Chapter 46

ANALYSIS OF THE RESPONSE OF OFFSHORE-MOORED SHIPS TO WAVES

Jan J. Leendertse*
The RAND Corporation
Santa Monica, California

INTRODUCTION

A vessel moored at sea will experience complicated series of translational and rotational oscillations due to sea waves. These motions can be considered as the summation of six components, three translational and three rotational.

In the presently available analyses of motions of unmoored ships, differential equations can be written for each mode of movement. Unfortunately, motions in one of these modes are coupled to motions of other modes, and the analysis becomes rather complicated. Generally, the problem is simplified by neglecting some of the coupling effects and by specifying the position of the vessel in the wave system.

This study develops and analyzes a model for a moored ship restrained by mooring lines, using the presently available mathematical models for the free ship and the force-displacement relationship of the cable-holding points on the ship.

The coupled movement (three degrees of freedom) in a vertical plane through the longitudinal axis of the vessel and the generated mooring-line forces are considered in detail. The general case of six degrees of freedom in arbitrary heading is discussed briefly in general terms.

MOTIONS OF AN UNRESTRAINED VESSEL IN HARMONIC WAVES

Referring to the analyses by Weinblum and St. Denis (1950), the movement of a vessel unrestrained by mooring lines in harmonic waves may be expressed with certain approximations by the second-order linear differential equation

$$M_{ss} \frac{ds^2}{dt^2} + N_{ss} \frac{ds}{dt} + K_{ss} s + R_t = A\bar{F}^s_{ex} \, e^{j\omega t} \qquad (1)$$

The first subscript of the mass, damping, and stiffness coefficients refers to the considered force or moment equation; the second subscript, to the mode of movement to which the coefficient belongs (see Appendix for symbols). The first term on the left in Eq. (1) represents the inertia force; the second term represents the damping force; the third term is the restoring force; and the fourth term is the force due to other modes of movement. The term on the right expresses the periodic force of the waves.

*This research is sponsored by the United States Air Force under Project RAND. This is an abridgment of RAND Memorandum RM-3368-PR. The views, conclusions, and recommendations expressed herein do not necessarily reflect the official views or policies of the United States Air Force.

Extensive literature is available concerning the calculations of the mass and damping coefficients for a ship of particular dimensions and the periodic wave force. Weinblum and St. Denis (1950), and Korvin-Kroukovsky (1961), particularly, present readily applicable data for calculating these coefficients and the wave forces. However, in many instances it will be necessary to obtain these coefficients from model tests. It is noted here that the magnitude of the wave force depends on the direction of the ship to the waves.

Information about the coupling of the different modes of movement is limited, and only a few incidental cases have been investigated; for example the coupled heave and pitch by Korvin-Kroukovsky and Jacobs (1957). Weinblum and St. Denis neglect the coupling in their analyses of ship motion, and in this study, the coupling term will also be neglected initially.

For the unrestrained ship, the restoring forces and moments in the different modes are caused by the displacement of the ship from the position of rest; if the ship is moored, the forces in the mooring line will, of course, cause additional restoring forces and moments.

MOORING-LINE CHARACTERISTICS

The forces exerted on a ship or vessel by mooring it with a long single chain or cable that has an embedded anchor at its other end are functions of the weight of the chain or line and the location of the holding point in the ship relative to the anchor. If it is assumed that the cable is lying partly on a flat bottom as in (a) of Fig. 1, then the horizontal and the vertical forces on the ship are nonlinear functions of the horizontal and vertical displacement. Based on the analyses of single mooring lines by use of catenary equations, (b) of Fig. 1 presents the total tension and its horizontal and vertical components as a function of the displacement in nondimensional parameters.

In a particular condition of the mooring chain, for example, as presented in (a) of Fig. 1, a rectangular-coordinate system is fixed to this point, with the x-axis horizontal and z-axis vertical. For small displacements around the holding point (o,o) the horizontal and the vertical components of the force in the chain at this point may be assumed to be linear with the displacement and may be expressed by

$$H_{(x,z)} = H_{(o,o)} + ax + bz \tag{2}$$

$$V_{(x,z)} = V_{(o,o)} + cx + dz \tag{3}$$

The coefficients a, b, c, and d can be obtained directly from Fig. 2, which is based upon an analysis of the catenary equations. It will be noted that $b < a$ and $d < c$.

If a chain with a sinker is used, the forces can again be expressed by Eqs. (2) and (3), but the calculation of the coefficients becomes cumbersome.

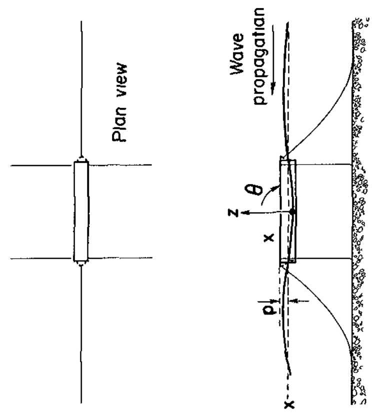

FIG 3 SPREAD-MOORED SHIP

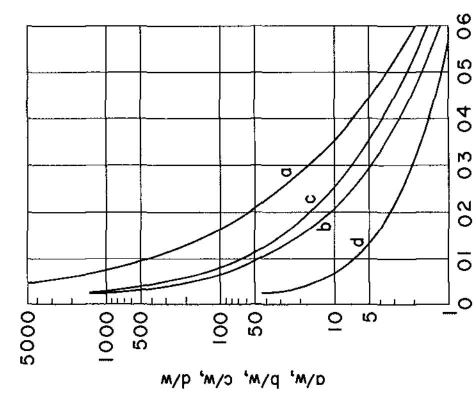

FIG 2 LINEAR COEFFICIENTS FOR SMALL DISPLACEMENT OF HOLDING POINT

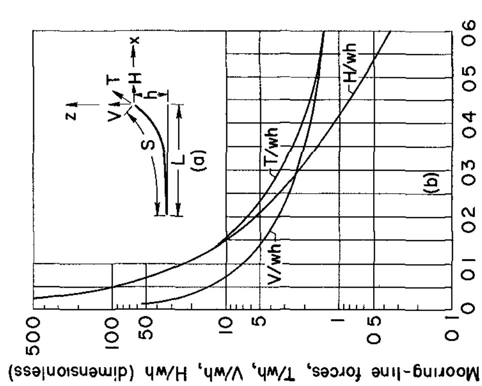

FIG 1 NONDIMENSIONAL REPRESENTATION OF MOORING-LINE FORCES AS FUNCTION OF DISPLACEMENTS

SPREAD-MOORED SHIP

Spread-mooring is used presently in the oil industry for mooring tender-barges near offshore drilling platforms. The layout of a simple mooring is represented in Fig. 3. It is assumed that the waves approach the ship head-on. Initially, it is assumed that the ship is subjected to uniform waves; later on, the effect of irregular waves will be introduced.

