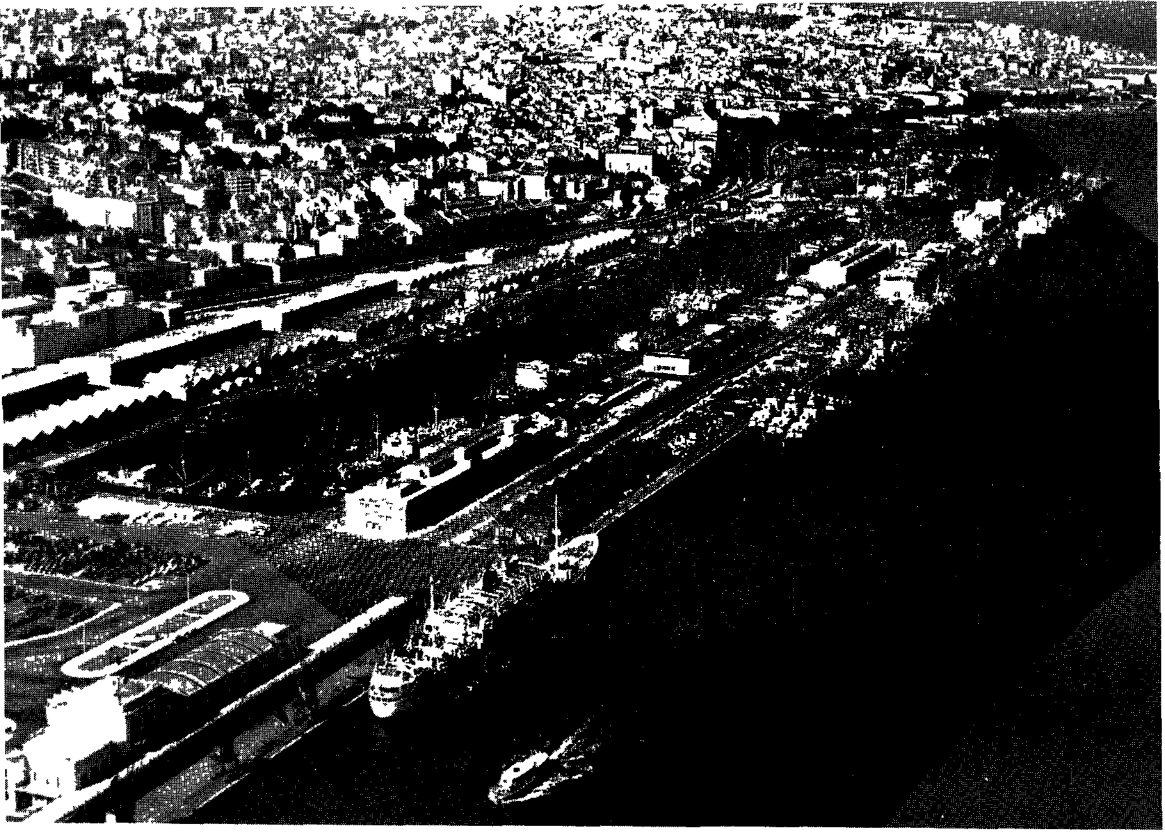
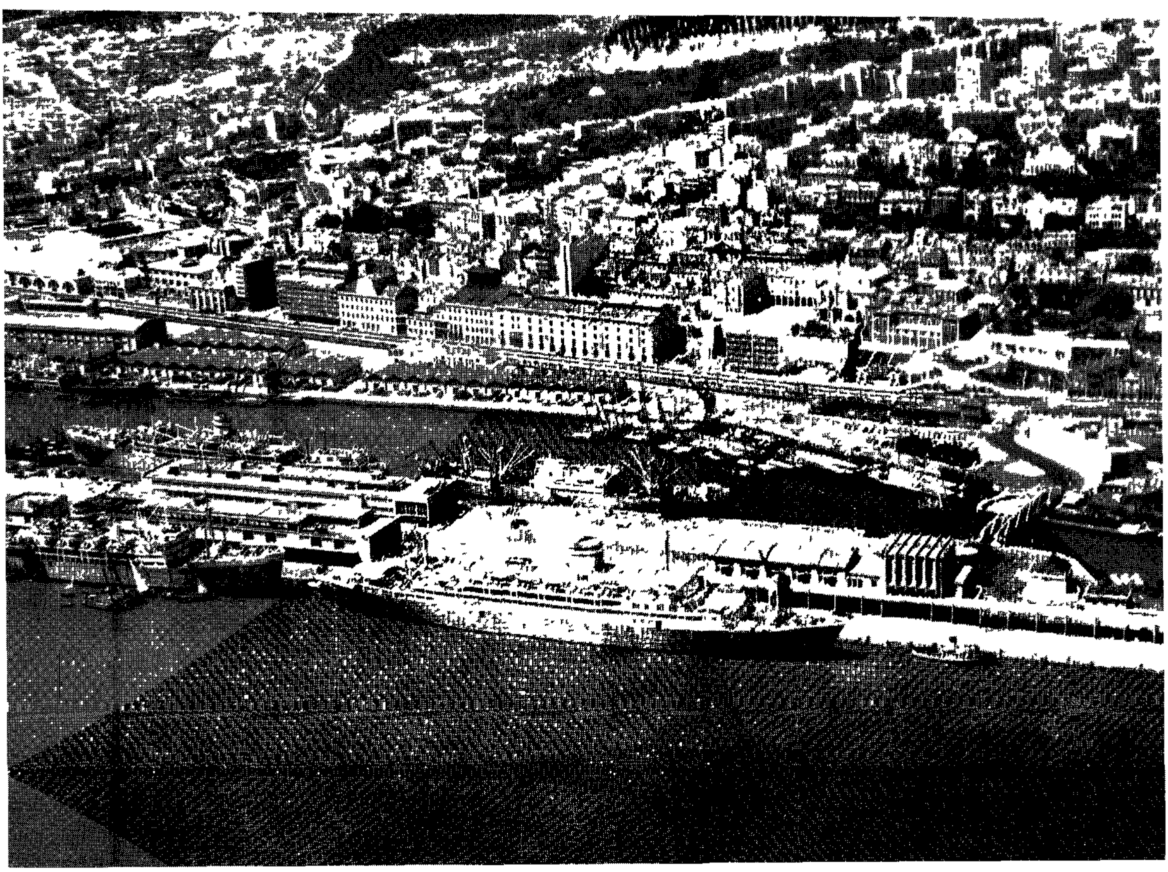


Lisbon Harbor

Part 1
THEORETICAL AND OBSERVED WAVE CHARACTERISTICS

Lisbon Harbor



Chapter 1

CNOIDAL WAVES IN SHALLOW WATER

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ABSTRACT

The propagation of long waves of finite amplitude in water with depth to wavelength ratios less than about one-tenth and greater than about one-fiftieth can be described by cnoidal wave theory. To date little use has been made of the theory because of the difficulties involved in practical application. This paper presents the theory necessary for predicting the transforming characteristics of long waves based on cnoidal theory. Basically the method involves calculating the power transmission for a wave train in shallow water from cnoidal theory and equating this to the deep water power transmission assuming no reflections or loss of energy as the waves move into shoaling water. The equations for wave power have been programmed for the range of cnoidal waves, and the results are plotted in non-dimensional form.

INTRODUCTION

The cnoidal wave theory developed in 1895 by Korteweg and deVries describes a class of permanent type long waves of finite amplitude. This theory which yields the solitary wave and the sinusoidal wave as its two limiting cases is useful for describing the propagation of periodic waves in shallow water with depths less than about 1/10 the wave length. The theory for cnoidal waves is based on the assumption that the square of the slope of the water surface is small in relation to unity. The properties of the waves are given in terms of the Jacobian elliptic functions and the complete elliptic integrals of the first and second kind.

