CHAPTER 6

ON NON-SATURATED BREAKERS AND THE WAVE RUN-UP

Bernard Le Méhauté, D. Sc.
Senior Staff
National Engineering Science Co.
Washington 6, D. C.

ABSTRACT

Some theoretical results pertaining to the physical behavior of gravity waves on a sloped plane are presented. The notion of "saturated" breakers and "non-saturated" breakers which follow the breaking index curve is introduced. Criteria for different kinds of breaking and successive breaking of waves are presented. Some considerations on the wave run-up are deduced.

Then a critical analysis of the method of characteristics is presented, with some possible refinements. Path curvature effect is taken into account and the problem of waves climbing on a dry bed is solved. Criteria for determining saturated and non-saturated breakers and the wave run-up by the method of characteristics are proposed.

INTRODUCTION

It is commonly admitted that breakers on a beach can be separated into spilling breakers on a very flat slope and plunging breakers on a steeper slope. (Plunging breakers are sometimes called surging breakers on a very steep slope.) This separation of breakers into these two (or three) categories is based on visual observations rather than on some hydrodynamical criterium. However, the essential hydrodynamical characteristics of these breakers are recalled.

The profile of a spilling breaker remains, for the most part, almost symmetrical and the wave breaks by curling over slightly at the crest (Figure 1). As long as the foam of the breaker is small by comparison with the "bulk" water, which happens on a very gentle slope, the wave presents roughly the main characteristics of a solitary wave, even after breaking inception. But due to the spilling breaker a given amount of energy is dissipated in such a way that the wave crest follows the breaking index curve defined by $H = 0.78\, d$. Then the spilling breaker is transformed into a bore when the slope becomes steeper. When the slope is steep before breaking inception, the wave profile first losts its symmetrical shape, then a plunging breaking wave generates a bore directly.

In the following an attempt is made to analytically investigate these described phenomena. As usual two methods exist. The first method -- the energy method -- is only approximate but gives a great amount of information from relatively simple calculations. The second method -- the analytical method -- is more accurate but requires tedious computations for each particular case. Then it will be seen that the method of characteristics requires some refinements for analyzing the surf motion.

THE ENERGY METHOD

The energy method of investigation consists of first determining the wave motion on a horizontal bottom independent of the friction forces. This work has already been accomplished. Then the wave motion is assumed to keep its essential characteristics on a sloped bottom with friction forces with only a simple change in wave height (and wave length). This method is valid provided the bottom slope is gentle enough. Then the cnoidal wave theory or the solitary wave theory is used for very shallow water.

Even after the inception of breaking the essential characteristics of the motion of "bulk" water will be assumed to be also those of a solitary wave. In practice these assumptions are valid for swell waves which give rise to slightly spilling breakers on a very gentle beach. The case under study is presented schematically in figure 1a. Such assumptions do not hold true for steep waves (sea waves) on steep beaches which give rise to plunging breakers, or even when the foam of a spilling breaker becomes too important.

Then it will be written that the variation of transmitted wave energy over a length dx, namely $\frac{d(EC)}{dx}$, is equal to the rate at which this energy is destroyed: $-\frac{dE}{dt}$ which is written:

$$\frac{d(EC)}{dx} = -\frac{dE}{dt}$$

Now the problem arises of how to evaluate EC. Because of the great simplicity of the solitary wave theory one assumes that the wave characteristics are those of a solitary wave, even if the energy is distributed otherwise over a "wave length".

