangency, while the free sag theory presumes no shear at the center of sag and yields a finite bending moment at that point.

SAG OF A TWENTY-INCH PIPE

The specifications of a 20.5 inch O.D. steel pipe (thickness 3/4 inch) are given in Table III. The value of the product EI_r for this pipe is 275×10^6 lb. ft.^{2}. If this pipe has a 5/8 inch coating of protective Somastic and a 1 inch coating of reinforced concrete, the total weight in air (gas filled) would be about 269 lbs. per ft. The over-all diameter of the protected pipe would be 23.75 inches.

It will be presumed that this pipe sags into a weak material identical to that discussed in the case of the ten inch nominal pipe. This implies that if the bearing capacity of the sediment is proportional to the diameter of the pipe, as equation (2) indicates, then

$$P_b \doteq P_m = \frac{23.75}{13.00} \times 70 = 128 \text{ lbs./ft.}$$

Furthermore the bouyancy due to moisture content of the sediment (for an entrenched pipe) would be 134 lbs./ft. (B = 43.7 lbs./cu. ft. as before). Thus the net vertical force w per unit length would be only 7 lbs. per ft. for a gas filled pipe. This is of the same order of magnitude as the errors involved in the estimate of P_u and/or B . Consequently one should investigate the influence of the possible error in w on the resulting stress, in order to see if the computation leads to an unqualified decision regarding the vertical stability of the pipe.

If the pipe is to transport liquid having a specific gravity approaching that of water, then the value of w would be about 130 lbs. per ft. For a given span length of sag, it is evident that there will be considerable difference in the maximum pipe stresses induced by these two limiting conditions of loading. Computations based on these two cases, for a span length of 500 feet (for comparison with the example of free sag of a ten inch pipe), are summarized below.

Free Sag of a Gas Filled Pipe

In this case w = 7 lbs./ft. and ℓ_o = 500 ft., which yields a value of q equal to 1590. The corresponding value of n^2 from Figure 3 is 19.2 for the conditions: $s = 0$, $n_o^2 = 0$, and $\theta_1 = 0$. The maximum sag, y_m, under these conditions is only 1.6 feet. Furthermore from Figure 4, $m_1 = -.065$, and the computed stresses are:

$$S_b = 6{,}200 \text{ psi},$$

$$S_t = 800 \text{ psi},$$

and

$$S_b + S_t = 7{,}000 \text{ psi}.$$

The maximum combined stress of 7,000 psi would occur if the supporting material at the ends were rigid; otherwise the maximum stress would be less than this. However any slippage at the ends would tend to increase the stress (as found in the case of the ten inch pipe).

If the probable error in w is taken as $\pm$ 7 lbs./ft., it is evident that the value of expected stress ranges between the limits of zero and some upper limit. Figure 6 can be used to facilitate the computation of this upper limit. By doubling the value of w, and hence q the dimensionless combined stress parameter (for $\theta_1 = 0$) is increased from about 170 to 290. The combined stress itself will be increased in the same proportion, consequently the upper limit is about 12,000 psi. This indicates that an error of even $\pm$ 100 per cent in the estimated w, in this case, would not invalidate a qualitative decision with regard to the safety of this pipe line. Such clear cut results, however, appear to be the exception rather than the rule, and the decision regarding safety usually must be qualified by a statement regarding the degree of risk involved, unless positive steps are taken to avoid, partially or completely, the conditions of sag which are anticipated.

Free Sag of a Liquid Filled Pipe

In this case $w = 130$ lbs./ft. and $\ell_o = 500$ ft., yielding a value of q of 29,800. The value of n^2 is found to be 270 for the conditions $\rho = 0$, $n_o^2 = 0$, and $\theta_1 = 0$. Corresponding to this, the maximum sag is 5.98 feet, which is very nearly the same as for the water filled ten inch pipe.