The cnoidal wave theory has been studied more recently by Benjamin and Lighthill (1945), Patterson (1948), Keulegan and Patterson (1949), Littman (1957), Wehausen and Laitone (1960), Laitone (1960), (1961), (1962), (1963), and Sandover and Taylor (1962). Although this class of waves has received rather extensive theoretical study, little use has been made of the theory. Wiegel (1960) summarized much of the existing work on cnoidal waves and presented the leading results of Korteweg and deVries and Keulegan and Patterson in a more useable form. However, solutions of Wiegel's equations for the wave characteristics are still complex and involve either trial and error type computations or require extensive use of graphs of the cnoidal functions. To further facilitate the application of cnoidal wave theory, Masch and Wiegel (1961) computed several of the cnoidal wave characteristics such as celerity, wave length, and wave period based on the results of Korteweg and deVries, and presented these results in tabular form over a range applicable to water waves.

WAVE TRANSFORMATION

As waves propagate from deep water into shallow water ($d/L < 1/2$), the geometric properties of the waves such as height and length change with decreasing depth. It is customary to assume that as a wave train moves into shallow water, the wave period as defined under deep water conditions remains essentially constant. Generally speaking as the bottom begins to affect the wave motion, the phase velocity is reduced. Since the period for the shoaling waves remains nearly constant, the wave length is reduced and in shallow water the waves can be thought of as stacking up behind one another.

If the waves move perpendicular to the shoreline with their crests parallel to the bottom contours and it is further assumed that energy dissipation and reflections are negligible, then the power transmitted per unit crest width is constant at all points along the path the wave follows. If the waves refract as they move into shallow water, it can be assumed that the power is constant between adjacent orthogonals drawn normal to successive wave crests.

If such deep water wave characteristics as height, period, or length are known, it is possible to compute the rate of energy transmission or power transmission per unit width of the wave crest in deep water. Equating the deep water power transmission to that in shallow water as computed from a suitable finite amplitude wave theory and by making use of the fact that the period remains constant as the waves move into shallow water, the wave characteristics in the shallow water can be determined.

When evaluated for deep water conditions, small amplitude wave theory reduces to the following well known equations:

$$C_o = (g L_o / 2\pi)^{1/2} \tag{1}$$

$$L_o = gT^2 / 2\pi \tag{2}$$

and

$$P_o = \frac{1}{16} \frac{\gamma H_o^2 L_o}{T} \tag{3}$$

where C_o, L_o, H_o, P_o are the deep water wave velocity, length, height and power respectively. Based on trochoidal theory, the deep water wave power given by Mason (1951) is

$$P_o = \frac{\gamma H_o^2}{16} \sqrt{g L_o / 2\pi} \left(1 - 4.93 \frac{H_o^2}{L_o^2} \right) \tag{4}$$

The zero subscript is the conventional notation for deep water wave conditions.

Using the concepts of Rayleigh (1877) the power or rate of energy transmission in deep water can be equated to that in shallow water to give

$$P_o = (nEC)_o = (nEC) = P \tag{5}$$

where E is the wave energy, and n is the ratio of group velocity to phase velocity and has a value approaching one-half in deep water and unity in shallow water. For waves of small steepness, energy is directly proportional to the square of the wave height. Using eq. (5), the wave height is then given by

$$\frac{H}{H_o} = \left(\frac{1}{2}\cdot\frac{1}{n}\cdot\frac{C_o}{C}\right) \tag{6}$$

where

$$n = \frac{1}{2}\left[1 + \frac{4\pi d/L}{\sinh(4\pi d/L)}\right] \tag{7}$$

If it is assumed that the phase velocity of waves propagating over a sloping bottom is the same as that for waves moving over a horizontal bottom at the corresponding depth, then for waves of small steepness, eq. (6) becomes

$$\frac{H}{H_o} = \left[\frac{2\cosh^2(2\pi d/L)}{4\pi d/L + \sinh(4\pi d/L)}\right]^{1/2} \tag{8}$$

Similarly it can be shown that

$$\frac{L}{L_o} = \tanh(2\pi d/L) \tag{9}$$

The variability of wave height and length as given by eqs. (8) and (9) is usually related to the relative depth d/L_o, and these equations are shown graphically in Fig. 1. It is seen from eqs. (8) and (9) that shallow water wave characteristics can be predicted theoretically at any depth from a knowledge of specified deep water wave conditions.

Wiegel (1950), Iverson (1952), (1953), Eagleson (1956) and others have performed experiments on shoaling waves and have made comparisons of measured wave characteristics with those computed from small amplitude wave theory. They have found the small amplitude theory satisfactorily predicts the phase velocity of shoaling waves. This is as one would anticipate since the wave velocity changes only slightly when the effect of finite amplitude is taken into account. On the other hand, predicted wave heights were usually found to be smaller than measured heights when compared on the basis of small amplitude theory.

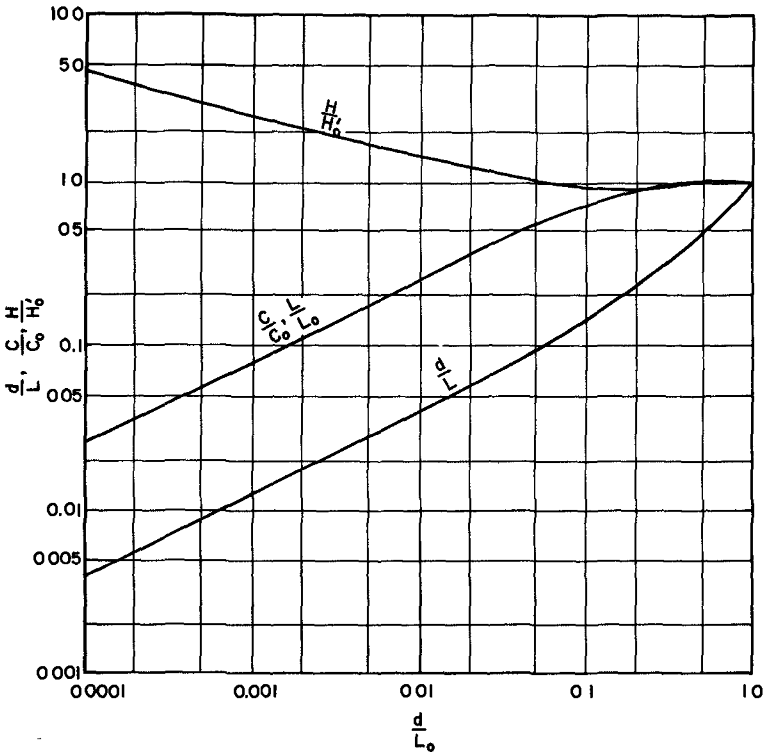

FIG 1. WAVE TRANSFORMATION
(after Wiegel, 1959)

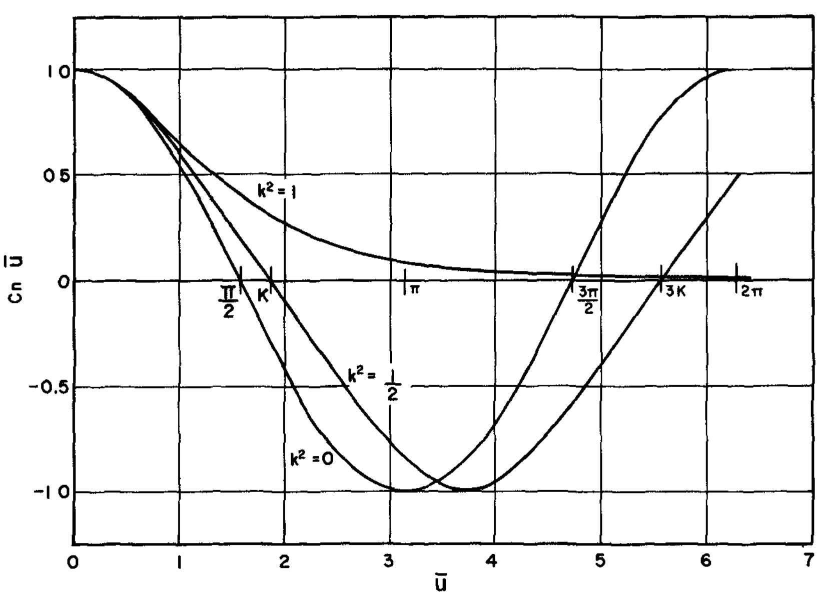

FIG 2 CNOIDAL FUNTION
(after Milne-Thompson)

In order to determine a useable expression for the transformation of waves in shallow water which takes into account the effect of finite amplitude waves, it remains to determine the power transmission in terms of the wave properties from a suitable shallow water wave theory. The following sections of this paper are devoted to this end in which the power transmission for shallow water waves is computed according to the cnoidal theory. Before computing the actual power transmission, a brief resume of cnoidal wave theory is included.