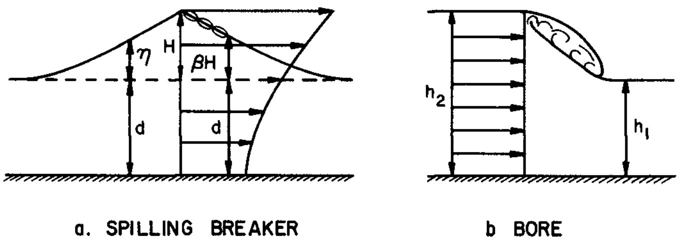

Fig. 1

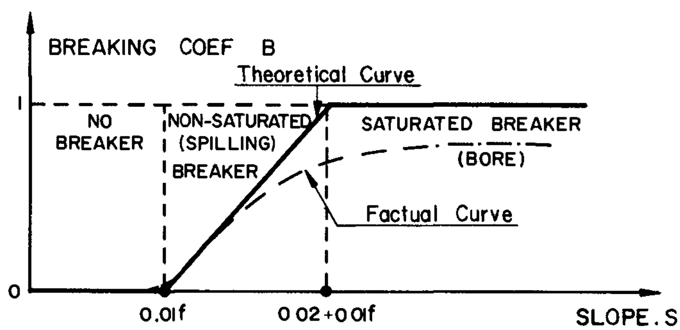

Fig. 2

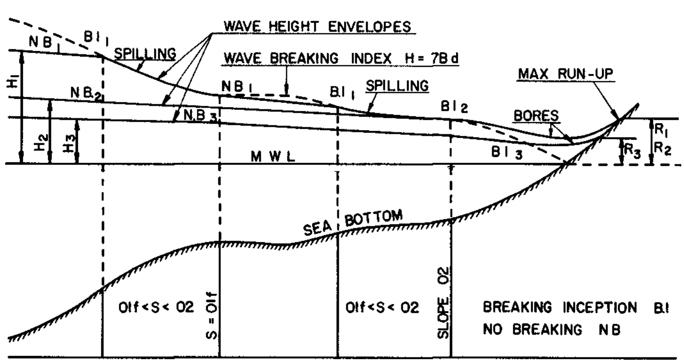

Fig. 3

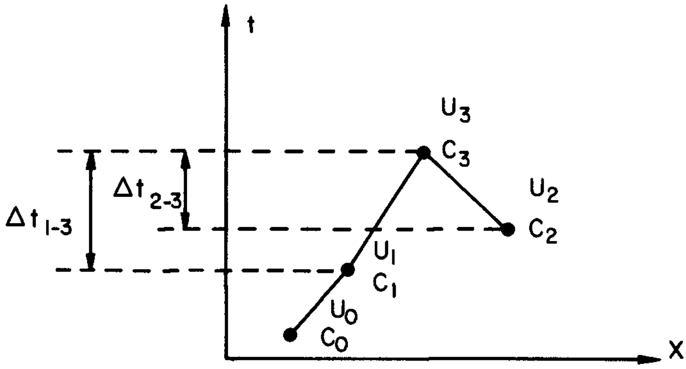

Fig. 4

Then it is known (Munk(1)*):

$$\frac{d(EC)}{dx} = \frac{8}{3\sqrt{3}} \rho g \frac{d}{dx}\left(H^{3/2} d^{3/2} C\right) \quad \ldots (1)$$

where $C = g(d + H)^{1/2}$.

The rate at which the energy is lost is due to bottom friction $\left.\frac{dE}{dt}\right|_b$ and to spilling breaker $\left.\frac{dE}{dt}\right|_s$. Hence:

$$\frac{d(EC)}{dx} = -\left[\left.\frac{dE}{dt}\right|_b + \left.\frac{dE}{dt}\right|_s\right] \quad \ldots (2)$$

A number of studies has been carried out on the damping of solitary waves due to viscous bottom friction. However, despite the difficult encountered in evaluating a friction coefficient, it is more realistic for practical purposes to assume the motion to be turbulent. Then the unit shear τ is quadratic: $\tau = \rho f u^2$ where f is a friction coefficient and u the horizontal velocity.