The ship's motions in the plane considered involve surging, heaving, and pitching. For the unrestrained (free-floating) ship, surge does not have important effects on the heave and pitch and consequently may be considered uncoupled. In the case of the moored ship, however, coupling will enter into the system due to the mooring lines. For example, the position of the bow, which is determined by heave and pitch, influences the horizontal component of the mooring-line force, and hence the surge.

Referring to Eq. (1), Weinblum and St. Denis (1950), and Wilson (1959) the linearized equation of motion in surge for the center of gravity of the unrestrained ship, compared to a fixed coordinate system taken in the cente of the ship in still water, takes the form

$$M_{xx}\ddot{x} + N_{xx}\dot{x} = A\overline{F}^{x}_{ex}\, e^{j\omega t} \tag{4}$$

where

$M_{xx} = M + M''_x$ = virtual mass of the ship in x direction

M = mass of ship

M''_x = added mass in x direction

Generally, the drag is small and may be neglected. However, in some cases, moored crafts may be built specially for mooring purposes, and in that case, no emphasis may be placed on towing or propulsion characteristic Then, N_{xx} (damping coefficient) is not necessarily small, and estimates of values may be obtained from the propulsion characteristics and a linearization process as developed by Havelock (described in Ref. 1) for the heaving motion. For the time being, the drag term will be maintained, being important even when small in cases of a resonance condition.

In addition to the inertia and drag forces, a restraining force exists for the moored vessel, and the equation of motion becomes

$$M_{xx}\ddot{x} + N_{xx}\dot{x} + F_h = A\overline{F}^{x}_{ex}\, e^{j\omega t} \tag{5}$$

where F_h is the resultant horizontal component of the restoring force of the mooring cables. With reference to Eqs. (2) and (3), taking the direction of the x-axis toward the left in the direction of wave propagation, the horizontal force of the left cable is

$$H_{(x,z)_{stern}} = H_{(o,o)_{stern}} - a(x - p\theta) + b(z + L\theta) \tag{6}$$

and for the cable on the upstream side

$$H_{(x,z)_{bow}} = - H_{(o,o)_{bow}} - a(x - p\theta) - b(z - L\theta) \tag{7}$$

The other four mooring lines have no significant component in the x direction. Consequently, the total restoring force is

$$- F_h = - 2ax + 2(bL = ap)\theta \tag{8}$$

Thus, Eq. (5) becomes

$$M_{xx}\ddot{x} + N_{xx}\dot{x} + 2ax - 2(bL + ap)\theta = A\bar{F}^{x}_{ex}\, e^{j\omega t} \tag{9}$$

Introducing the stiffness coefficients

$$K_{xx} = 2a \tag{10}$$

and

$$K_{x\theta} = - 2(bL + ap) \tag{11}$$

Eq. (9) becomes

$$M_{xx}\ddot{x} + N_{xx}\dot{x} + K_{xx}x + K_{x\theta}\theta = A\bar{F}^{x}_{ex}\, e^{j\omega t} \tag{12}$$

In this analysis, following the presentation by Kriloff (1898), Weinblum and St. Denis (1950), and Wilson (1959), the coupling effects as induced on the free-floating ship are neglected. Tests on ship models and computation of coupled and uncoupled motions indicated that neglecting the coupling terms is of minor significance for the pitching motion but is more important for the heaving motions. As will appear, since the effects of heave on the mooring-line forces are relatively minor compared with those of pitch, neglecting these coupling terms in the motion equation of the free-floating ship seems justified and simplifies the analysis significantly.

For the moored ship, the heaving motion is influenced by the restoring force of the chains. The restoring force F_v for the bow and stern chains can be calculated from the vertical mooring-line force

$$V_{(x,z)_{stern}} = - V_{(o,o)_{stern}} + c(x - p\theta) - d(z + L\theta) \tag{13}$$

$$V_{(x,z)_{bow}} = - V_{(o,o)_{bow}} - c(x + p\theta) - d(z - L\theta) \tag{14}$$

Addition of Eq. (13) and Eq. (14) gives

$$F_v = - V_{(o,o)_{stern}} - V_{(o,o)_{bow}} - 2dz \tag{15}$$

The constant forces $V_{(o,o)_{stern}} + V_{(o,o)_{bow}}$ act downward on the ship and increase its displacement. Generally, this increase is very small and may be neglected. Consequently, the stiffness coefficient in heave for the moored ship becomes

$$K_{zz} = \left(\rho g A_s + 2d\right)z \tag{16}$$

where A_s is the horizontal cross-sectional area of a ship at the still-water surface. The first term on the right side of Eq. (16) represents the vertical force due to the displaced bolume of water; the second term, the force due to the mooring lines on the bow and stern of the vessel.

The coefficient d appears to be very small compared with $\rho g A_s$, and consequently the bow and stern mooring lines have an insignificant effect on the heaving motion. Likewise, the other mooring lines already neglected in Eq. (16) have no effect on the heaving motion.

Following Eq. (1), the equation of motion in pitch of a free-floating vessel may be written, if coupling with other modes of movement is neglected as

$$M_{\theta\theta}\ddot{\theta} + N_{\theta\theta}\dot{\theta} + K_{\theta\theta}\theta = A\bar{F}^{\theta}_{ex}\ e^{j\omega t} \tag{17}$$

where

$K_{\theta\theta} = \rho g J_Y$

The restoring moment ($K_{\theta\theta}\theta$) of the free-floating ship is increased when the ship is moored.

The total moment of the vertical components of the bow and stern line is

$$M_v = -\ (L - p\theta)\ V_{(x,z)_{stern}} + (L + p\theta)\ V_{(x,z)_{bow}} \tag{18}$$

$$= 2cLx = 2cpL\theta - 2dL^2\theta$$

$$+ p\theta\left(V_{(o,o)_{stern}} + V_{(o,o)_{bow}}\right) + 2dz \tag{19}$$

The moments due to the horizontal forces in the stern and bow lines are

$$M_h = -\ (p + L\theta)\ H_{(x,z)_{stern}} + (p - L\theta)\ H_{(x,z)_{bow}} \tag{20}$$

$$= +\ 2apx - 2ap^2\theta - 2pbL\theta$$

$$-\ L\theta\left(H_{(o,o)_{stern}} + H_{(o,o)_{bow}}\right) - 2bzL\theta \tag{21}$$

the moments due to vertical forces in the mooring lines perpendicular to the long axis of the ship are

$$M_p = 4\left(d_1L^2 + pV_{(o,o)_p} + pd_1z\right)\theta \tag{22}$$

where

d_1 = coefficient determining the influence of the vertical movement

Neglecting the higher-order terms, the resultant moment due to all mooring-line forces is

$$M_h + M_v + M_p = 2(ap + cL)\, x - \Big[2ap^2 + 2(b + c)pL + \Big(H_{(o,o)_{stern}} + H_{(o,o)_{bow}} + 2dL + 4d_1L\Big)L - \Big(V_{(o,o)_{stern}} + V_{(o,o)_{bow}} + 4\, V_{(o,o)_p}\Big)p\Big]\,\theta \tag{23}$$