RESUME OF CNOIDAL WAVE THEORY

In shallow water where cnoidal wave theory is applicable, the wave profile, y_s, measured above the bottom is given by

$$y_s = y_t + H\,cn^2(\bar{u}, k) \tag{10}$$

where y_t is the distance from the bottom to the trough, H is the wave height, and $\bar{u}$ and k are the argument and parameter respectively of the elliptic cosine denoted hereafter as cn. The elliptic cosine is a periodic function of $\bar{u}$ whose amplitude is equal to unity. However the period is not a fixed constant as in the case of the circular functions but rather depends on the modulus, k, where k is defined over the range $o \leq k < 1$. The argument, $\bar{u}$, is defined by the definite integral

$$\bar{u} = \int_0^{\phi} \frac{d\phi}{\sqrt{1 - k^2 \sin^2\phi}} \tag{11}$$

which is an elliptic integral of the first kind, and is a function of k and the upper limit, ϕ. When evaluated over a quarter period, eq. (11) becomes

$$\int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2\phi}} = K\left(\frac{\pi}{2}, k\right) \tag{12}$$

which is the complete elliptic integral of the first kind. This is analogous to defining the quarter period of the circular functions by the complete circular integral.

Thus the period of the cn function is $4K(k)$ and the cn^2 function is $2K(k)$. It can be noted that when $k = 0$, $cn(\bar{u}, o) = \cos(\bar{u})$ and $K = \frac{\pi}{2}$, so that the elliptic cosine reduces to the circular cosine with a fixed quarter period of $\pi/2$ and a fixed period of 2π. Similarly when $k = 1$, the elliptic cosine degenerates to the sech $(\bar{u})$, the quarter period, $k(1) = \infty$, and the wave profile becomes essentially that of a solitary wave. The cn function is plotted in Fig. 2 for several values of k.

Equation (10) is often written in the form

$$y_s = y_t + H\,cn^2(\ ) \tag{13}$$

where $cn^2(\ )$ denotes $cn^2\left[2K(k)\ (x/L - t/T),\ k\right]$ where $\bar{u}$ has been repla by the more conventional notation of $2K(k)\ (x/L - t/T)$. Values of the c function have been tabulated over limited ranges of k by Spenceley and Spenceley (1947), Milne-Thompson, (1950), and Schuler and Gabelein (1955 Masch and Wiegel (1961) have extended the range by tabulating values of the sn, cn, and dn functions for $1\text{-}10^{-2} \leq k^2 \leq 1\text{-}10^{-40}$.

The distance from the bottom to the wave trough as used in eq. (13) is defined by the relation

$$\frac{y_t}{d} = \left(\frac{y_c}{d} - \frac{H}{d}\right) = \frac{16}{3}\frac{d^2}{L^2}\left\{K(k)\left[K(k) - E(k)\right]\right\} + 1 - \frac{H}{d} \tag{14}$$

where y_c is the distance from the bottom to the wave crest, d is the still water depth, L is the wave length, and K(k) and E(k) are the complete elliptic integrals of the first and second kind respectively. The wave length is given by the equation

$$L = \sqrt{\frac{16d^3}{3H}}\ k\,K(k) \tag{15}$$

Based on the work of Korteweg and deVries, the wave period is defined by the equation

$$T\sqrt{g/d} = \sqrt{\frac{16d}{3H}}\left[\frac{k\,K(k)}{1 + \left(\frac{H}{d}\right)\left(\frac{1}{k^2}\right)\left(\frac{1}{2} - \frac{E(k)}{K(k)}\right)}\right] \tag{16}$$

and the phase or propagation velocity is given by

$$\frac{C}{\sqrt{gd}} = \left[1 + \frac{H}{d}\cdot\frac{1}{k^2}\left(\frac{1}{2} - \frac{E(k)}{K(k)}\right)\right] \tag{17}$$

These equations are in the form given by Wiegel (1961) and are the relat used by Masch and Wiegel for computing their Tables of Cnoidal Wave Func tions.

If it is further assumed as an approximation that the pressure dist bution is linear, then the pressure at any point, y, above the bottom is

$$p = \rho g\,(y_s - y) \tag{18}$$

where y_s is defined by eq. (13).

The horizontal and vertical components of water particle velocity based on the equations of Keulegan and Patterson (1948) are

$$\frac{u}{\sqrt{gd}} = \left[\frac{h}{d} - \frac{h^2}{4d^2} + \left(\frac{d}{3} - \frac{y^2}{2d}\right)\frac{\partial^2 h}{\partial x^2}\right] \tag{19}$$

and

$$\frac{v}{\sqrt{gd}} = -g\left[\left(\frac{1}{d} - \frac{h}{2d^2}\right)\frac{\partial h}{\partial x} + \frac{1}{3}\left(d - \frac{y^2}{2d}\right)\frac{\partial^3 h}{\partial x^3}\right] \tag{20}$$

where u and v are the horizontal and vertical components of water particle velocity and h is defined by

$$h = y_s - d = -d + y_t + H\,cn^2(\ ) \tag{21}$$

POWER TRANSMISSION IN CNOIDAL WAVES

In considering power or the rate at which energy is transmitted across a vertical plane in the direction of wave propagation, it is convenient to define power as the product of energy per unit volume of fluid and the volume rate of movement. Using the bottom as a datum for y = o, and defining terms as in Fig. 3, the energy per unit volume with respect to the still water level is

$$E_{\forall} = p + \rho g y - \rho g d \tag{22}$$

Substituting the approximate pressure distribution relation of eq. (18) gives

$$E_{\forall} = \rho g (y_s - d) \tag{23}$$

The instantaneous rate, P, at which work is done across a section of unit width is

$$P = \int_0^{y_s} \rho g (y_s - d)\, u\, dA \tag{24}$$

The average power over one wave period is

$$\overline{P} = \frac{1}{T}\int_0^T P\,dt = \frac{1}{T}\int_0^T\int_0^{y_s} \rho g (y_s - d)\, u\, dA \tag{25}$$