The resulting stresses induced by sag are:

$$S_b = 47{,}700 \text{ psi},$$

$$S_t = 11{,}000 \text{ psi},$$

and

$$S_b + S_t = 58{,}700 \text{ psi}.$$

An error of $\pm$ 7 lbs./ft. in w, in this case, represents a relative error of only $\pm$ 5.4 per cent; and the corresponding error in the combined stress is only about 2/3 of this or 3.6 per cent. Under these conditions the maximum combined stress above is presumably accurate to within $\pm$ 2,100 psi. This would not influence the qualitative interpretation of this result, nor would the effect of an error in w of as much as $\pm$ 20 lbs./ft.

The magnitudes of stresses involved in these two examples bear out, quantitatively, the extremely different situations which might be realized for a gas filled pipe on one hand and a liquid filled pipe on the other.

SUMMARY AND CONCLUSIONS

In the planning and installation of a marine pipe line, the question of vertical stability can be important enough to warrant serious consideration especially if the pipe is to be laid upon or entrenched within a marine sediment of the type existing on the Gulf shelf. The important points pertaining to this question are set forth below.

1. A knowledge of the structure and strength of the sediments along the path of a proposed pipe line is essential if definite conclusions are to be reached regarding the question of vertical stability of the pipe. Regions in which the sediment bearing capacity is not sufficient to support the submerged weight of a pipe line can be disclosed only by appropriate investigation in the field.

2. The difference in strength characteristics between extremely weak zones and adjacent supporting material, together with the horizontal and vertical dimensions of the zone of weak sediment along the path of the pipe line, are the prime environmental factors in the determination of differential sinking of a pipe and consequently in the determination of the maximum stresses associated with such sag.

3. In general the determination of the stresses induced by sag depends upon the following basic parameters:

w	determined by pipe weight and by the weak sediment characteristics
l_o, h_o	dimensions of the weak zone
E, I, A_s, r	pipe specifications (actually $r = \sqrt{I/A_s}$)
N_o, f_r, f_c, E_e	initial and end conditions (rigidity factors of supporting material are f_c and E_e)

For the case of free, symmetrical sag, the following functional relations hold

$$n^2 \equiv \frac{N l_o^2}{EI} = F_1(q, s, n_o, \text{end rigidity}),$$

$$m \equiv \frac{M}{w l_o^2} = F_2(n),$$

$$\theta_1 \frac{l_o}{r} \doteq 2.24 \frac{q}{f(n)} n,$$

and

$$\frac{y_m}{r} \doteq 0.626 \frac{q}{f(n)} n,$$

$$S_b + S_t = E\left(\frac{r}{l_o}\right)^2 \left[\left(\frac{R_o}{r}\right) m q + n^2\right],$$

where

$$q \equiv \frac{w l_o^4}{EIr}, \quad \text{Flexibility parameter;}$$

$$s \equiv \frac{w}{f_r}\left(\frac{EI}{wr^3}\right)^{1/4}, \quad \text{End slippage parameter;}$$

$$n_o^2 \equiv \frac{N_o l_o^2}{EI}, \quad \text{Initial tension parameter.}$$

If the end slippage is zero then

$$\theta_1 \doteq 2.24\sqrt{\frac{(N - N_o)}{E A_s}},$$

and

$$y_m \doteq 0.626\sqrt{\frac{(N - N_o)}{E A_s}}\, l_o .$$

If in addition $N_o = 0$, then

$$f(n) = q$$

This implicit equation for n in terms of q is represented graphically in Figure 3, and in tabulated form in Table VIII, for two different end conditions. The relationship $m = F_2(n)$ is shown graphically in Figure 4.