Consequently, the total restoring moment is a function of x and θ, and the equation of motion may be written

$$M_{\theta\theta}\ddot{\theta} + N_{\theta\theta}\dot{\theta} + K_{\theta\theta}\theta + K_{\theta x}x = A\overline{F}^{\theta}_{ex}\; e^{j\omega t} \tag{24}$$

where

$$K_{\theta\theta} = \rho g J_\gamma + \Big[2ap^2 + 2(b + c)pL + \Big(H_{(o,o)_{stern}} + H_{(o,o)_{bow}} + 2dL + 4d_1L\Big)L - \Big(V_{(o,o)_{stern}} + V_{(o,o)_{bow}} + 4\, V_{(o,o)_p}\Big)p\Big] \tag{25}$$

$$K_{\theta x} = -\ 2(cL + ap) \tag{26}$$

Thus, the three equations of motion are

$$M_{xx}\ddot{x} + N_{xx}\dot{x} + K_{xx}x + K_{x\theta}\theta = A\overline{F}^{x}_{ex}\; e^{j\omega t} \tag{27}$$

$$M_{\theta\theta}\ddot{\theta} + N_{\theta\theta}\dot{\theta} + K_{\theta\theta}\theta + K_{\theta x}x = A\overline{F}^{\theta}_{ex}\; e^{j\omega t} \tag{28}$$

$$M_{zz}\ddot{z} + N_{zz}\dot{z} + K_{zz}z = A\overline{F}^{z}_{ex}\; e^{j\omega t} \tag{29}$$

It will be noticed that Eqs. (27) and (28) are coupled. Anticipating a solution

$$x = \overline{A}\, e^{j\omega t} \qquad \theta = \overline{B}\, e^{j\omega t} \tag{30)(31}$$

Where $\overline{A}$ and $\overline{B}$ are complex quantities, then

$$\dot{x} = j\omega\overline{A}\; e^{j\omega t} \qquad \ddot{x} = -\omega^2\, \overline{A}\; e^{j\omega t} \tag{32)(33}$$

$$\dot{\theta} = j\omega\overline{B}\; e^{j\omega t} \qquad \ddot{\theta} = -\omega^2\, \overline{B}\; e^{j\omega t} \tag{34)(35}$$

Introducing these complex quantities in place of the real quantities in Eqs. (27) and (28) gives

$$\Big(-\omega^2 M_{xx} + j\omega N_{xx} + K_{xx}\Big)\overline{A} + K_{x\theta}\overline{B} = A\overline{F}^{x}_{ex} \tag{36}$$

$$K_{\theta x}\overline{A} + \left(-\omega^2 M_{\theta\theta} + j\omega N_{\theta\theta} + K_{\theta\theta}\right)\overline{B} = A\overline{F}^{\theta}_{ex} \qquad (37)$$

We now introduce the impedances

$$\overline{Z}_{xx} = -\omega^2 M_{xx} + j\omega N_{xx} + K_{xx} \qquad (38)$$

and

$$\overline{Z}_{\theta\theta} = -\omega^2 M_{\theta\theta} + j\omega N_{\theta\theta} + K_{\theta\theta} \qquad (39)$$

which simplifies Eqs. (36) and (37) to

$$\overline{Z}_{xx}\overline{A} + K_{x\theta}\overline{B} = A\overline{F}^{x}_{ex} \qquad (40)$$

$$K_{\theta x}\overline{A} + Z_{\theta\theta}\overline{B} = A\overline{F}^{\theta}_{ex} \qquad (41)$$

Solving for $\overline{A}$ and $\overline{B}$ gives

$$\overline{A} = \frac{\begin{vmatrix} \overline{F}^{x}_{ex} & K_{x\theta} \\ \overline{F}^{\theta}_{ex} & \overline{Z}_{\theta\theta} \end{vmatrix}}{\begin{vmatrix} \overline{Z}_{xx} & K_{x\theta} \\ K_{\theta x} & \overline{Z}_{\theta\theta} \end{vmatrix}} A \qquad \text{and} \qquad \overline{B} = \frac{\begin{vmatrix} \overline{Z}_{xx} & \overline{F}^{x}_{ex} \\ K_{\theta x} & \overline{F}^{\theta}_{ex} \end{vmatrix}}{\begin{vmatrix} \overline{Z}_{xx} & K_{x\theta} \\ K_{\theta x} & Z_{\theta\theta} \end{vmatrix}} A$$

(42) (43)

by which amplitudes and phase lags with the exciting periodic waves can be calculated.

For the vertical motion, a complex solution is anticipated

$$z = \overline{C}e^{j\omega t} \qquad (44)$$

where $\overline{C}$ is a complex quantity. Following the method for x and θ, we obtain

$$\overline{Z}_{zz}\overline{C} = A\overline{F}^{z}_{ex} \qquad (45)$$

where

$$\overline{Z}_{zz} = -\omega^2 M_{zz} + j\omega N_{zz} + K_{zz} \qquad (46)$$

Thus

$$\bar{C} = \frac{\bar{F}^z_{ex}}{\bar{Z}_z} A \qquad (47)$$

The fluctuations in the mooring cables may now be determined. For example, rewriting Eq. (6)

$$H_{(x,z)_{stern}} = H_{(o,o)_{stern}} - ax + (ap + bL)\theta + bz$$

If we introduce the following expression for the force fluctuation in the mooring cable, which is a function of wave amplitude and frequency

$$H_{stern}(A,\omega) = - ax + (ap + bL)\theta + bz \qquad (48)$$

then

$$H_{stern}(A,\omega) = \mathrm{Re}\left[- a\bar{A} + (ap + bL)\bar{B} + b\bar{C}\right]Ae^{j\omega t} \qquad (49)$$

In many instances, the term $K_{x\theta}$ in Eq. (40) is very small compared to $\bar{Z}_{xx}$, and $K_{\theta x}$ in Eq. (41) is very small compared to $\bar{Z}_{\theta\theta}$. Then the pitch and surge of the moored ship are essentially uncoupled, and

$$\bar{A} \approx \frac{\bar{F}^x_{ex}}{\bar{Z}_{xx}} A \qquad (50)$$

$$\bar{B} \approx \frac{\bar{F}^\theta_{ex}}{\bar{Z}_{\theta\theta}} \qquad (51)$$

The coupling is important, however, for the resonance movement in surge, which is generally not significantly damped, and in that case Eqs. (42) and (43) have to be used.