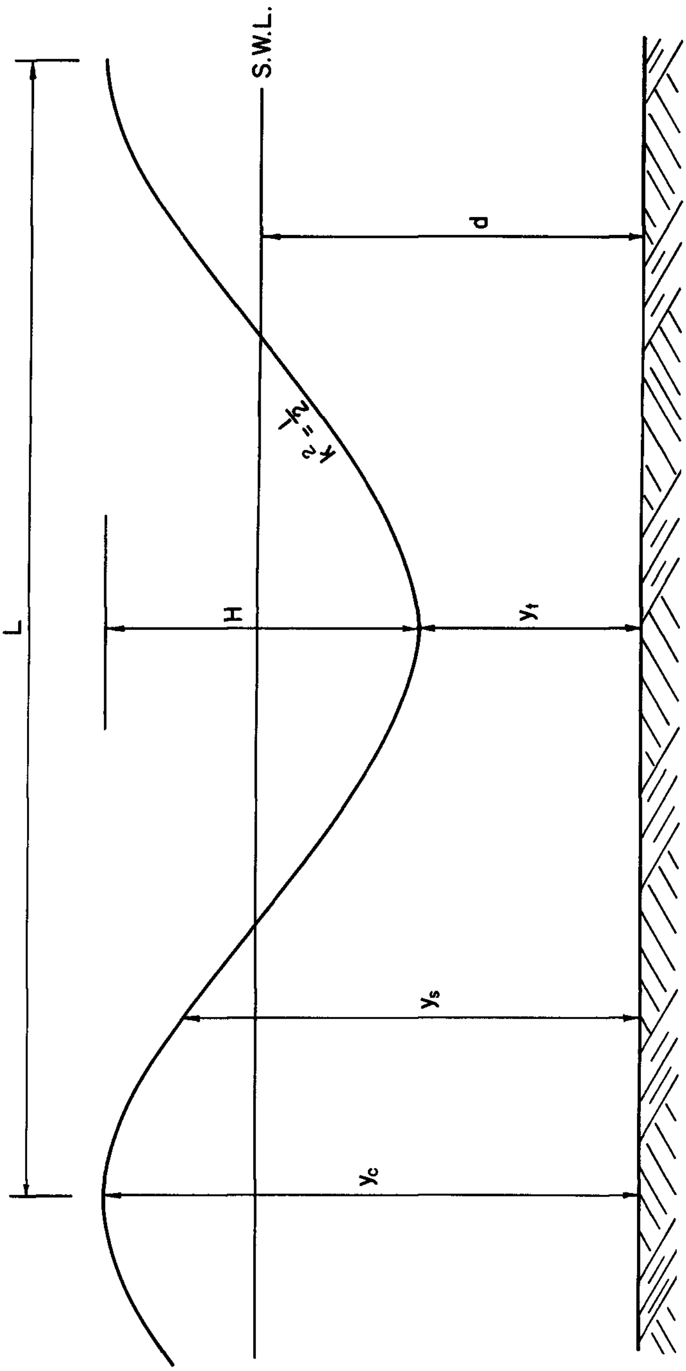

FIG. 3. DEFINITION SKETCH

Substituting the expressions of eqs. (13) and (19) for cnoidal waves into eq. (25) the average power is given by

$$\overline{P} = \frac{1}{T}\int_0^T \int_0^{y_t + H\,cn^2(\ )} \rho g \left[-d + y_t + H\,cn^2(\ )\right] u \, dy\, dt \tag{26}$$

where

$$u = \sqrt{gd}\Big[-\frac{5}{4} + \frac{3}{2}\frac{y_t}{d} - \frac{y_t^2}{4d^2} + \left(\frac{3H}{2d} - \frac{y_t H}{2d^2}\right) cn^2(\ )$$
$$- \frac{H^2}{4d^2} cn^4(\ ) - \frac{8H}{L^2} K^2(k)\left(\frac{d}{3} - \frac{y_t}{2d}\right)\big(k^2 sn^2(\ )\,cn^2(\ )$$
$$+ cn^2(\ )\,dn^2(\ ) - sn^2(\ )\,dn^2(\ )\big)\Big]$$

Now integrating first with respect to y, substituting the limits, and rearranging, the average power can be written as

$$\frac{\overline{P}}{(\rho g)(gd)^{1/2}} = \frac{1}{T}\int_0^T \Big\{\left(\frac{5dy_t}{4} - \frac{11}{4}y_t^2 + \frac{7}{4}\frac{y_t^3}{d} - \frac{y_t^4}{4d^2}\right) + cn^2(\ )\Big(\frac{5dH}{4}$$
$$- \frac{11}{2}Hy_t + \frac{21}{4}\frac{Hy_t^2}{d} - \frac{y_t^3 H}{d^2}\Big) + cn^4(\ )\Big(-\frac{11H^2}{4} + \frac{21}{4}\frac{H^2 y_t}{d} - \frac{3}{2}\frac{y_t^2 H}{d}$$
$$+ cn^6(\ )\left(\frac{7H^3}{4d} - \frac{H^3 y_t}{4d^2}\right) + cn^8(\ )\frac{H^4}{4d^2} + \frac{8HK^2(k)}{L^2}\,\Omega\Big[\Big(\frac{d^2 y}{3}$$
$$- \frac{d y_t^2}{3} - \frac{y_t^3}{6} + \frac{y_t^4}{6d}\Big) + cn^2(\ )\Big(\frac{d^2 H}{3} - \frac{y_t^2 H}{2} - \frac{2dHy_t}{3}$$
$$+ \frac{2Hy_t^3}{3d}\Big) + cn^4(\ )\left(-\frac{dH^2}{3} + \frac{y_t^2 H^2}{d} - \frac{y_t H^2}{2}\right)$$
$$+ cn^6(\ )\left(\frac{2y_t H^3}{3d} - \frac{H^3}{6}\right) + cn^8(\ )\left(\frac{H^4}{6d}\right)\Big]\Big\} \qquad (2$$

where $\Omega = -k^2\, sn^2(\ )\, cn^2(\ ) + cn^2(\ )\, dn^2(\ ) - sn^2(\ )\, dn^2(\ ) \neq \Omega(y)$.

Before performing the integration of the various even powers of the cn( ) function with respect to time, the elliptic cosines were expanded in terms of the elliptic integrals to determine the value of the integrals, and to prove that the elliptic integrals become complete elliptic integrals when integrated over a cnoidal wave period. Masch (1964) gives the details of these expansions as well as the evaluation of the various integral formulas. When evaluating the integrals with respect to time, it is also important to recognize that the period of the $cn^2(\ )$ function is equal to 2K(k), i.e., two times the first complete elliptic integrals. Since only tables of K(k) and E(k) are available, it becomes expedient to make use of the symmetry of the cnoidal function, and to write the identity

$$\int_{\bar{u}=0}^{\bar{u}=2K} cn^2(\bar{u}, k)\, d\bar{u} = 2\int_{\bar{u}=0}^{\bar{u}=K} cn^2(\bar{u}, k)\, d\bar{u} \qquad (28)$$

Integrating eq. (27) and after some rearrangement and collecting of terms, the average power transmission of a cnoidal wave over one wave period is

$$\begin{aligned}
\frac{\bar{P}}{(\rho g)\sqrt{gd}} = \frac{1}{T}\Bigg\{ & A_0 T + \frac{A_2 T}{k^2}\left[\frac{E(k)}{K(k)} + k^2 - 1\right] + \frac{A_4 T}{3k^4}\left[(4k^2-2)\frac{E(k)}{K(k)}\right. \\
& \left. + 3k^4 - 5k^2 + 2\right] + \frac{A_6 T}{15 k^6}\left[(23k^4 - 23k^2 + 8)\frac{E(k)}{K(k)} + 15k^6 - 23k^4\right. \\
& \left. + 27k^2 - 8\right] + \frac{A_8 T}{105 k^8}\left[(176k^6 - 264k^4 + 184k^2 - 48)\frac{E(k)}{K(k)} + 105k^8\right. \\
& \left. - 298k^6 + 353k^4 - 208k^2 + 48\right] + {}^{8H}\!/_{L^2}\,(K(k))^2\Bigg\{\frac{B_0 T}{k^2}\left[(4k^2\right. \\
& \left. - 2)\frac{E(k)}{K(k)} + 3k^4 - 5k^2 + 2\right] + \frac{2B_0 T(1-2k^2)}{k^2}\left[\frac{E(k)}{K(k)} + k^2 - 1\right] \\
& + B_0 T(k^2-1) + \frac{2B_2 T}{3k^4}\left[(4k^2-2)\frac{E(k)}{K(k)} + 3k^4 + 27k^2 - 8\right] + \frac{B_2 T}{k^2}(k^2-1) \\
& \left[\frac{E(k)}{K(k)} + k^2 - 1\right] + \frac{B_2 T}{5k^4}\left[(23k^4 - 23k^2 + 8)\frac{E(k)}{K(k)} + 15k^6 - 34k^4\right. \\
& \left. + 27k^2 - 8\right] + \frac{B_4 T}{3k^6}\left[(176k^6 - 264k^4 + 184k^2 - 48)\frac{E(k)}{K(k)} + 105k^8\right. \\
& \left. - 298k^6 + 353k^4 - 208k^2 + 48\right] + \frac{2B_4 T}{15k^6}(1-2k^2)\left[(23k^4 - 23k^2\right. \\
& \left. + 8)\frac{E(k)}{K(k)} + 15k^6 - 34k^4 + 27k^2 - 8\right] + \frac{B_4 T}{3k^4}(k^2-1)\left[(4k^2-2)\frac{E(k)}{K(k)}\right. \\
& \left. + 3k^4 - 5k^2 + 2\right] + \frac{B_6 T}{315 k^8}\left[(1689k^8 - 3378k^6 + 3537k^4 - 1848k^2\right.
\end{aligned}$$