4. The dependence of maximum combined stress on flexibility is shown conveniently by the non-dimensional plots of Figure 6. The dimensionless stress factor,

$$\sigma_b + \sigma_t \equiv \frac{S_b + S_t}{E} \cdot \left(\frac{l_o}{r}\right)^2,$$

is presented graphically for the two limiting end conditions $\theta_1 = 0$ and $M_1 = 0$ and for the case of no initial tension and no end slippage. The stress parameter increases in direct proportion to q for very small values of q, corresponding to the theory of simple flexure ($S_t = 0$ or $N = 0$). For very large values of q, the stress factor becomes proportional to $q^{2/3}$, corresponding to a flexible cable. The proportionality coefficient in either limiting case depends upon the end condition. It is important to note that the condition of no end moment leads to the greater stress for low values of q ; but the situation is reversed for large values of q ($>$ 280) since the condition of rigid ends in this case leads to greater stress. In summary, for small flexibility:

$$\left(S_b + S_t\right)_m = m w \ell_o^2 \frac{R_o}{I}, \begin{Bmatrix} |m| = \frac{1}{12}, \text{ for } \theta_1 = 0 \\ |m| = \frac{1}{8}, \text{ for } m_1 = 0 \end{Bmatrix};$$

while for large values of q :

$$\left(S_b + S_t\right)_m = k\left[E\left(\frac{w\ell_o}{A_s}\right)^2\right]^{1/3}, \begin{Bmatrix} k = 1.548, \text{ for } \theta_1 = 0 \\ k = 0.347, \text{ for } m_1 = 0 \end{Bmatrix}.$$

5. In general the sagging pipe section will carry its load by the vertical component of tension as well as by cross-sectional shear. For small flexibility, however, the tension is negligible and the load is carried entirely in shear. As the flexibility is increased the tension becomes more and more important, and in the limiting case of extremely large flexibility ($> 10^6$), the pipe carries its entire load by tension, except in the case of rigid end conditions, where the shear is still important but only immediately adjacent to the ends of the sagging section.

6. The situation of free sag is subject to the condition that $y_m < h$. If this is not the case then the pipe can receive additional support at the base of the weak zone. The problem of such restricted sag can be solved, in the first approximation, by considering the unsupported portion of the pipe at the ends of the sagging section as a situation of symmetrical free sag. The length of this equivalent free sag section is unknown in this case, but can be determined if h is known.

7. Two clear cut end conditions have been examined in some detail:

(I) No end slope, representing the condition for which the supporting material is nominally rigid and free of any significant deformation, implying that

$$\left| f_c / w \right| \text{ is very large } (> 1000),$$

and

$$|E_e/E| \text{ is of the order of unity}$$

(II) No end moment, representing the condition for which the supporting material is plastically deformed at the ends by the amount $h_o - h = y_m$, which requires that

$$|f_c/w| = 1,$$

and

$$E_e = 0 .$$

The intermediate case of quasi-rigid conditions, representing that of finite elastic distortion of the supporting material, has also been examined. Values of f_c and E_e intermediate between the values above can be taken into account in a correction factor to be applied to the bending moment for a rigid end condition (equation 80). The effect of distortion of the supporting material at the ends of the sagging section of pipe is always to reduce the maximum bending moment induced in the material. In the examples worked out for the 10 inch nominal pipe, the limiting case of plastic distortion which was examined gave a value of maximum stress which is about 25 per cent of that which would be realized for rigid ends. The effect of elastic distortion of the supporting material is less pronounced.

8. The effect of end slippage is to exaggerate the severity of the maximum combined stress. The tension in this case is reduced but this, in turn, is associated with a greater proportionate increase in bending moment.

9. The effect of initial tension is to decrease the severity of the maximum combined stress, since the bending moment is decreased in greater proportion than the increase in tension.

10. The effect of initial <u>compression</u> has not been examined in the examples, but would lead to an increase in the severity of the maximum combined stress. Although the equations for free sag given here are restricte to the case of positive N (real values of n), the same restriction is <u>not</u> imposed upon N_o . That is, negative values of N_o (or initial compression) <u>can</u> be taken into account provided that this compression is not too great (see Appendix). Such compression could be induced by a sudden temperature increase in the pipe line, if the latter is inhibited from expanding in the longitudinal direction. In the event of sag, this longitudinal compression is released and in turn a tension will be developed, provided that the loading is great enough.

