CHAPTER 26

ON A COEXISTENCE SYSTEM OF FLOW AND WAVES

by

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INTRODUCTION

It is a well known experimental fact that the undulation such as sand-ripples or antidunes are formed on the bed surface composed of fine sand, corresponding to the flow characteristics of open channel flow.[1] In this case, as the mechanical effects of the bed undulationes stated above, a kind of periodic motion is superposed on the flow, and accordingly the water surface undulates periodically. On the other hand, the mechanical effects of this surface undulationes are surperposed on these undulating bed surfaces as another kind of periodic motion. The wave generated in open sea propagates upstream through an estuary. Accordingly the incoming wave is superposed on the flow stated above as a forced oscillation. Both of these phenomena are in the coexistence system of flow and waves in the open channel flow.

In this paper, as the first step to study the subjects stated above, the author treats the problem of the coexistence system in the case when the forced oscillation of water surface is superposed on the open channel flow with fixed bed, and analyzes theoretically and experimentally the mechanical properties of the reciprocal action between flow and waves.

1 THEORETICAL CONSIDERATION

(1) THE FIRST ORDER APPROXIMATE SOLUTION [2]

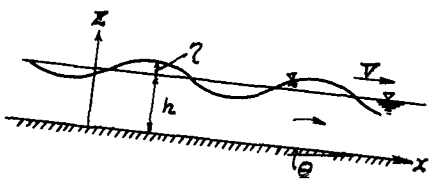

Fig. 1. Ideal Model under Consideration.

Let us consider the two dimensional phenomenon as shown in Fig.1, which expresses an ideal model in the case when surface wave is superposed on the open channel flow with fixed bed. For the sake of simplicity, let the scale of motion of water particles in the phenomenon be so feeble that the whole condition of the motion can be regarded as laminar flow of viscous fluid. Therefore, neglecting the non-linear terms the Navier-

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Stokes equations yield

$$\frac{\partial u_1}{\partial t} = g\sin\theta - \frac{1}{\rho}\frac{\partial p_1}{\partial x} + \nu\left(\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial z^2}\right), \tag{1}$$

$$\frac{\partial w_1}{\partial t} = g\cos\theta - \frac{1}{\rho}\frac{\partial p_1}{\partial z} + \nu\left(\frac{\partial^2 w_1}{\partial x^2} + \frac{\partial^2 w_1}{\partial z^2}\right), \tag{2}$$

$$\frac{\partial u_1}{\partial x} + \frac{\partial w_1}{\partial z} = 0, \tag{3}$$

in which x is the distance measured along the bed surface in the direction of downstream, z the upward distance vertical to the bed surface, t the time, u_1 and w_1 the velocity components parallel to the axes of x and z respectively, ν the kinematic coefficient of viscosity, g the gravity acceleration, p_1 the pressure, ρ the density of water, and $\sin\theta$ the bed slope.

To obtain the solution of these equations, the velocity components u_1, w_1, and the pressure p_1 are assumed to be given by the sum of a periodic function and a non-periodic one as follows:

$$u_1 = u_{11}(z)e^{im_1(x-V_1t)} + u_{12}(z), \tag{4}$$

$$w_1 = w_{11}(z)e^{im_1(x-V_1t)} + w_{12}(z), \tag{5}$$

$$p_1 = \rho p_{11}(z)e^{im_1(x-V_1t)} - \rho p_{12}(z), \tag{6}$$

in which u_1, w_1, and p_1 denote the first order approximate solutions of the coexistence system, and u_{ij}, w_{ij}, and p_{ij} denote arbitrary functions of z, $m_1 = 2\pi/L_1$, and $V_1 = L_1/T_1$. L_1 and T_1 are the wave-length of surface waves and the period of waves respectively.