$$\left.\frac{dE}{dt}\right|_b = \int_{-\infty}^{\infty} \tau u \, dx$$

Then by inserting some classical relationships from the solitary wave theory

$$u = C\frac{\eta}{d}, \qquad \eta = \frac{H}{\cosh^2 \alpha}$$

where

$$\alpha = \frac{\sqrt{3}}{2}\left(\frac{H}{d}\right)^{1/2} \frac{x - Ct}{d}$$

it is found that

$$\left.\frac{dE}{dt}\right|_b = \frac{2}{\sqrt{3}} \rho f C^3 \left(\frac{H}{d}\right)^{3/2} H \int_{-\infty}^{\infty} \frac{d\alpha}{\cosh^6 \alpha}$$

By use of the guddermannian of α it is found that the integral is equal to 16/15. Hence, finally

$$\left.\frac{dE}{dt}\right|_b = \frac{32}{15\sqrt{3}} \rho f \frac{H^{5/2} C^3}{d^{3/2}} \quad \ldots (3)$$

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* Numbers refer to references listed at end of paper.

In a very first approximation, f can be taken to be equal to g/C_h^2 where C_h is the Chezy coefficient and n is the Manning coefficient.

$$f = \frac{g}{C_h^2} = \frac{g\,n^2}{(1486)^2\, d^{1/3}} = 14.6\,\frac{n^2}{d^{1/3}} \qquad \ldots (4)$$

This assumption involves that the vertical velocity distribution due to bottom friction in a solitary wave appears as that of a succession of steady flows. In practice such an assumption is really valid for periodic, very long waves in shallow water, which themselves are considered as a succession of solitary waves.

The rate of loss of energy due to a spilling breaker is very similar to that of a tidal bore (which is a shock wave). In the case of a shock wave it is known that (see figure 1b) (Stoker[2]):

$$\frac{dE}{dt} = \rho g\, Q\, \frac{(h_2 - h_1)^3}{4\, h_1 h_2} \qquad \ldots (5)$$

where h_1 and h_2 are the depths before and after the front of the bore, respectively, and Q is the discharge due to the moving bore. It is recalled that the above formula is based on the assumption that the vertical distribution of the horizontal velocity u is uniform.

In the case under study the spilling breaker is due to the fact that the horizontal velocity at the crest becomes greater than the wave celerity C. By analogy (see figure 1a) $h_2 = d + H$ and $h_1 = d + \beta H$ where β is always smaller than unity and can be zero at the limit. The vertical velocity distribution, and consequently the discharge, is directly related to the average horizontal velocity. Hence the discharge could be written:

$$Q = (d+H)u_2 - (d+\beta H)u_1 = C\,\frac{H}{d}\left[d + H - \beta(d+\beta H)\right]$$

Inserting these values into equation (5) and defining B as follows:

$$\left.\frac{dE}{dt}\right|_s = \rho g C \frac{H}{d}\left[d + H - \beta(d+\beta H)\right] \frac{H^3(1-\beta)^3}{4(d+H)(d+\beta H)} = \rho g\, B\, \frac{C H^4}{4 d^2} \qquad \ldots (6)$$

B will be called the "breaking coefficient". The breaking coefficient B is the ratio of the rate of energy dissipated by the spilling breaker to the rate of energy which could be dissipated by a bore of front height equal to the height of the solitary wave which generated it. $B = 0$ corresponds to no breaking ($\beta = 1$). A small value for B corresponds to a little spilling breaking near the crest (β close to unity). It is a partial break ing or a non-saturated breaker. It is difficult to ascertain the maximum value for B by the energy method. However, it is certain that B canno be larger than unit (β = 0). Then there is total breaking and the breake is a saturated breaker. Further consideration will be given to the physical meaning of B later in this paper.