A ten inch nominal pipe under the same conditions of sag as presumed for the data of column 6 of Table VII except that n_o^2 is -1000 (corresponding to 145,000 lbs. initial compression) would yield a maximum combined stress of 46,200 psi. The maximum sag in this case would be 7.2 feet, which is about twice that found for the case of 145,000 lbs. initial tension. However the combined stress is only about 10 per cent greater.

11. The examples of sag worked out for the liquid filled 10 inch and 20 inch nominal pipes serve to demonstrate that considerable stresses can be induced by sag of the pipe into a zone of extremely soft sediment (such as found in the Atchafalaya Bay region) if such a zone is contained between material which is considerably stiffer.

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APPENDIX I

Figure 3 is not in a form that is adequate for actual use. In order to make it convenient for the plotting of the functions $f_1(n)$ and $f_2(n)$ on a detailed, large scale logarithmic grid, the computed values of these functions for the selected values of n used in the construction of Figure 3 are given in Table VIII. These values, obtained from equations (32) and (33) have been evaluated by computing machine for the lower range of n and by slide rule for the values of n greater than 4. The values are considered accurate to within $\pm$ 0.5 per cent.

In construction of the logarithmic plots of the functions $f_1(n)$ and $f_2(n)$ it is convenient to construct the straight line asymptotes given by equations (35a,b) and (36) as a guide.

Isolines for both positive and negative values of n_o^2 can be plotted on the n^2 versus q diagram by making use of equation (28) for the case of $\delta = 0$:

$$q = \sqrt{1 - n_o^2/n^2}\; f(n)$$

The isolines for the case of negative n_o^2 will lie to the right of the curve for $n_o^2 = 0$, i.e. the curve $q = f(n)$, and will be asymptotic to the latter curve for large values of q. For small values of n^2 the isoline for negative n_o^2 will approach a constant value of q, having the value:

$$246\,|n_o^2|\;, \text{ for } \theta_1 = 0,$$

or

$$4.90\,|n_o^2|\;, \text{ for } m_1 = 0.$$

Thus for a given value of q it is seen that there is an upper limit of initial compression beyond which the present theory fails to give a solution, because n^2 itself becomes negative.

If n_o^2 is just at the critical value for a given q then the tension parameter is zero, and the stress in the pipe is purely flexural and is given by the simple bending theory. This stress is always greater than that applying to the case of zero initial compression, as can be seen from Figure 6.

TABLE VIII

COMPUTED VALUES OF THE FUNCTIONS $f_1(n)$ AND $f_2(n)$
FOR SELECTED VALUES OF n

n	n^2	$f_1(n)$ ($\theta_1 = 0$)	$f_2(n)$ ($m_1 = 0$)
.01	.0001	2.46	.49
.1	.01	24.6	4.8
.3	.09	73.9	14.5
1.0	1.00	252	53.8
1.3	1.69	335	74.2
2	4	540	137
3	9	903	275
4	16	1,374	511
5	25	1,990	858
6	36	2,780	1,380
7	49	3,790	2,060
8	64	5,040	2,905
10	100	8,410	5,410
15	225	23,200	17,320
20	400	50,200	40,200
30	900	155,000	133,800
50	2,500	672,000	615,000
100	10,000	5,120,000	4,900,000
1,000	1,000,000	4,900,000,000	4,900,000,000

In the case where the initial tension is greater than the critical limit, then N is negative and the equations for pipe distortion are modified. The distortion of the pipe given by equation (8) takes on the form

$$y = \frac{1}{N^*}\left\{\left(M_o + w\lambda^{*2}\right)\left(1 - \cos x/\lambda^*\right) + \frac{1}{2}wx^2\right\}$$

where

$$N^* = -N \quad \text{and} \quad \lambda^* = \sqrt{\frac{EI}{N^*}}$$

In the special case where $w = 0$, it can be shown that a critical compression,

$$N_c^* = EI\left(\frac{2\pi}{l_o}\right)^2$$

exists. If the initial compression is less than this then the pipe is absolutely stable with regard to transverse deflection. If the initial compression is greater than N_c^* then the pipe will deflect transversely, but in so doing, the initial compression is partially relieved and a definite equilibrium with an associated maximum bending moment is developed. This situation would represent a condition of quasi-buckling since there is but one form which can be assumed by the pipe for a given value of N_o^*. The transverse deflection is not severe unless N_o^* is considerably greater than N_c^*.