Substituting the equations (4), (5) and (6) into the equation (1), (2) and (3) yields

$$\left(im_1V_1u_{11} + im_1p_{11} - m_1^2\nu u_{11} + \nu\frac{d^2u_{11}}{dz^2}\right)e^{im_1(x-V_1t)} + g\sin\theta + \nu\frac{d^2u_{12}}{dz^2} = 0, \tag{7}$$

$$\left(im_1V_1w_{11} + \frac{dp_{11}}{dz} - m_1^2\nu w_{11} + \nu\frac{d^2w_{11}}{dz^2}\right)e^{im_1(x-V_1t)} - g\cos\theta + \frac{dp_{12}}{dz} + \nu\frac{d^2w_{12}}{dz^2} = 0, \tag{8}$$

$$\left(im_1u_{11} + \frac{dw_{11}}{dz}\right)e^{im_1(x-V_1t)} + \frac{dw_{12}}{dz} = 0. \tag{9}$$

Assuming that these equations must be always satisfied regardless of the values, x and t, the following equations, (10), (11), (12) and (13), are obtained:

$$\left.\begin{aligned} &\left(\frac{d^2}{dz^2} - m_1^2 + \frac{im_1V_1}{\nu}\right)u_{11} = -\frac{im_1p_{11}}{\nu} \\ &\left(\frac{d^2}{dz^2} - m_1^2 + \frac{im_1V_1}{\nu}\right)w_{11} = -\frac{1}{\nu}\frac{dp_{11}}{dz} \\ &im_1u_{11} + \frac{dw_{11}}{dz} = 0 \end{aligned}\right\}, \tag{10}$$

$$g\sin\theta + \nu\frac{d^2u_{12}}{dz^2} = 0, \tag{11}$$

$$-g\cos\theta+\frac{dp_{12}}{dz}+\nu\frac{d^2w_{12}}{dz^2}=0\ ,\quad \frac{dw_{12}}{dz}=0. \tag{12),(13}$$

Refering to the research results by S. S. Hough,[3] the general solutions of the equation (10) are given as follows:

$$u_{11}=-\frac{1}{V_1}(Ae^{m_1z}+Be^{-m_1z})+\frac{ik}{m_1}(Ce^{kz}-De^{-kz}), \tag{14}$$

$$w_{11}=\frac{i}{V_1}(Ae^{m_1z}-Be^{-m_1z})+(Ce^{kz}+De^{-kz}), \tag{15}$$

$$p_{11}=Ae^{m_1z}+Be^{-m_1z}, \tag{16}$$

in which A, B, C and D are the arbitrary constants to be determined by the boundary conditions, and

$$k^2=m_1^2-im_1V_1/\nu. \tag{17}$$

The general solution of the equations, (11), (12) and (13), are

$$u_{12}=-\frac{g\sin\theta}{2\nu}z^2+k_1z+k_2\ ,\quad w_{12}=k_3, \tag{18),(19}$$

$$p_{12}=g\cos\theta\cdot z+k_4, \tag{20}$$

in which k_1, k_2, k_3 and k_4 are arbitrary constants.

Now, let consider the boundary conditions. If η, the height of the free surface above the plane z = h, is expressed in the form,

$$\eta=a\cdot e^{im_1(x-V_1t)},$$

the conditions are given in the following items, a) and b).

a) SURFACE CONDITIONS

If F and G denote the components of the stresses parallel to the axes x and z respectively acting on a plane z = const., these are given as follows:

$$\begin{aligned}F&=\rho\nu\left(\frac{\partial w_1}{\partial x}+\frac{\partial u_1}{\partial z}\right)\\&=\rho\nu\left(im_1w_{11}+\frac{du_{11}}{dz}\right)e^{im_1(x-V_1t)}+\rho\nu\left(-\frac{g\sin\theta}{\nu}z+k_1\right),\\G&=-p_1+2\rho\nu\frac{\partial w_1}{\partial z}\\&=\left(p_{11}+2\nu\frac{dw_{11}}{dz}\right)\rho e^{im_1(x-V_1t)}+\rho(g\cos\theta\cdot z+k_4),\end{aligned}$$

and at the surface z = h the following stress-conditions must be satisfied:

$$[F]_{z=h}=0, \quad [G]_{z=h}=-\rho g\eta \,. \qquad (21),(22)$$

Substituting the values F and G stated above into the equations (21) and (22) respectively yields

$$\left(im_1 w_{11}+\frac{du_{11}}{dz}\right)_{z=h}=0, \quad \left(-\frac{g\sin\theta}{\nu}z+k_1\right)_{z=h}=0, \qquad (23),(24)$$