Now, by introducing equalities (1), (3) and (5) into equation (2), it is found that

$$\frac{d}{dx}\left[H^{3/2} d^{3/2} C\right] = -\left[\frac{4}{5}\frac{f}{g}\frac{H^{5/2}}{d^{3/2}} + \frac{3\sqrt{2}}{32}\frac{BCH^4}{d^2}\right] \quad \ldots (7)$$

which gives after division by $H^{3/2}d^{3/2}C$, integration between a small interval $\Delta x = x_2 - x_1$, and since $e^{-ax} \cong 1 - ax$

$$H_2 = H_1\left(\frac{d_1}{d_2}\right)\left(\frac{C_1}{C_2}\right)^{2/3}\left[1 - \frac{8}{15}\frac{fH_1C_1^2\Delta x}{g\,d_{1-3}^3} - \frac{\sqrt{2}}{16}\frac{BH_1^{5/2}\Delta x}{d_{1-2}^{7/2}}\cdots\right] \quad (8)$$

When all friction effects are neglected (f = 0) and there is no breaking (B = 0), the classical law

$$\frac{H_2}{H_1} = \frac{d_1}{d_2}\left(\frac{d_1+H_1}{d_2+H_2}\right)^{1/3} \cong \left(\frac{d_1}{d_2}\right)^{4/3} \quad \ldots (9)$$

is easily recognized. It is known that such a law is not too well verified experimentally. The variation of wave height with distance depends, in fact, on the relative depth $\frac{d_1 + d_2}{d_b}$ where d_b is the depth of breaking(1) and the slope (Ippen and Kulin(5)). This has thrown some doubt on the validity of using the solitary wave theory for analyzing the wave motion on a slope. It is also known that equation (9) should be replaced by the Green Law $\frac{H_2}{H_1} = \left(\frac{d_1}{d_2}\right)^{1/4}$ or again by $\frac{H_2}{H_1} = \left(\frac{d_1+H_1}{d_2+H_2}\right)^{1/4}$

Despite these limitations, the physical interpretation of this study will be based on equations (7) and (8) because the spilling breaker effect tends to replace the variation of wave height by a simple H = 0.78 d . When H $<$ 0.78 d, there is no breaking and the breaking coefficient B = 0 . Then $C = [g(d+H)]^{1/2}$. Moreover, assuming H is small by comparison with d ,

$$H_2 = H_1 \left(\frac{d_1}{d_2}\right)^{4/3} \left[1 - \frac{B}{15} \frac{f H_1 \Delta x}{d_{1-2}^2}\right] \quad \ldots (10)$$

It is interesting to note that in the case of long periodic waves, a calculation based on similar assumptions gives

$$H_2 = H_1 \left(\frac{d_1}{d_2}\right)^{1/4} \left[1 - \frac{2}{3\pi} \frac{f H_1 \Delta x}{d_{1-2}^2}\right]$$

When $H_b = 0.78\, d_b$, there is inception of breaking and the breaking coefficient B becomes > 0. In the case of a small spilling breaker, C keeps its value $C = [g(d + H)]^{1/2}$ Then, replacing these values for H and C in equation (7):

$$\frac{d}{dx}\left(d^{7/2}\right) = 1.1 \frac{f}{g} d^{5/2} + 0.07\, B\, d^{5/2}$$

i.e. the slope $S = \frac{d}{dx}(d) = 0.01 f + 0.02 B$ or within the known limits:

$$\underline{0 \leq B = 50 S - 0.5 f < 1} \quad \ldots (11)$$

It is seen that the breaking coefficient B increases with the slope: the steeper the slope, the greater the rate at which the energy is dissipated by the spilling breaker.

It may occur that due to bottom friction B always retains a zero value despite the shoaling when $S < 0.01 f$ as is easily seen from equation (11). (This result can also be found directly from equation (7) when replacing H by $0.78 d$ and equating B with zero.)

Inserting the value (4) for f, a criterium for damping without breaking is proposed:

$$S < \frac{14.6\, n^2}{100\, d^{1/3}}$$

i.e. with the Manning coefficient $n = 0.02$

$$\underline{S < \frac{6.10^{-5}}{d^{1/3}}} \quad (d \text{ in feet}) \quad \ldots (12)$$

On the other hand, it has been seen that B cannot exceed unity. This happens when $S = 0.02 + 0.01 f \cong 0.02$. When $S = 0.02$, then the breaker is "saturated." Figure 2 illustrates these considerations.