$$+384)\frac{E(k)}{K(k)}+945k^{10}-3207k^{8}+5043k^{6}-4437k^{4}+2040k^{2}$$

$$-384\Big]+\frac{2B_6T(1-2k^2)}{105k^8}\Big[(176k^6-264k^4+184k^2-48)\frac{E(k)}{K(k)}$$

$$+105k^8-298k^6+353k^4-208k^2+48\Big]+\frac{B_6T(k^2-1)}{15k^6}\Big[(23k^4$$

$$-23k^2+8)\frac{E(k)}{K(k)}+15k^6-34k^4+27k^2-8\Big]+\frac{B_8T}{3465k^{10}}\Big[(19524k^{10}$$

$$-48810k^8+68232k^6-53538k^4+22272k^2-3840)\frac{E(k)}{K(k)}$$

$$+10395k^{12}-40947k^{10}+80199k^8-93729k^6+64434k^4-24192k^2$$

$$+3840\Big]+\frac{2B_8T(1-2k^2)}{945k^{10}}\Big[(1689k^8-3378k^6+3537k^4-1848k^2$$

$$-3840)\frac{E(k)}{K(k)}+945k^{10}-3207k^8+5043k^6-4437k^4+2040k^2$$

$$-384\Big]+\frac{B_8T(k^2-1)}{105k^8}\Big[(176k^6-264k^4+184k^2-48)\frac{E(k)}{K(k)}+105k^8$$

$$-298k^6+353k^4-208k^2+48\Big]\Big\}\Big\}\qquad(29)$$

where

$$A_0=\left(\frac{5dy_t}{4}-\frac{11y_t^2}{4}+\frac{7y_t^3}{4d}-\frac{y_t^4}{4d^2}\right)$$

$$A_2=\left(\frac{5dH}{4}-\frac{11Hy_t}{2}+\frac{21Hy_t^2}{4d}-\frac{y_t^3H}{d^2}\right)$$

$$A_4=\left(-\frac{11H^2}{4}+\frac{21H^2y_t}{4d}-\frac{3y_t^2H^2}{2d^2}\right)$$

$$A_6=\left(\frac{7H^3}{4d}-\frac{H^3y_t}{4d^2}\right)$$

$$A_8=\left(\frac{H^4}{4d^2}\right)$$

$$B_0=\left(\frac{d^2y_t}{3}-\frac{dy_t^2}{3}-\frac{y_t^3}{6}+\frac{y_t^4}{6d}\right)$$

$$B_2 = \left(\frac{d^2 H}{3} - \frac{Y_t^2 H}{2} - \frac{2 d H Y_t}{3} + \frac{2 H Y_t^3}{3 d}\right.$$

$$B_4 = \left(-\frac{dH}{3} + \frac{Y_t^2 H^2}{d} - \frac{Y_t H^2}{2}\right)$$

$$B_6 = \left(\frac{2}{3}\frac{Y_t H^3}{d} - \frac{H^3}{6}\right)$$

$$B_8 = \left(\frac{H^4}{6d}\right)$$

For any given value of the modulus, k, the groups of terms which resulted from the integrations of the cn( ) functions and involve k, K(k), and E(k) can be evaluated. These terms are dimensionless and will be denoted by $\bar{c}_{A2}$, $\bar{c}_{A4}$,.... and $\bar{c}_{B2}$, $\bar{c}_{B4}$,.... Equation (29) for the average power can also be put into a more convenient non-dimensional form by dividing through by the square of the still water depth to give

$$\begin{aligned}
\frac{\bar{P}}{(\rho g)\sqrt{gd}\; d^2} = & \left[\frac{5}{4}\left(\frac{Y_t}{d}\right) - \frac{11}{4}\left(\frac{Y_t}{d}\right)^2 + \frac{7}{4}\left(\frac{Y_t}{d}\right)^3 - \frac{1}{4}\left(\frac{Y_t}{d}\right)^4\right] \\
& + \left(\frac{H}{d}\right)\left[\frac{5}{4} - \frac{11}{2}\left(\frac{Y_t}{d}\right) + \frac{21}{4}\left(\frac{Y_t}{d}\right)^2 - \left(\frac{Y_t}{d}\right)^3\right]\bar{c}_{A2} \\
& + \left(\frac{H}{d}\right)^2\left[-\frac{11}{4} + \frac{21}{4}\left(\frac{Y_t}{d}\right) - \frac{3}{2}\left(\frac{Y_t}{d}\right)^2\right]\bar{c}_{A4} \\
& + \left(\frac{H}{d}\right)^3\left[\frac{7}{4} - \frac{1}{4}\left(\frac{Y_t}{d}\right)\right]c_{A6} + \left(\frac{H}{d}\right)^4\left(\frac{1}{4}\right)\bar{c}_{A8} \\
& + \frac{3}{16}\frac{1}{k^2}\left\{\left(\frac{H}{d}\right)^2\left[\frac{8}{3}\left(\frac{Y_t}{d}\right) - \frac{8}{3}\left(\frac{Y_t}{d}\right)^2 - \frac{4}{3}\left(\frac{Y_t}{d}\right)^3 + \frac{4}{3}\left(\frac{Y_t}{d}\right)^4\right]\bar{c}_{B0}\right. \\
& + \left(\frac{H}{d}\right)^3\left[\frac{8}{3} - \frac{16}{3}\left(\frac{Y_t}{d}\right) - 4\left(\frac{Y_t}{d}\right)^2 + \frac{16}{3}\left(\frac{Y_t}{d}\right)^3\right]\bar{c}_{B2} \\
& + \left(\frac{H}{d}\right)^4\left[-\frac{8}{3} - 4\left(\frac{Y_t}{d}\right) + 8\left(\frac{Y_t}{d}\right)^2\right]\bar{c}_{B4} \\
& + \left(\frac{H}{d}\right)^5\left[-\frac{4}{3} + \frac{16}{3}\left(\frac{Y_t}{d}\right)\right]\bar{c}_{B6} \\
& + \left(\frac{H}{d}\right)^6\left[\frac{4}{3}\right]\bar{c}_{B8}
\end{aligned} \qquad (3$$

Using a functional notation, the average power transmission thus becomes

$$\frac{\overline{P}}{(\rho g)\sqrt{gd}\, d^2} = f_1\left(\frac{H}{d}, \frac{Y_t}{d}, k\right) \qquad (31)$$

The number of parameters in eq. (31) may be reduced by substituting eq. (15) into eq. (14) which gives

$$\frac{Y_t}{d} = \left(\frac{H}{d}\right)\frac{1}{k^2K^2(k)}\left[K(k) - E(k) + 1 - \frac{H}{d}\right] \qquad (32)$$

Since Y_t/d can be written in terms of H/d and k, eq. (32) can be substituted into eq. (30) giving the dimensionless power relation in terms of two parameters

$$\frac{\overline{P}}{(\rho g)\sqrt{gd}\, d^2} = f_2\left(\frac{H}{d}, k\right) \qquad (33)$$

While the power remains constant as the wave propagates into shallow water, the period also remains essentially constant. For cnoidal waves, the period is as given by eq. (16) or again using a functional notation

$$T\sqrt{g/d} = f_3\left(\frac{H}{d}, k\right) \qquad (34)$$

Assuming then that the specific wave characteristics, $\overline{P}$ and T, are known or can be calculated in deep water, the left-hand side of eqs. (33) and (34) are known for any shallow water depth, d. This leaves two equations with two unknowns which can be solved simultaneously for k and H/d.