APPENDIX II

LIST OF SYMBOLS

Symbol		Dimensions
a	A characteristic quantity in the modified rigid end equation	lb. ft.
A_s	Cross-sectional area of the steel pipe	sq. ft.
b	Magnitude of maximum bending moment for the case of rigid ends	lb. ft.
B	Bouyant force acting on pipe per unit displaced volume in the sediment .	lbs./cu.ft.
c	Value of x' at which $y' = 0$ for elastic deformation of supporting material	ft.
D	Over-all diameter of the protected pipe	ft.
D_i	Inside diameter of the steel pipe	ft.
D_o	Outside diameter of the steel pipe	ft.
E	Elastic modulus of the steel pipe	lbs./sq.ft.
E_e	Effective modulus of the supporting material at the ends of the sagging section	lbs./sq.ft.
f	The net upward force per unit length at position x' exerted on the pipe by the supporting material	lbs./ft.
f_c	Limit of f beyond which the deformation of the supporting material becomes plastic	lbs./ft.
f_r	The longitudinal restraint per unit length of pipe exerted by the supporting material	lbs./ft.
$f(n)$	A function of n and the end condition	dimensionless
$f_1(n)$	A function of n only, for the condition $\theta_1 = 0$	dimensionless
$f_2(n)$	A function of n only, for the condition $m_1 = 0$	dimensionless
F	Total net vertical reaction exerted by the supporting material	lbs.

Symbol		Dimensions
F_{ult}	Extreme value of F which can be sustained by an elastically distorted sediment	lbs.
F^*	A characteristic quantity used in the modified rigid end equation	lbs.
g	Acceleration of gravity, 32.2	ft./sec.^{2}
h	Vertical distance between the base of the weak zone and the vertical position of the pipe at $x = \ell_o/2$ (Figure 2)	ft.
h_o	Depth of the weak zone below equilibrium position of the pipe (Figure 2)	ft.
I	Cross-sectional moment of inertia of the pipe, taken about the neutral surface	ft.^{4}
k_b	Coefficient of proportionality between P_b and $\tau_u D$	dimensionless
k_1	Coefficient dependent upon n_o^2/n^2	dimensionless
k_2	Coefficient dependent upon $\Delta, q, n^2 - n_o^2$	dimensionless
k_3	Coefficient having the approximate value 2.24	dimensionless
k_4	Coefficient having the approximate value 0.626	dimensionless
ℓ	Length of pipe between points (1′) and (1) after sag occurs (Figure 1)	ft.
ℓ_e	Wave length of the non-critically damped elastic deformation curve	ft.
ℓ_i	Length of pipe between points (1′) and (1) before sag occurs (Figure 1)	ft.
ℓ_m	Distance from end of weak zone at which maximum moment is attained	ft.
ℓ_o	Length of weak sediment zone measured along the pipe line	ft.
ℓ_o'	Effective length of free sag in the restricted sag problem	ft.