If coupled motion for the free-floating ship in pitch and heave are important--for example, for a ship with the center of mass not approximately in the middle of the ship as described by Korvin-Kroukovsky (1961)--the equation of motion of this vessel when moored becomes

$$M_{xx}\ddot{x} + N_{xx}\dot{x} + K_{xx}x + K_{x\theta}\theta = A\bar{F}^x_{ex}\, e^{j\omega t} \qquad (52)$$

$$K_{\theta x}x + M_{\theta\theta}\ddot{\theta} + N_{\theta\theta}\dot{\theta} + K_{\theta\theta}\theta + M_{\theta z}\ddot{z} + N_{\theta z}\dot{z} + K_{\theta z}z = A\bar{F}^\theta_{ex}\, e^{j\omega t} \qquad (53)$$

$$M_{z\theta}\ddot{\theta} + N_{z\theta}\dot{\theta} + K_{z\theta}\theta + M_{zz}z + N_{zz}\dot{z} + K_{zz}z = A\bar{F}^z_{ex}\, e^{j\omega t} \qquad (54)$$

or using the mechanical impedances $\bar{Z}_{\theta z}$ and $\bar{Z}_{z\theta}$, similarly $\bar{Z}_{xx}$ as in Eq. (38) the equations of motion may be expressed:

$$\bar{Z}_{xx}\,\bar{A} + K_{x\theta}\,\bar{B} = A\bar{F}^x_{ex} \qquad (55)$$

$$K_{\theta x}\,\overline{A} + \overline{Z}_{\theta\theta}\,\overline{B} + Z_{\theta z}\,\overline{C} = A\overline{F}^{\theta}_{ex} \tag{56}$$

$$\overline{Z}_{z\theta}\,\overline{B} + \overline{Z}_{zz}\,\overline{C} = A\overline{F}^{z}_{ex} \tag{57}$$

This set of linear equations may be solved by using Cramer's rule, writing for the determinant of the system

$$\Delta = \begin{vmatrix} \overline{Z}_{xx} & K_{x\theta} & 0 \\ K_{\theta x} & \overline{Z}_{\theta\theta} & \overline{Z}_{\theta z} \\ 0 & \overline{Z}_{z\theta} & Z_{zz} \end{vmatrix} \tag{58}$$

The unique solutions are given by

$$\overline{A} = \frac{\Delta_x}{\Delta} A \qquad \overline{B} = \frac{\Delta_\theta}{\Delta} A \qquad \overline{C} = \frac{\Delta_z}{\Delta} A \qquad (59)(60)(61)$$

where Δ_x, Δ_θ, Δ_z are the determinant forms obtained by replacing the elemen of the first, second, or third columns, respectively, of Eq. (58) by $\overline{F}^{x}_{ex}$, $\overline{F}^{\theta}_{ex}$, $\overline{F}^{z}_{ex}$.

For a unit wave amplitude, the complex numbers $\overline{A}$, $\overline{B}$, and $\overline{C}$, which are frequenty dependent, are generally called complex response operators and written as $T_{x\eta}(\omega)$, $T_{\theta\eta}(\omega)$, and $T_{z\eta}(\omega)$. The real part of these complex oper tors is that part of the response,which is in phase or 180 deg out of phase with the wave and often indicated as $c_{x\eta}(\omega)$, $c_{\theta\eta}(\omega)$, and $c_{z\eta}(\omega)$. The imaginary part of the complex operators is that part which is 90 deg or 270 deg out of phase with the wave and is indicated as $q_{x\eta}(\omega)$, $q_{\theta\eta}(\omega)$, and $q_{z\eta}(\omega)$ (see Korvin-Kroukovsky (1961).

SHIP MOORED BY BUOYS WITH UNIFORM WAVES HEAD-ON

The equations of motion for a ship using mooring buoys can be derived in a fashion similar to that for the ship using mooring cables only. In this case, the motions of the buoys have to be considered in addition to the motions of the ship.

Considering a mooring configuration in Fig. 4, it will be noted that the relative vertical motions between the buoys and the ship will induce small horizontal displacements between the buoys and the ship, thus relatively small force fluctuations in the lines between ship and buoy. Consequently, the heaving and pitching motions of the ship are considered as of no importance to the forces in these lines. This is naturally not the case for the heaving motions of the buoys.

Assuming again a linear relationship between forces and movements, and neglecting the pitching of the buoys, the equations of motion for waves with this height of the system neglecting damping in surge become

$$M_{1_{xx}}\,\ddot{x}_i + N_{1_{xx}}\,\dot{x} + a\,x_1 + K_1\,(x_1 - x_2) - b\,z_1 = \overline{F}^{x}_{1_{ex}}\,e^{j\omega t} \tag{62}$$

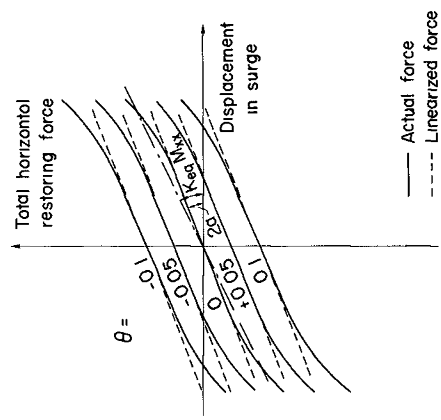

FIG 5 TYPICAL PLOT FOR HORIZONTAL RESTORING FORCE VERSUS DISPLACEMENT FOR DIFFERENT PITCH ANGLES

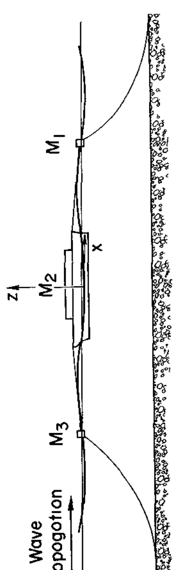

FIG 4 BUOY MOORING

$$M_{2_{xx}} \ddot{x}_2 + K_1 (x_2 - x_1) + K_2 (x_2 - x_3) = \overline{F}^x_{2_{ex}} \; e^{j\omega t} \tag{63}$$

$$M_{3_{xx}} \ddot{x}_3 + N_{3_{xx}} \dot{x} + K_2 (x_3 - x_2) + a\, x_3 + b\, z_3 = \overline{F}^x_{3_{ex}} \; e^{j\omega t} \tag{64}$$

$$M_{1_{zz}} \ddot{z}_1 + N_{1_{zz}} \dot{z} + \left[\rho g \;(2\, L_1 B_1) + d\right] z_1 - cx_1 = \overline{F}^z_{1_{ex}} \; e^{j\omega t} \tag{65}$$

$$M_{3_{zz}} \ddot{z}_3 + N_{3_{zz}} \dot{z} + \left[\rho g \;(2\, L_3 B_3) + d\right] z_3 + cx_3 = \overline{F}^z_{2_{ex}} \; e^{j\omega t} \tag{66}$$