COMPUTATION OF CNOIDAL WAVE POWER

In order to evaluate the functional relationship of eq. (33) for application to shallow water waves, the terms involving k, K(k) and E(k) (i.e., $\overline{c}_{A2}, \ldots$ and $\overline{c}_{B2}, \ldots$) must be evaluated for the range $0.05 \leq k^2 \leq 1-10^{-40}$, the range of modulus applicable to water waves. Because of the repetitive nature of the somewhat complex expressions involving k, these calculations can best be done by digital computer. Still other repeated computations are necessary. For example, the parameter H/d has a range extending from about 0.01 to 0.78 such that for each value of H/d, a range of k's may exist. To enable the dimensionless power term to be written in terms of two parameters, use must also be made of eq. (14) for y_t/d, the value of which also depends upon k and H/d.

Equations related to the power transmission of cnoidal waves were computed on a CDC 1604 digital computer. Because it was desirable to check various parts of the calculations, the computer program was dividec into two parts. In the first part of the program, those terms involving k, K(k), and E(k) were computed for values of k^2 up to $1\text{-}10^{-40}$. The coefficients evaluated from this program were checked and used as input for the second part of the program which computed the actual power.

The second part of the overall computer program involves the calculation of eqs. (14), (33), and (34) for the range of cnoidal waves. For each value of H/d, the computations have been truncated to include only those values for C^2/gd (as computed from eq. (17)) greater than 0.8200. This is the same range of computations used by Masch and Wiegel (1961). Since the values of C^2/gd have already been computed and given in tabular form, these values were not printed out in the program for power.

CNOIDAL WAVE CHARACTERISTICS

Using the computed values for eqs. (33) and (34), the simultaneous solution for these two equations is given in Fig. 4 with the non-dimensic power transmission term plotted against the ratio, H/d. This graph covers the range of power calculations and utilizes $T\sqrt{g/d}$, which is a function of the modulus of the elliptic integrals, as a parameter. The parametric values of $T\sqrt{g/d}$ have been selected to be representative of the range applicable to cnoidal waves. The solution of eqs. (33) and (34) have als been plotted in Fig. 5 with the non-dimensional power term plotted agains $T\sqrt{g/d}$ for different values of H/d up to the limiting value of 0.78.

The simultaneous solution of eqs. (33) and (34) enables the ratio H/d to be determined from which it is possible to compute the transformin wave height. Assuming that either deep water wave conditions are known or can be predicted so that the period and the deep water power are speci fically known or that the wave power transmission is known at some depth, then for any other shallow water depth, the left hand sides of eqs. (33) and (34) are known, and the wave height at the specified shallow water depth can be determined. These calculations can be carried out with the aid of either Fig. 4 or 5. On the basis of deep water conditions, an initial point may be located for example on Fig. 5. As the wave progress into shallow water, the average power and period are assumed to remain constant, while the depth decreases. For a depth $d_1 < d_o$, the depth used to locate the initial point, the ordinate of Fig. 5 increases defining a new level for the non-dimensional power term. Similarly, for $d_1 < d_o$, the period parameter increases, and defines the horizontal position on the ne power level. From this newly defined point, the value of H_1/d_1 can be re from the graph, and the wave height, H_1 can be obtained from the depth d_1