Now a complete physical interpretation can be drawn from the previous considerations. If the slope is always smaller than $6.10^{-5}/d^{1/3}$, then the wave height is completely damped by bottom friction. There is no breaking and no run-up. This occurrence is very rare.

On a steeper slope, there is a maximum amount of wave energy that a solitary wave can transmit towards the shoreline over a given depth. This maximum energy is reached when $H = 0.78\ d$. If the amoun of energy passing through a given plane tends to be larger than this maximum value, a spilling breaker will dissipate the difference. This is on a relatively gentle slope and is a non-saturated breaker, in which case the wave height is directly related only to the depth. Then the run-up is negligible.

Considering the usual range of variation of the bottom slope S and a possible range of variation for f, a reason for successive wave breaking due to change of slope and depth is found and its criterium established.

In fact other reasons also exist for successive wave breaking. First, by effect of "hydraulic hysteresis" or inertia for the free turbulenc due to the breaker being damped, more wave energy is spent by a breaker than indicated by equation (11). Then another non-breaking wave could be reformed, even if equation (12) is not fully satisfied.

Moreover, successive breaking may also be due to the superimposition of crests of irregular waves for which the following formula is proposed: (Le Méhauté (3))

$$\sum^{n} \left(\frac{H}{L}\right)^2 = \frac{0.02}{n} \sum^{n} \tanh^2 \frac{2\pi d}{L}$$

(It is interesting to note that this formula gives good results whatever the wave direction: two-dimensional irregular waves, clapotis, short-crestec waves.) However, since in very shallow water all waves tend to travel to a constant velocity, the above criterium is valid more for the whitecaps at sea than in the vicinity of the shoreline.

Now the case of total breaking is considered.

It is seen, also, that there is a limiting amount of energy which could be dissipated by a breaker over a given length. Hence, when the slope becomes steeper and steeper, the regulating effect of the spilling breaker reaches its limit when $B = 1$. Then the breaking index curve is surpassed by the height of the front bore. There is run-up. The words "saturated" and "non-saturated" breakers are now defined, explained and justified.

A very important conclusion is also drawn: On a beach having its curvature upwards, the maximum possible wave run-up is given by the wave which breaks at a depth where the slope is equal to 0.02. It is known that if d_b is the depth over that slope, the corresponding wave height is $H_b = 0.78\, d_b$. Any wave having a greater height breaks sooner, dissipating its energy following the breaking index curve up to the plane where the slope becomes larger than 0.02.

In fact the theoretical value 0.02 for the critical slope (corresponding to $B = 1$) may be replaced by a more factual and conservative value 0.01. The exact determination of this value requires further investigation by the method of characteristics.

The results of this section are summarized in figure 3 by three typical cases. It must be noted that the run-up in cases I and II is the same despite their different deep water wave heights.

ON THE METHOD OF CHARACTERISTICS

First the usual presentation of the method of characteristics is recalled. Then it will be seen that the application of this method to the problem under study requires a number of modifications and refinements.

It seems that the first application of the method of characteristics to the problem of a wave breaking over a beach was due to Stoker[2]. (Other references are given by Ho and Meyer.[4]) In fact the same method has been applied for a long time in studying tidal motion and bore formation in estuaries. In both cases the vertical velocity distribution is assumed to be uniform and the pressure distribution hydrostatic. Moreover, the friction forces (neglected by Stoker) take great importance in the study of tidal motion in estuaries. Then the two basic equations are:

momentum: $$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -g\frac{\partial \eta}{\partial x} - g\frac{u|u|}{C_h^2(d+\eta)}$$

continuity: $$\frac{\partial \eta}{\partial t} = -\frac{\partial u(d+\eta)}{\partial x}$$

where u is the average horizontal velocity along OX, η the elevation of the free surface above the still water level, $(d + \eta)$ the depth, and C_h the Chezy coefficient.