Symbol		Dimensions
$(\ell_o)_c$	Upper limit of ℓ_o for a given w and given pipe specifications, beyond which the simple bending theory is not valid	ft.
m	Bending moment factor at position x	dimensionless
m_o	Bending moment factor at $x = o$	dimensionless
m_1	Bending moment factor at $x = \ell_o/2$	dimensionless
m'	Bending moment factor for length ℓ_o'	dimensionless
M	Bending moment at position x	lb. ft.
M_o	Bending moment at position $x = o$	lb. ft.
M_1	Bending moment at position $x = \ell_o/2$	lb. ft.
M_{max}	Maximum bending moment developed in the pipe for given end conditions	lb. ft.
M_{ult}	Extreme bending moment which can be developed in a pipe in an elastically deformed sediment	lb. ft.
n^2	Tension factor	dimensionless
n_o^2	Initial tension factor	dimensionless
n_1	Characteristic wave number, elastic theory	ft.$^{-1}$
n_2	Characteristic wave number, elastic theory	ft.$^{-1}$
$(n')^2$	Tension factor for the length	dimensionless
$(n_o')^2$	Initial tension factor for the length	dimensionless
N	Axial tensile force in the pipe	lbs.
N_o	Initial axial tension prior to sag	lbs.
p	End shear factor	dimensionless
P_b	Ultimate load bearing capacity of the sediment per unit length of pipe	lbs./ft.
P_m	Maximum vertical restraint exerted on pipe by the plastically deformed weak sediment	lbs./ft.

Symbol		Dimensions
q	Flexibility parameter	dimensionless
q_c	Critical value of q beyond which the simple theory of bending is not valid	dimensionless
Q	Moisture content of sediment expressed as per cent of dry weight	dimensionless
r	Radius of gyration of cross-section of steel pipe taken about the neutral surface	ft.
R_i	Inside radius of steel pipe	ft.
R_o	Outside radius of steel pipe	ft.
s	End slippage coefficient	dimensionless
S_b	Flexural stress in the steel pipe farthest from the neutral surface of bending	lbs./sq.ft.
S_h	Hoop stress in the pipe due to internal pressure	lbs./sq.ft.
$\bar{S}_s$	Shear stress associated with V	lbs./sq.ft.
S_t	Tensile stress associated with N	lbs./sq.ft.
$S_{s,m}$	Maximum combined shear stress	lbs./sq.ft.
$S_{t,m}$	Maximum combined normal stress	lbs./sq.ft.
V	Cross-sectional shear force at x	lbs.
V_1	Cross-sectional shear at $x = \ell_o/2$	lbs.
w	Net downward force per unit length exerted on pipe in the weak sediment zone	lbs./ft.
w_p	Unit weight of pipe in air (including weight of contained fluid)	lbs./ft.
x	Horizontal distance measured from the center of sag along the pipe line	ft.
x'	Horizontal distance measured from the end of the weak zone into the supporting material	ft.

Symbol		Dimensions
y	Vertical distance measured upwards from the position of maximum sag	ft.
y'	Vertical distance measured upwards from the equilibrium level of the pipe	ft.
y_c'	Critical value of vertical deflection beyond which the supporting material becomes plastically deformed	ft.
y_m	Maximum sag of the pipe in the weak zone, for the length ℓ_o	ft.
y_m'	Value of y' at which $\theta = 0$	ft.
y_1'	Vertical deformation of the pipe at the position $x' = 0$, or $x = \ell_o/2$	ft.
α	A characteristic wave number, elastic wave theory	ft.$^{-1}$
β	A characteristic wave number, elastic wave theory	ft.$^{-1}$
γ	A tension parameter in the elastic theory	dimensionless
Δ	Horizontal distance between the position for which $y' = 0$ and $\theta = 0$ in the elastic wave	ft.
θ	Slope of the pipe at position x	dimensionless
θ_1	Slope of the pipe at $x = \ell_o/2$	dimensionless
θ_1'	Effective slope of the pipe at ends of the length ℓ_o'.	dimensionless
λ	A characteristic length in the free sag theory	ft.
ν	A characteristic wave number, elastic wave theory	ft.$^{-1}$
ν'	A characteristic wave number, elastic wave theory	ft.$^{-1}$
π	3.1416...	dimensionless

Symbol		Dimensions
ρ_s	Wet density of the sediment	slugs/cu.ft.
σ_b	Flexural stress factor	dimensionless
σ_t	Tensile stress factor	dimensionless
τ_u	Ultimate shear strength of the weak sediment	lbs./sq.ft.
τ_u'	Ultimate shear strength of the supporting material	lbs./sq.ft.