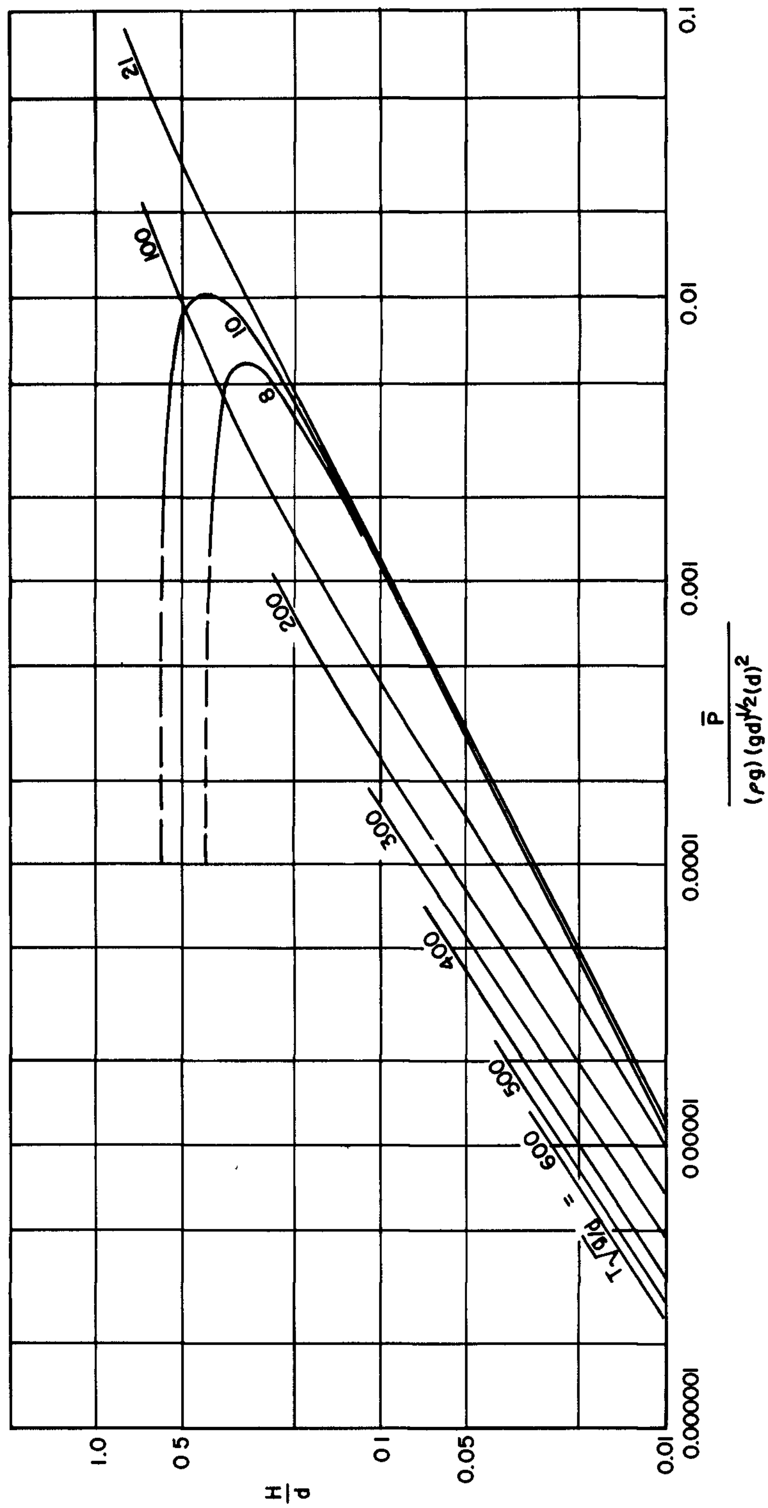

FIG. 4. $\frac{H}{D}$ vs. $\frac{\bar{P}}{(\rho g)\ (gd)^{1/2}\ (d)^2}$

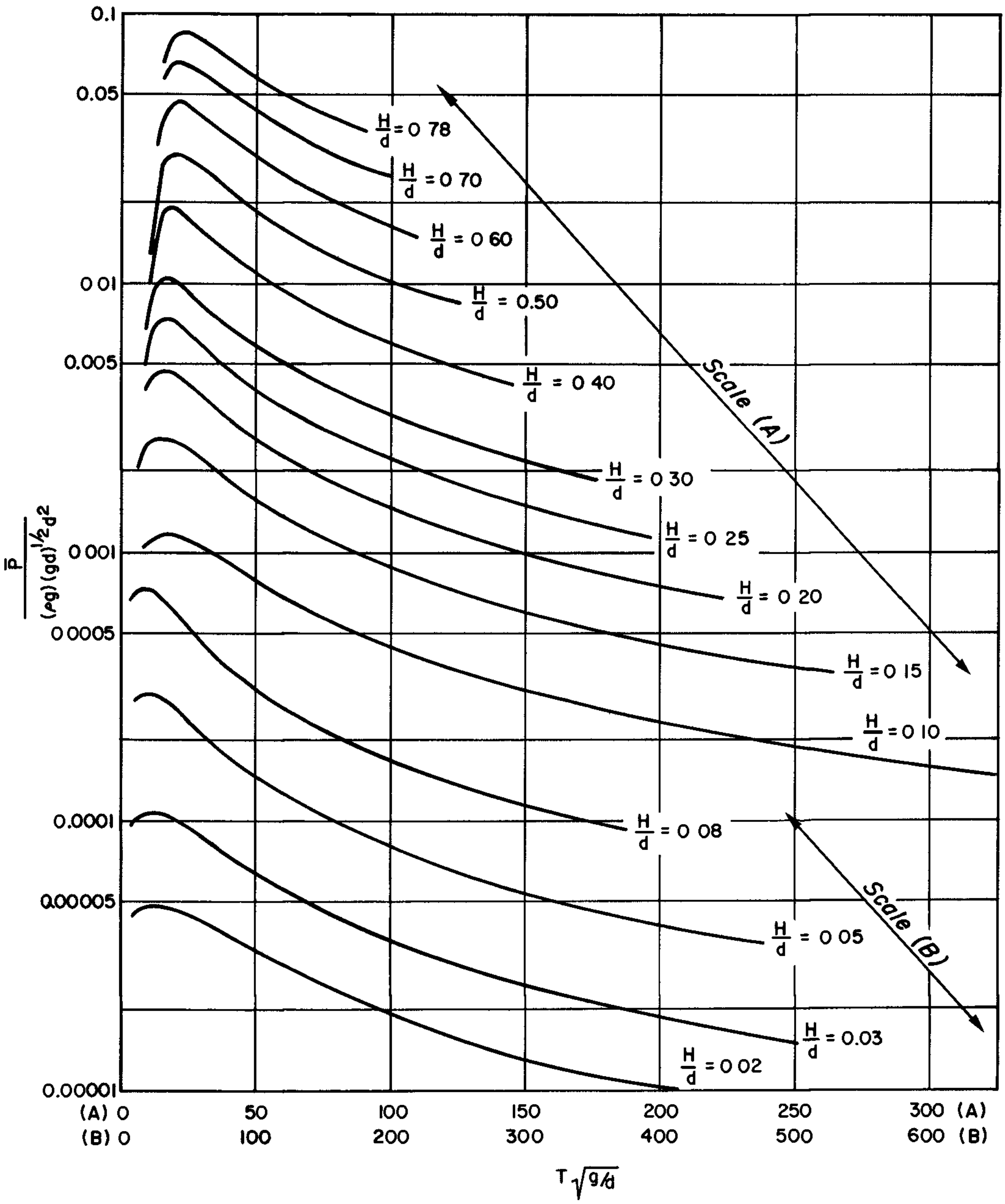

FIG. 5. $\frac{\overline{P}}{(\rho g)(gd)^{1/2}\,d^2}$ vs $T\sqrt{g/d}$

Figure 6 is a plot showing the relation of y_t/d to $T\sqrt{g/d}$ for several different values of H/d. This plot clearly shows the steepening of the crests and the flattening of the troughs. Since an increase in the value of $T\sqrt{g/d}$ reflects an increase in the modulus, k, then as the depth becomes smaller, the modulus, k, approaches unity and the wave form approaches that of the solitary wave. Thus as $T\sqrt{g/d}$ increases, the distance from the bottom to the wave trough, y_t, approaches the still water depth, d, and the ratio, y_t/d approaches unity.

The transformation of wave length as cnoidal waves move into shallow water can be determined directly from wave steepness considerations. Equation (15) which defines cnoidal wave length can be rearranged to the form

$$\frac{H}{L} = \left[\left(\frac{H}{d}\right)^3 \cdot \frac{3}{16}\right]^{1/2} \frac{1}{k\,K(k)} \tag{35}$$

which is an expression for the conventional wave steepness. Using the Tables of Cnoidal Wave Functions by Masch and Wiegel (1961), eq. (35) has been evaluated and the steepness has been plotted against $T\sqrt{g/d}$ for different values of H/d in Fig. 7. As the wave moves into shoaling water, it has been assumed that the period remains constant, so that the quantity $T\sqrt{g/d}$ increases. At the same time, it can be seen from Figs. 4 and 5 that for a fixed period, the wave height also increases as the depth becomes shallower, and the ratio H/d increases up to the limiting value of 0.78. Thus from Fig. 7, the wave steepness is seen to increase as the wave propagates into shallow water. It can also be seen from eq. (35) that for the range of cnoidal wave theory, the wave length decreases as the wave moves into shallower water.

From Fig. 7, it can be noted that cnoidal waves are not very steep. In general the value of wave steepness is less than 0.1, and for the greater part of the range of cnoidal waves, the shallow water steepness is actually less than 0.01. These values of steepness are considerably less than most of those reported in laboratory studies on shoaling waves. However this is not unexpected since the theory is limited to a class of long waves in water with depths less than about one-tenth the wave length.

SUMMARY

The study described in this paper has been devoted to the application of cnoidal wave theory to the transformation of long waves in shoaling water. The method used involved the calculation of the power transmission of cnoidal waves in terms of the geometric wave characteristics and the complete elliptic integrals of the first and second kind. The power and wave period equations were evaluated on a digital computer for the range of cnoidal waves assuming no loss of energy and no reflections as the waves propagate into shallow water. Variations of wave parameters such as height length and steepness were determined from theoretical considerations. It was found that both the wave height and steepness increases as the depth becomes shallower, whereas the wave length decreases with decreasing water