In many instances, in mooring with buoys, the connection between the ship and buoy is made with a cable that is relatively light in comparison with the heavy chains used on the buoys. If these cables are placed in high tension, the horizontal movements of the buoys and the ship are practically the same, and it may be assumed that $x_1 = x_2 = x_3$. Then Eqs. (62) through (66) reduce to

$$\left(M_{1_{xx}} + M_{2_{xx}} + M_{3_{xx}}\right) \ddot{x} + \left(N_{1_{xx}} + N_{2_{xx}}\right) \dot{x} + 2ax - bz_1 + bz_3$$

$$= \left(\overline{F}^x_{1_{ex}} + \overline{F}^x_{2_{ex}} + \overline{F}^x_{3_{ex}}\right) e^{j\omega t} \tag{67}$$

$$M_{1_{xx}} \ddot{z}_1 + N_{1_{zz}} \dot{z}_1 + \left[\rho g (A_\pi) + d\right] z_1 - cx = \overline{F}^z_{1_{ex}} \; e^{j\omega t} \tag{68}$$

$$M_{3_{zz}} \ddot{z}_3 + N_{3_{zz}} \dot{z}_3 + \left[\rho g (A_\pi) + d\right] z_3 + cx = \overline{F}^z_{2_{ex}} \; e^{j\omega t} \tag{69}$$

In Eq. (67), the virtual masses of the buoys are small compared with the mass of the ship, and also the horizontal wave forces are small compared with the wave force acting on the ship; consequently, the effects of the buoys in this horizontal movement of the ship may be neglected. Generally, the natural frequency in heave of the buoys is higher than the frequencies of the waves, thus the terms $M\ddot{z}$ and $N\dot{z}$ are small compared with the term $\left[\rho g (2LB) + d\right]$ and may be neglected in our initial investigation of the ship's movement. Disregarding the above-mentioned terms, introduction of Eqs. (68) and (69) into Eq. (67) gives

$$M_{2_{xx}} \ddot{x} + \left(N_{1_{xx}} + N_{2_{xx}}\right) \dot{x} + \left\{2a - \frac{2bc}{(\rho g A_\pi + d)}\right\} x = \left(+ \frac{b}{(\rho g A_\pi + d)} \overline{F}^z_{1_{ex}}\right.$$

$$\left. - \frac{b}{(\rho g A_\pi + d)} \overline{F}^z_{3_{ex}} + \overline{F}^x_{2_{ex}}\right) e^{j\omega t} \tag{70}$$

This result is important, since in principle it enables the design of a mooring which, at the resonance frequency

$$\omega_{o_x} = \left[\frac{2a - \dfrac{2bc}{\rho g A_\pi + d}}{M_{2_{xx}}} \right]^{1/2} \tag{71}$$

the excitation term on the right side of Eq. (70) becomes small by proper placement of the buoys.

SPREAD-MOORED SHIP IN LONG-CRESTED IRREGULAR WAVES

It appears that the actual wave condition in the ocean can best be represented by use of the model of a random process as derived by Neumann and described by Pierson, Neumann, and James (1960). Statistical values such as average wave height are given, not values of the environment as a function of time. The sea is taken as a summation of a large number (or as an integral of an infinite number) of uniform wave trains, each with different amplitudes and directions superimposed in random phase. The profiles of the individual waves are assumed to be sine curves according to Airy's Theory (Johnson (1951)).

Techniques are available to predict the amplitudes of the waves and their distribution over the frequency range from wind velocity, wind duration, and the fetch. Generally, the result can be presented in the form of a wave spectrum, which is the distribution of the mean squares of the wave amplitudes in a given increment of the frequency (spectral density) over the wave frequencies.

In the following analysis, it is assumed that the waves are unidirectional and meet the ship or submerged vessel head on. This case is realistic, as it represents the crafts moored in swell.

Following the work by St. Denis and Pierson (1953), the relation between the spectral density of wave and ship responses is given by

$$S_r(\omega) = S_w(\omega)\Big[T(\omega)\Big]^2 \tag{72}$$

where

$S_r(\omega)$ = spectral density of the response in a particular variable (displacement, strain, etc.)

$S_w(\omega)$ = spectral density of the wave

$T(\omega)$ = ratio of response in a particular variable to wave amplitude (complex frequency factor)

If the spectrum of the waves if given, the spectrum of the response can be calculated by Eq. (70). The mean square of the response is then given by

$$\sigma^2 = \int_o^\omega S_r(\omega)d\omega = \int_o^\omega S_w(\omega)\Big[T(\omega)\Big]^2 d\omega \tag{73}$$

It has been shown by Longuet-Higgins (1952) that for a relatively narrow band of wave frequencies, such as is the case with swells being assumed here, the probability distribution of the wave amplitudes tends to be Gaussian if the frequency factor has nonzero values in the range of wave frequencies. Consequently, it may be expected that the probability distribution of the response amplitudes is also Gaussian.

Longuet-Higgins calculated important statistical relationships for the narrow-frequency spectrum, which were consequently tabulated by Pierson, Neumann, and James (1960); for example

$R_{av} = 1.25\ \sigma$ (Average response amplitude)

$R_{1/3} = 2.0\ \sigma$ (Average response amplitude of the 1/3 highest responses equals significant response amplitude)

In many instances, the response spectrum may not be considered to be narrow, and the expected number (M_α) of maxima of the response per unit time exceeding the value of the response $R(t) = \alpha$ can be expressed after Bendat (1958) as

$$M_\alpha = \frac{1}{2\pi}\left(\frac{E\left[R'(t)^2\right]}{\sigma^2}\right)^{1/2} e^{\left(\frac{-\alpha^2}{2\sigma^2}\right)} \tag{74}$$

where

$$E\left[R'(t)^2\right] = \int_0^\infty \omega^2 S_w \left[T(\omega)\right]^2 d\omega \tag{75}$$

Thus, this presentation introduces the probability concept into the calculation of movements and cable stresses.