Defining the quantity $c = \left[g(d+\eta)\right]^{1/2}$ it is found after some transformation that

$$\left\{\frac{\partial}{\partial t} + (u \mp c)\frac{\partial}{\partial x}\right\}(u \mp 2c) = G$$

where $G = -gS - \frac{g}{C_h^2} \frac{u|u|}{(d+\eta)}$

i.e. $\frac{d}{dt}(u + 2c) = G$ along $\frac{dx}{dt} = u + c$

$\frac{d}{dt}(u - 2c) = G$ along $\frac{dx}{dt} = u - c$

It is recalled that G is considered as a constant over a small interval Δ t.

Then, by knowing u and c at two points defined by their position x and time t (figure 4), it is possible to calculate the location of a third point by drawing the lines of slope $1/u_1 + c_1$ and $1/u_2 - c_2$ in a (t,x) diagram and to calculate u_3 and c_3 from the equations:

$$\begin{cases} u_3 + 2c_3 = u_1 + 2c_1 + G\,\Delta t_{1-3} \\ u_3 - 2c_3 = u_2 - 2c_2 + G\,\Delta t_{2-3} \end{cases}$$

Hence, this powerful method permits the complete analysis of the wave motion as a function of time and space. However, as it is presently used, it has some limitations. One of them is rightly pointed out by Stoker:

The method of characteristics gives a marked steepening of the wave front and a very unsymmetrical shape for the wave at breaking. In a word, the method of characteristics gives directly a bore or saturated breaker while it is well observed that spilling breakers remain almost symmetrical in shape.

The case of a solitary wave on a horizontal bottom without friction is of particular interest because it is known that a solitary wave must travel without deformation. When this problem is treated by the method of characteristics, the wave profile becomes quickly deformed and even generates a bore, despite the fact that the bottom is horizontal. This discrepancy is due to the fact that the flow curvature, particularly important near the crest of a wave, is neglected in the method of characteristics while the profile of a solitary wave is obtained by integrating the continuity and momentum equations in which a term for the flow curvature has been introduced. It is because of this term that the pressure distribution in a solitary wave is actually smaller than the hydrostatic pressure and much smaller (40%) under a near breaking wave crest.

This term is presented in many books. It is therefore judged unnecessary to reproduce the calculations here. It is sufficient to know that it is obtained by considering that the vertical component of velocity is assumed to be linearly distributed from the bottom to the free surface. This is a very realistic assumption. It is also assumed that the bottom slope has a negligible effect on the path curvature. This assumption holds true only for a gentle slope near the wave crest. But, if these conditions are not satisfied, the path curvature correction becomes small, in any case, by comparison with other terms such as ($-g\,S$). Hence a more complex calculation of the path curvature effect taking account of the bottom slope would be easy to perform but not worthwhile for practical purposes.

Then the momentum equation is written

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -g\frac{\partial \eta}{\partial x} - g\frac{u|u|}{C_h^2(d+\eta)} - \frac{d+\eta}{3}\frac{\partial^3 \eta}{\partial t^2 \partial x}$$

Hence the only modification introduced by the flow curvature in the method of characteristics is to give G the value

$$G = -gS - \frac{g\,u|u|}{C_h^2(d+\eta)} - \frac{d+\eta}{3}\frac{\partial^3 \eta}{\partial t^2 \partial x}$$

This expression can easily be expressed as a function of u and c for each point of the x, t diagram along the characteristics since according to the continuity equation and the definition for c

$$G = -gS - \frac{g^2}{C_h^2}\left(\frac{u}{c}\right)^2 + \frac{c^2}{3g^2}\frac{\partial^3(uc^2)}{\partial t\,\partial x^2}$$

Hence the previous values for G, G_1 and G_2 will be corrected by terms ΔG_1 and ΔG_2 .