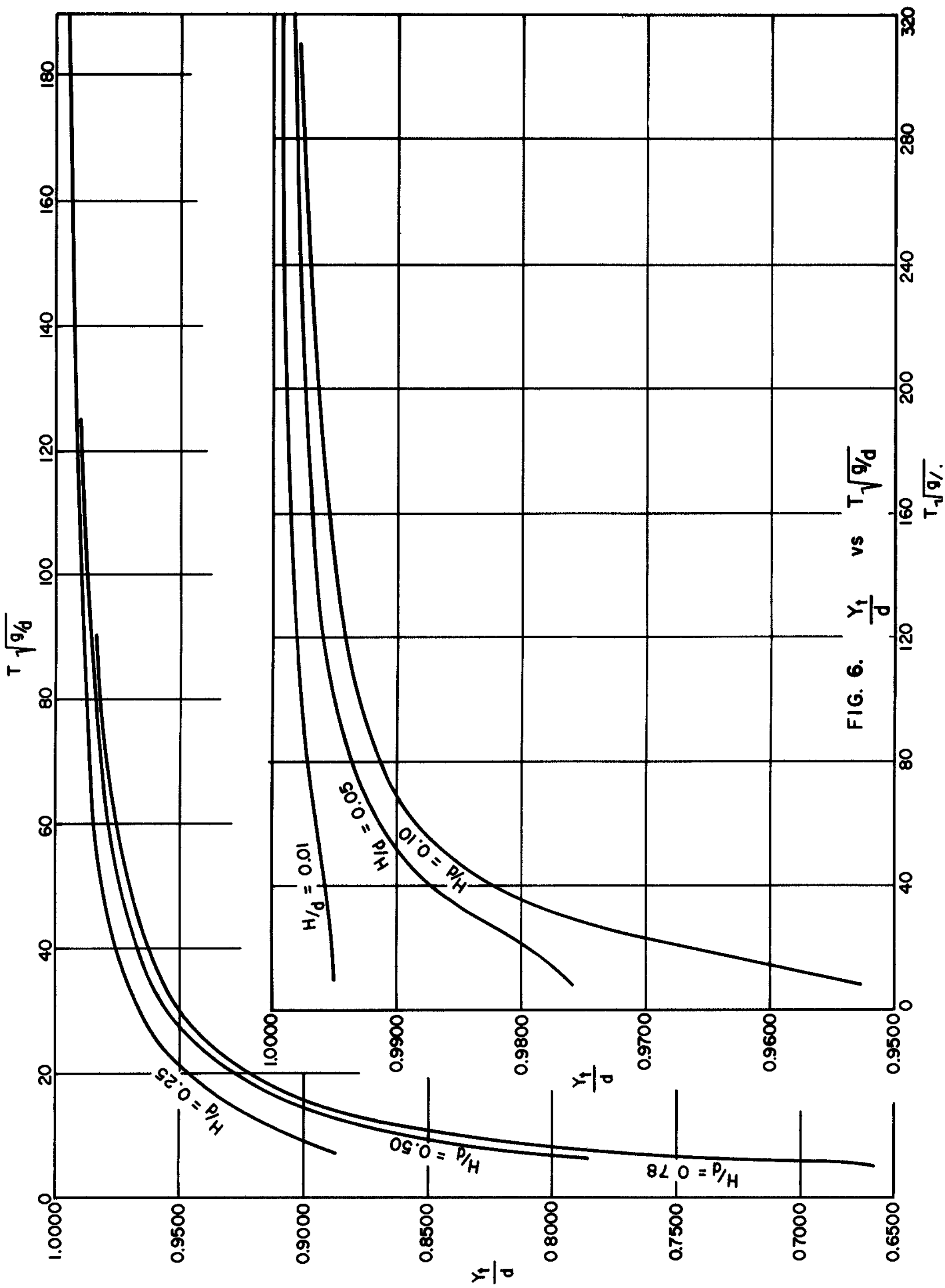

FIG. 6. $\frac{Y_t}{d}$ vs $T\sqrt{g/d}$

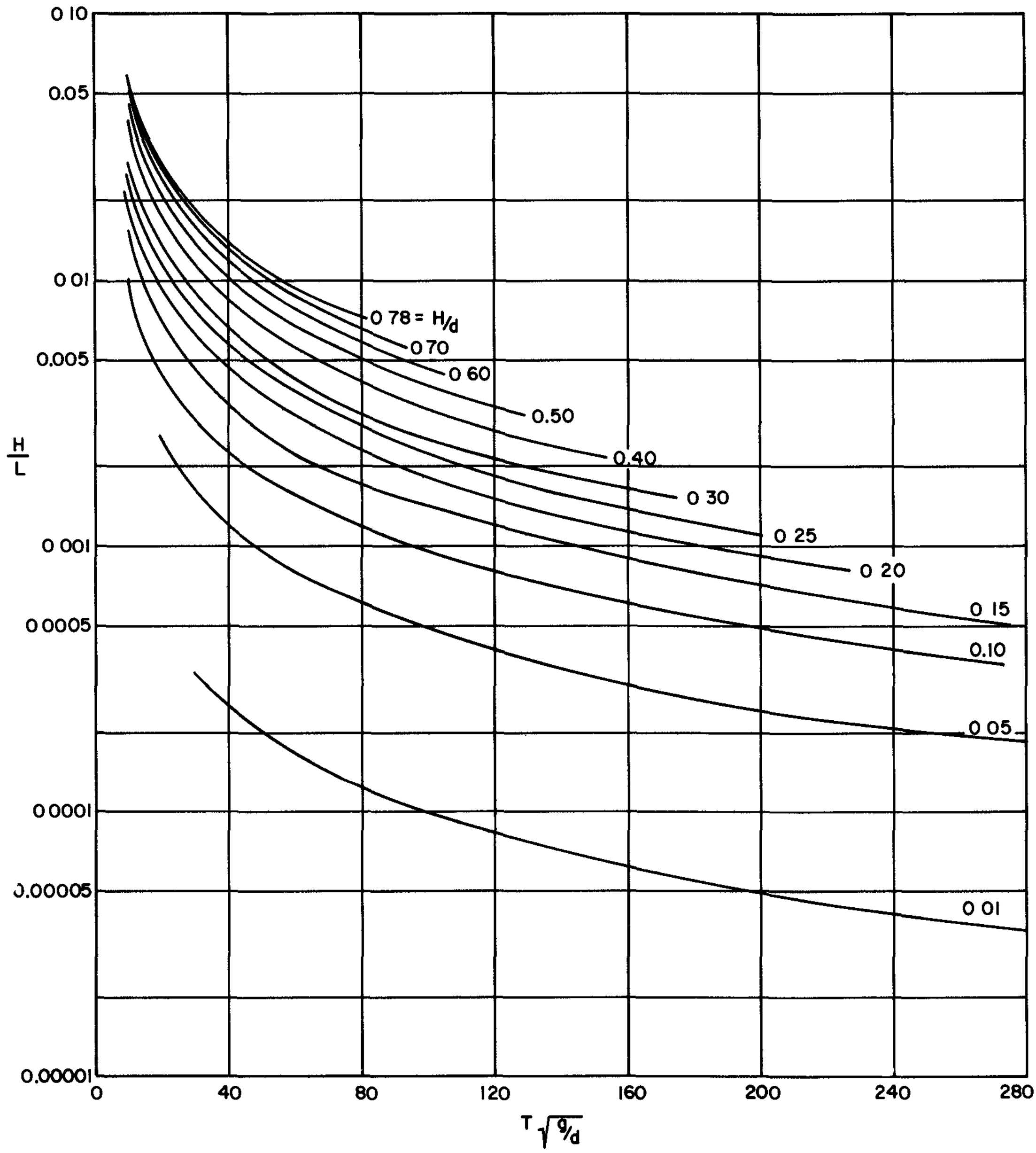

FIG. 7. $\frac{H}{L}$ vs $T\sqrt{g/d}$

depth. These results are compatible with those found by other wave theories although the range of steepnesses for cnoidal waves is much lower than that for other finite amplitude waves.

Generally speaking cnoidal wave characteristics cannot be represented as neatly as can the properties of waves determined by other theories. According to linear wave theory, the phase velocity, wave length, and wave period are independent of the wave height. Even to the second order Stokes theory, the phase velocity, wave length and steepnes are still independent of the wave height. On the other hand, the relati ships for cnoidal wave properties, eqs. (15), (16), (17), and (35) all involve the wave height in addition to the still water depth. Hence sin graphs such as Fig. 1 which show wave height, phase velocity, steepness, etc. related directly to d/L_0 cannot be constructed. Since most cnoidal wave equations are three variable equations, families of curves are necessary to represent the results.

It still remains to apply the theoretical computations of this stu to experimental or field data. Although a thorough search of the litera has not been completed, there appears to be only meager data reported fc shoaling waves in water with d/L less than 0.1. Of the data available f d/L less than 0.1, wave steepnesses are relatively large and wave perioc are short so that the waves are out of the range of cnoidal theory.

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