EFFECT OF THE NONLINEAR MOORING-LINE FORCES

In the analyses of the response, it has been assumed that the restoring forces of the cables are linear with the displacement by use of Eqs. (2) and (3). This assumption will introduce certain errors in the calculated response and the mooring-line forces.

Considering the spread-moored ship, it has been seen that the pitch and surge are coupled because of the bow and stern mooring lines.

If the total horizontal restoring force of a system is plotted as a function of the horizontal displacement for different pitch angles, a graph of the type presented in Fig. 5 will be obtained. In this graph the linearization calculated by Eqs. (2) and (3) is also plotted.

The nonlinearity of the total restoring force is much smaller than that of the individual cables.

It will be seen from such graphs that force-displacement curves for different pitch angles are essentially parallel for equal distances over the expected range of pitch angles.

It is assumed that movements in surge extend into the nonlinear range. The horizontal restoring force may now be written, following the procedures of Crandall (1961), by extending the linear Eq. (8)

$$F_h = 2a\left[x + \epsilon g(x)\right] - 2(bL + ap)\theta \tag{76}$$

where

ϵ = small parameter modifying the nonlinear function

$g(x)$ = odd single-valued power function of x

The values ϵ and $g(x)$ are chosen in such a manner that for zero pitch angle, Eq. (76) is identical with the force-displacement curve obtained by use of calculations of the catenary equations.

The coupled equations of motion in surge and pitch for the ship in irregular waves can now be written by introducing the nonlinearity in Eq. (27).

$$\ddot{x} + \frac{N_{xx}}{M_{xx}}\dot{x} + \frac{2a}{M_{xx}}\left[x + \epsilon g(x)\right] + \frac{K_{x\theta}\theta}{M_{xx}} = I_x(t) \tag{77}$$

$$\ddot{\theta} + \frac{N_{\theta\theta}}{M_{\theta\theta}}\dot{\theta} + \frac{K_{\theta\theta}}{M_{\theta\theta}}\theta + \frac{K_{\theta x}\theta}{M_{\theta\theta}} = I_\theta(t) \tag{78}$$

where $I_x(t)$ and $I_\theta(t)$ are random functions, both derived from the wave spectrum.

Equation (77) may be rewritten by introducing the equivalent linear stiffness coefficient K_{eq}

$$\ddot{x} + 2\alpha\dot{x} + K_{eq}x + k_{x\theta}\theta = I_x(t) + \Xi \tag{79}$$

where

$$\Xi = \left(K_{eq} - \omega_{o_x}^2\right)x - \epsilon\omega_{o_x}^2 g(x) \tag{80}$$

$$\omega_{o_x}^2 = 2a/M_{xx} \tag{81}$$

Assuming that Ξ is zero, the mean square response of the system to an irregular sea with a particular spectrum is found by Eq. (73)

$$\bar{\sigma}_x^2 = \int_o^\infty S_w(\omega)\left[T(\omega)\right]^2 d\omega \tag{82}$$

The spectral density $S_w(\omega)$ is given from the assumed sea condition, and the square of the absolute value of the complex-frequency factor $\left[T(\omega)\right]^2$ is obtained from Eq. (42)

$$\left[T_{x,\eta}(\omega)\right]^2 = \left| \frac{\begin{vmatrix} \bar{F}^x_{ex} & K_{x\theta} \\ \bar{F}^\theta_{ex} & \bar{Z}_{\theta\theta} \end{vmatrix}}{\begin{vmatrix} \bar{Z}_{xx} & K_{x\theta} \\ K_{\theta x} & \bar{Z}_{\theta\theta} \end{vmatrix}} \right|^2 \tag{83}$$

where

$$\bar{Z}_{xx} = -\omega^2 M_{xx} + j\omega N_{xx} + M_{xx} K_{eq} \tag{84}$$

Introducing Eqs. (83) and (84) into Eq. (82)

$$\bar{\sigma}_x^2 = G\ f\left(K_{eq}\right) \tag{85}$$

for small variation of K_{eq} from $\omega_{o_x}^2$, Eq. (85) may be expressed

$$\bar{\sigma}_x^2 = G_w\left[1 + \gamma\left(K_{eq} - \omega_{o_x}^2\right)\right] \tag{86}$$

where $G_w = \bar{\sigma}_o^2$ = spectral energy of the response for $\epsilon = o$

$$\gamma = \frac{d\ f\left(K_{eq}\right)}{d\ K_{eq}} \text{ at } K_{eq} = \omega_{o_x}^2 \tag{87}$$

In the analysis with Eqs. (82) through (86) it was assumed that the remainder function Ξ equals zero, which is naturally not the case; Ξ is again a stationary random process just like $I_x(t)$ and depends on the value of the equivalent stiffness coefficient. A measure of its value is its expected mean square $E\left[\Xi^2\right]$

The mean square of the remainder function Ξ can be expressed by use of Eq. (80)

$$E\left[\Xi^2\right] = K_{eq}^2\ E\left[x^2\right] - 2K_{eq}\omega_o^2 E\left[x^2 + \epsilon x g(x)\right] + \omega_o^4 E\left[\left\{x + \epsilon g(x)\right\}^2\right] \tag{88}$$

This will be a minimum for fixing K_{eq} when

$$\frac{d\left(E\left[\Xi^2\right]\right)}{dK_{eq}} = 0 \tag{89}$$

which results in

$$K_{eq} = \omega_{o_x}^2 \left(1 + \varepsilon \frac{E\left[xg(x)\right]}{E\left[x^2\right]} \right) \tag{90}$$

Inserting Eq. (90) into Eq. (86) results in

$$\frac{\bar{\sigma}_x^2}{\bar{\sigma}_o^2} = 1 + \gamma\omega_{o_x}^2 \epsilon \frac{E\left[xg(x)\right]}{E\left[x^2\right]} \tag{91}$$

The probability density of a random variable Y with zero mean value is

$$f(Y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-Y^2/2\sigma^2} \tag{92}$$

where σ = standard deviation.

The expectation value $E\left[xg(x)\right]$ in Eq. (91) is for the nonlinear system, which would require knowing the response of the nonlinear system. Fortunately, the term in Eq. (91) is to be multiplied by the small parameter ϵ, and the expectation value $E\left[xg(x)\right]$ of the linear system instead of the nonlinear system will induce errors of the second order.

Consequently

$$\frac{\bar{\sigma}_x^2}{\bar{\sigma}_o^2} = 1 + \frac{\gamma\epsilon\omega_{o_x}^2}{\sqrt{2\pi}\,\bar{\sigma}_o^3} \int_{-\infty}^{+\infty} xg(x) e^{-x^2/2\sigma_o^2} dx \tag{93}$$

by which the effect of the linearization can be investigated. The term γ may be positive or negative.