For more generality it is of great interest to work with dimensionless terms. Defining $C_* = [g\,d_1]^{1/2}$ where d_1 is an arbitrary depth

$$U = \frac{u}{C_*}\ ,\quad C = \frac{c}{C_*}\ ,\quad X = \frac{x}{d_1}\ ,\quad T = \frac{C_*\,t}{d_1}$$

The basic equations become

$$\frac{d}{dT}(U \mp 2C) = G_* = -S - \frac{g}{C_h^2}\left(\frac{U}{C}\right)^2 + \frac{C^2}{3}\frac{\partial^3(UC^2)}{\partial X^2 \partial T}$$

along $U \pm C = \frac{dX}{dT}$.

In view of determining the maximum possible run-up, it is seen from the first part of this study that the most convenient input is the profile of a limit solitary wave where the slope tends to become steeper than 0.01. Then d_1 can be taken as the depth at that particular location. It is at this location that the breaking index curve is no longer followed. The breaker is close to being saturated. The relative maximum possible wave run-up, $\frac{R_{Max}}{0.78\, d_1} = \frac{R_{Max}}{H_{Max}}$, will appear as a function of the slope only. Such a method of calculation often permits reduction of the number of calculations required. Consequently the cumulative errors are reduced in such a way that the final result is even better than that which would be given by starting directly from an input defined by a non-breaking wave in deep water. Also, as long as the wave travels on a slope smaller than 1/10, the curvature term has a non-negligible influence in computing the run-up.

Now another deficiency of the method of characteristics and its solution are analyzed. First it is recalled that there is bore formation when two characteristics of the same family cross each other. Then two values for c , and consequently for η , are obtained. It means physical that the wave breaks and forms a tidal bore.

Actually, as pointed out in the first section of this paper, a spillin breaker appears prior to bore formation. It has been seen resulting in a loss of energy, which is not taken into account by the method of character istics. This is due to the fact that the method of characteristics is based on the assumption that the vertical velocity distribution is uniform while the spilling breaker is due to the local high particle velocity near the cres

The method of characteristics can be corrected in order to take account of this important phenomenon. It is sufficient to impose to η or H, and consequently to c , a maximum value prior to the bore formation. This maximum value for H will be 0.78d , for example. Then

$$c_{max} = [1.78\, gd]^{1/2}.$$

Such computations define the area for non-saturated breakers. Then, again, when two characteristics of the same family cross each othe the bore appears and the non-saturated spilling breakers are transformed into saturated breakers. It is evident that on a steep slope this intermedi process of calculation does not appear.

Now the succeeding steps of the computations are given.

U and C can always be determined on the low side of the bore, say U_d and C_d. Along the bore line defined by $\frac{dX}{dT} = V$, where V is the speed of the bore, three unknowns must be determined: V itself, and U C on the high side of the bore, namely U_u and C_u. These three unknowns are determined from the momentum equation and the continuity equation for shock waves, and by the $U + C$ line which crosses the V line on the high side of the bore from a point o (U_o, C_o). (Its construction may require some interpolation.) Hence, the system of equations to be solved is:

$$C_u^2 - C_d^2 = 2C_d^2\left[V - U_d\right]\left[U_u - U_d\right]$$

$$U_u C_u^2 - U_d C_d^2 = V\left[C_u^2 - C_d^2\right]$$

$$U_u + 2C_u = U_o + 2C_o + G_* \Delta T_{o-u}$$

The solution of this system is given by the following set of equations:

$$\frac{X^4 - 1}{\sqrt{2}\, X\left[1 + X^2\right]^{1/2}} + 2X = K$$

where

$$X = \frac{C_u}{C_d}, \quad K = \frac{U_o + 2C_o - U_d + G_* \Delta T_{o-u}}{C_d}$$

which permits calculation of C_u. Then U_u is obtained directly and

$$V = U_d + \frac{C_u}{C_d}\left[\frac{C_u^2 + C_d^2}{2}\right]^{1/2}$$

Now the characteristics on the high side of the bore are easily determined from the obtained values for U_u and C_u.