DISCUSSION

The mathematical models presented here have shortcomings. The most important one is the assumed linear relationship between the restoring forces and the displacement of the ship. The effect of the nonlinearity of the mooring lines in the surge motion, which is particularly affected by the nonlinearity, was investigated in detail in the previous section of this paper, and a method was presented for calculating the ratio of the mean square of the nonlinear response and the linear response.

Naturally, the methods of analyzing the response of the moored ship has the limitations that are imposed on the analysis of a free-floating vessel, and the direct force-displacement relationship established in the section about mooring-line characteristics limits the method to mooring in a few hundred feet depth as only to that depth are the dynamic effects on the mooring line considered small.

Unfortunately, no experimental data are available in the literature to check the analysis in detail. A paper (Wiegel (1958)) describing model tests performed at the University of California presents no detailed information concerning the important characteristics of the vessel and its moorings, but by selecting a mooring with about the same characteristics in surge, one can obtain good agreement between experimental and calculated values of the response of an 880-ton vessel (Figs. 6 and 7 by Leendertse (1963)).

The design of moorings by using the formulas of this paper can be expedited considerably by graphical representation of the exciting forces and the impedances.

Since the impedance concept is introduced in the calculating of the responses, the procedures used for electrical circuit design and the design of servomechanisms appears to be a powerful tool for the numerical calculation of responses. Graphical representation of exciting forces and the impedances by use of complex plane diagrams enhances the understanding of the complicated phase relationship between waves and responses (Chestnut (1951)).

In practically all cases, the surge response of the vessel is the main contributor to high forces in the mooring lines. This is caused by the fact that very limited damping is available in this mode of movement. In principle, a reduction of the surge response is possible by two methods: namely, by increasing the damping or by mismatching the natural frequency in surge with the main range of frequencies of wave excitations. The application of the first method is limited because it is difficult. In an incidental case, surge movements have been limited by the introduction of damping devices in the mooring lines. Since wind and currents, whose effects are not discussed here, impose certain requirements on the mooring-line tensions, the applicability of the second method is often also limited.

In the previous sections of this paper, a few relatively simple but realistic cases where the ship was moored in the longitudinal plane of symmetry were considered. Mooring lines in directions other than the main axis will introduce coupling between many more modes of movement than is studied in this paper.

Weinblum and St. Denis (1950), in their now classical paper, presented a method for calculating the uncoupled motions of an unrestrained ship in its six degrees of freedom in regular waves with arbitrary heading. This work has been expanded by Pierson and St. Denis (1953) for the movement in irregular waves with a directional spectrum.

If the motions of the free-floating ship are considered uncoupled, the same ship in a moored condition will have coupled motions due to the mooring lines. For an arbitrary mooring, for example, the linearized equation of motion in surge becomes

$$M_{xx}x + N_{xx}x + K_{xx}x + K_{xy}y + K_{xz}z + K_{x\theta}\theta + K_{x\psi}\psi + K_{x\varphi}\varphi = \overline{F}^{x}_{ex}\,Ae^{j\omega t} \tag{94}$$

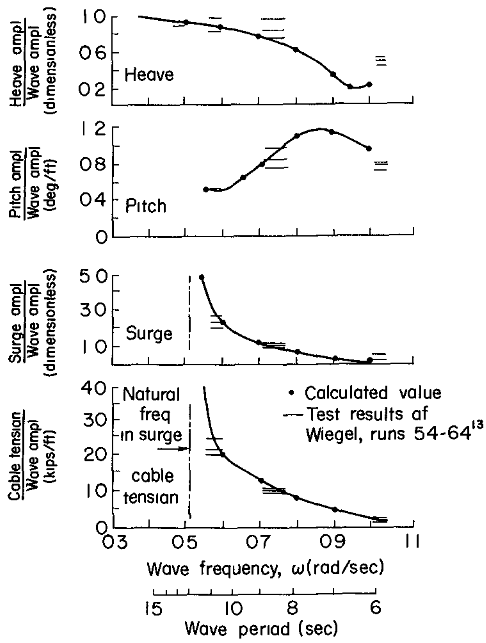

FIG 6 EXPERIMENTAL AND CALCULATED VALUES OF RESPONSE OF AN 880-TON VESSEL

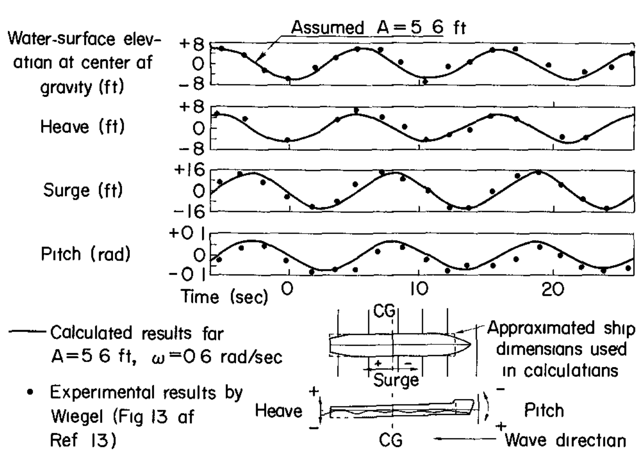

FIG 7 COMPARISON BETWEEN CALCULATED AND MEASURED RESPONSES OF A MOORED SHIP IN UNIFORM WAVES

The equations of motion in the other modes are similar, and the solution of response operators follows the pattern of Eqs. (52) to (61). The coefficients K_{xy}, K_{xz}, etc., depend on the mooring lines and can be calculated in a manner similar to that for the spread-moored ship in waves head-on (e.g., Eqs. (6) through (8)).

The procedure for the investigation of the nonlinear effects is not limited to unidirectional waves. For a directional wave spectrum, containing only energy in two quadrants such as the directional Neumann wind wave spectrum, Eq. (82) becomes

$$\bar{\sigma}_x^2 = \int_o^\infty \int_{-\pi/2}^{+\pi/2} S(\omega,\beta) \,|\, T\,(\omega,\beta - \chi)\,|^2 d\beta \;\; d\omega \qquad (95)$$

where

β = angle between direction of wave propagation and the coordinate system of the ship

χ = angle between center of the directional wave spectrum and the coordinate system of the ship

The directional complex response operators in Eq. (95) is calculated by use of Eq. (83) if one takes into account that the excitations F^x_{ex} and F^θ_{ex} are functions of the wave angle. The procedure in this section for nonlinear effects is otherwise generally valid.

In cases where the response operator of the linearized system is peaked within the frequency range of the maximum wave amplitudes, the gamma (γ) coefficient in Eq. (86) will be small and consequently the effect of the linearization used upon expectation values of the nonlinear response amplitudes will be very small.