When the depth tends to zero, $C_d \to 0$. Then it has been said that the above formula for V loses its physical meaning because V

tends to infinity. In fact it is pointed out that V can never exceed $U_u + C_u$. First, the wave elements have a tendency to catch up the front of the bore at a speed U + C. Then the energy at the front of the bore is dissipated by turbulence as it is in a shock wave. Hence C_u decreases up to the point where V given by the above formula equals $U_u + C_u$. The above equation for the bore is always valid but then C_u also tends to zero when C_d tends to zero, i.e. near the shoreline. It is a "depression wave." (But U retains a value different from zero.) Then the set of equations becomes:

$$V = U_u + C_u = U_d + \frac{C_u}{C_d}\left[\frac{C_u^2 + C_d^2}{2}\right]^{1/2}$$

and also, since the characteristics for V become

$$V = \frac{dX}{dT} = U_u + C_u \,, \quad U_u + 2C_u = (U_u + 2C_u)_o + G\,\Delta T_{ou-u}$$

index o indicating the previous values on the V line. This set of equation can easily be solved from the equations for C_u:

$$C_u^4 - AC_u^2 + BC_u - C = 0$$

where $A = C_d^2$

$$B = 4\left\{(U_u + 2C_u)_o + G\Delta T - U_d\right\} C_d^2$$

$$C = 2\left\{(U_u + 2C_u)_o + G\Delta T - U_d\right\}^2$$

Then the calculation of U_u and $V = U_u + C_u$ is obtained easily for the following step.

It is seen that the shock wave disappears at the shoreline and that it is an edge of water which climbs on the beach. Hydrodynamically speaking, it is not a bore (nor a shock wave) which climbs on a dry bed, but a "depression wave." In fact it is true that the extreme edge of water is cut and the front of the water is roughly at a 60^o angle with the vertical, presenting the aspect of a bore. At this extreme edge C always equals zero since then $\eta = -d$ and V = U. The maximum wave run-up is obtained when U also tends to zero.

The same mathematical process can be applied for studying the wave due to the breaking of a dam: V can never exceed U + C in shallow water and U on a dry bed.

CONCLUSION

New concepts such as saturated and non-saturated breakers and corresponding criteria have been established. It has been demonstrated that spilling breakers follow the breaking index curve as long as the bottom slope is not steeper than 0.02, at which point the breaker becomes saturated. Hence the maximum possible wave run-up is given by the wave which breaks over this slope. If the wave is higher it will dissipate its energy sooner and will finally give the same run-up. Because of this result, application of the method of characteristics in deeper water is without use. The input of the method of characteristics can be taken as a limit solitary wave where the slope is 0.02 or 0.01 by safety. Starting the method of characteristics on a gentler slope in deeper water will give more error due to the cumulative effect of errors.

It has been shown that on a slope smaller than 1/10, the path curvature has an important effect on wave deformation, which cannot be neglected The correcting term to be included in the method of characteristics has been established. It has also been shown that the dissipation of energy at the crest of a spilling breaker (without shock wave) has to be taken into account by imposing a maximum value for C .

Finally the method of computing the bore in shallow water and the climb of the water on a dry beach has been established. It has been demonstrated that the bore or shock wave stops at the shoreline and it is a "depression wave" or edge of water which climbs on the beach.

It can be concluded that the wave run-up (and associated problems such as waves due to the breaking of a dam) can now be completely solved by theory, whatever the complexity of the slope.

A computing program presently under development should permit investigation of the wave spectrum run-up taking into account the interaction of one wave on the following wave.

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