REFERENCES

Weinblum, Georg, and Manley St. Denis (1950). On the Motions of Ships at Sea: Trans. SNAME, vol. 58, pp. 184-231.

Korvin-Kroukovsky, B. V. (1961). Theory of Seakeeping: Society of Naval Architects and Marine Engineers, New York.

Korvin-Kroukovsky, B. V., and Winnifred R. Jacobs (1957). Pitching and Heaving Motions of a Ship in Regular Waves: Trans. SNAME, vol. 65, pp. 590-632.

Wilson, Dr. Basil W. (1959). The Energy Problem in the Mooring of Ships Exposed to Waves: Perm. Int. Assoc. of Nav. Congresses Bull. No. 50.

Kriloff, A. (1898). A General Theory of the Oscillations of a Ship on Waves, and On Stresses Experienced by a Ship in a Seaway: INA, vol. 40, pp. 135-212.

Weinblum, Georg (1954). Progress of Theoretical Investigations of Ship Motions in a Seaway: Ships and Waves, pp. 129-159.

Pierson, Williard J., Gerhard Neumann, and Richard W. James (1960). Observing and Forecasting Ocean Waves: Hydrographic Office Pub. 603.

Wiegel, R. L., and J. W. Johnson (1951). Elements of Wave Theory: Proc. of the First Conf. on Coastal Engineering, Council on Wave Research, Berkeley, California.

St. Denis, Manley, and Willard J. Pierson, Jr. (1953). On the Motions of Ships in Confused Seas: Trans. SNAME, vol. 61, pp. 280-357.

Longuet-Higgins, M. S. (1952). On the Statistical Distribution of the Heights of Sea Waves: J. Mar. Res., vol. 11, pp. 245-266.

Bendat, Julius S. (1958). Principles and Applications of Random Noise Theory: John Wiley and Sons, New York, pp. 130-133.

Crandall, Stephen (1961). Random Vibrations of Systems with Non-Linear Restoring Forces: Massachusetts Institute of Technology, AFOSR 708,

Wiegel, R. L. (1958). Model Studies of the Dynamics of an LSM Moored in Waves: Proc. of Sixth Conf. on Coastal Engineering, Council on Wave Research, Berkeley, California.

Leendertse, J. J. (1963). Analysis of the Response of Moored Surface and Subsurface Vessels to Ocean Waves: The RAND Corporation, RM-3368-PR, Santa Monica, California.

Chestnut, Harold, and Robert W. Mayer (1951). Servomechanism and Regulating System Design: J. Wiley and Sons, New York.

APPENDIX

A = wave amplitude
A_s = horizontal cross-sectional area of a ship at the still water surface
A_π = horizontal cross-sectional area of a buoy at the still water surface
$\bar{A}$ = complex value of the movement in surge for a wave with unit height
a, b, c, d = coefficients in linearized mooring-line equations
$\bar{B}$ = complex value of the movement in pitch for a wave with unit height
$\bar{C}$ = complex value of the movement in heave for a wave with unit height
d_1 = coefficient in linearized mooring-line equations
E = expectation value
F_n = resultant horizontal component of the restoring forces of the mooring cables
$\bar{F}^s_{ex}$ = complex value of the exciting force or moment in the s mode of movement for a wave of unit height

$f()$ = function

$\bar{f}^s_{ex}$ = complex value of the exciting force or moment per unit mass

G = spectral energy of the response for $\epsilon = o$

g = acceleration of gravity

$g(x)$ = odd-single valued power function of x

$H_{(o,o)}$ = horizontal force at the holding point (o,o)

$H_{(x,z)}$ = horizontal force at the holding point (x,z)

h = vertical distance between the holding point of a mooring line and the sea bottom

$I_x(t)$ = random (force) function, derived from the wave spectrum

$I_\theta(t)$ = random (moment) function, derived from the wave spectrum

Jy = inertia moment of the horizontal cross-sectional area of a ship around the y axis

K_{eq} = equivalent linear stiffness coefficient

$K_{s\tau}$ = stiffness coefficient in force equation of the s mode for the movement in the τ mode

k_{st} = stiffness coefficient per unit of mass

L = half-length of a ship

M_h = total moment of the horizontal components of the bow and stern lines

M_p = total moment due to the vertical forces in the mooring lines perpendicular to the long axis of the ship

$M_{s\tau}$ = virtual mass or mass inertia moment in the force on moment equation of the s mode for the movement in the τ mode

M_v = total moment of the vertical component of the bow and stern lines

M_α = expected number of maxima of the response per unit time exceeding the value of the response $R(t) = \alpha$

$M_{1_{xx}}$ = virtual mass in the x movement

$M_{2_{xx}}$ = virtual mass in the x movement

$M_{3_{xx}}$ = virtual mass in the x movement

M'' = added mass

$N_{s\tau}$ = linearized damping term in the force equation of the s mode for movement in the τ mode

p = vertical distance between the holding points of a mooring line and the mass center of the ship

R_{av} = average response amplitude

R_t = periodic force due to other modes of movement

$R(t)$ = response amplitude

$R_{1/3}$ = average response amplitude of the 1/3 highest responses (i.e., of the highest third of all amplitudes)

S = total length of a mooring line

$S_w(\omega)$ = spectral density of the response in a particular variable

s = general indication for mode of movement

T = total force in a mooring line

$T(\omega)$ = ratio of response in a particular variable to wave amplitude (complex frequency factor)

$\left[T(\omega)\right]^2$ = square of the absolute value of the complex frequency factor

t = time

$V_{(o,o)}$ = vertical force in a mooring line at the holding point o

$V_{(o,o)_p}$ = vertical component of the force in the mooring lines perpendicular to the long axis of the vessel

$V_{(x,z)}$ = vertical force in a mooring line at the holding point (x,z)

w = net weight of a mooring line per unit length

x,y,z = Cartesian co-ordinate axes

Y = random variable with zero mean value

$\bar{Z}_{ss} = -\omega^2 M_{ss} + j\omega N_{zz} + K_{zz}$ (impedance)

α = a value of the response

β = angle between direction of wave propagation and the co-ordinate system of the ship

γ = coefficient = $\dfrac{df(K_{eq})}{dK_{eq}}$

ϵ = small parameter modifying the nonlinear function

θ = pitch angle

Ξ = remainder function (Eq. 79)

σ = root mean square

σ_o = root mean square of the response of the assumed linear system in surge

σ_x = root mean square of the response of the nonlinear system in surge

τ = general indication for mode of movement (used only as a subscript)

φ = angle of roll

χ = angle between center line of the directional wave spectrum and the co-ordinate system of the ship

ω = wave frequency

ω_{o_s} = natural frequency in the s mode