$$\left(p_{11}+2\nu\frac{dw_{11}}{dz}+ga\right)_{z=h}=0, \quad \left(g\cos\theta\cdot z+k_4\right)_{z=h}=0. \qquad (25),(26)$$

b) BOTTOM CONDITIONS

As $u_1 = 0$ and $w_1 = 0$ at the bottom $z=0$,

$$(u_{11})_{z=0}=0, \quad (u_{12})_{z=0}=0, \qquad (27),(28)$$

$$(w_{11})_{z=0}=0, \quad (w_{12})_{z=0}=0. \qquad (29),(30)$$

Now, by using of these boundary conditions the integral constants A, B, C and D, and, k_1, k_2, k_3 and k_4 are determined as follows. First, substituting the equations,(14), (15) and (16), into the equations,(23), (25), (27) and (29), yields

$$\frac{i(k^2+m_1^2)}{m_1}(Ce^{kh}+De^{-kh})-\frac{2m_1}{V_1}(Ae^{m_1h}-Be^{-m_1h})=0, \qquad (31)$$

$$\left(1+\frac{2i\nu m_1}{V_1}\right)(Ae^{m_1h}+Be^{-m_1h})+2\nu k(Ce^{kh}-De^{-kh})=-ga, \qquad (32)$$

$$\frac{ik}{m_1}(C-D)-\frac{1}{V_1}(A+B)=0, \qquad (33)$$

$$C+D+\frac{i}{V_1}(A-B)=0. \qquad (34)$$

Next, introducing the value β ,

$$\beta=\sqrt{\frac{\pi}{\nu T_1}}=\sqrt{\frac{m_1V_1}{2\nu}},$$

and solving the equations (31)~(34) in regard to the values A, B, C and D, these integral constants are given as follows:

$$\left.\begin{aligned} &A=-\frac{ga\{2-(1+i)m_1/\beta\}}{4\cosh m_1h}, \quad B=\frac{-ga\{2+(1+i)m_1/\beta\}}{4\cosh m_1h}, \\ &C=0, \quad D=\frac{(1-i)gam_1}{2\beta V_1\cosh m_1h}, \end{aligned}\right\}, \qquad (35)$$

in these calculations, the following approximations are assumed:

1) $k=\pm\sqrt{\frac{-im_1V_1}{\nu}}=\pm(1-i)\beta,$

2) $e^{-kh} \ll \cosh m_1h,$

3) the quantities, $4m_1^2 e^{-m_1 h}$, $4m_1^2 e^{m_1 h}$, $(1+m_1/k)e^{-kh}$ and $(1-m_1/k)e^{-kh}$ are negligible as compared with the quantities, $(1-m_1/k)e^{kh}$ and $(1+m_1/k)e^{kh}$,

4) $4\nu^2 m_1^2 \ll V_1^2$,

5) $m_1^2 \sinh^2 m_1 h \ll 2\beta^2 \cosh^2 m_1 h$,

6) $(m_1/\beta)\sinh m_1 h \ll \cosh m_1 h$,

7) $2m_1\nu \ll V_1$,

and these approximations are verified by our experimental data, $m_1 = 6.0\times10^{-2}cm^{-1}$, $h=25cm$, $\nu=1.3\times10^{-2}cm^2/sec$, $V=160$ cm/sec.[4)]

In the next place, solving the equations (24), (26), (28) and (30) in regard to the integral constants k_1, k_2, k_3 and k_4 yields

$$\left.\begin{aligned} k_1 &= \frac{gh\sin\theta}{\nu}, \quad k_2 = 0, \\ k_3 &= 0, \quad k_4 = -gh\cos\theta \end{aligned}\right\}. \tag{36}$$

Summarizing the results obtained, the first order approximate solutions are given as follows:

$$u_1 = \left\{-\frac{1}{V_1}(Ae^{m_1 z}+Be^{-m_1 z}) + \frac{ik}{m_1}(Ce^{kz}-De^{-kz})\right\} e^{im_1(x-V_1 t)} + \left(-\frac{g\sin\theta}{2\nu}z^2 + k_1 z + k_2\right), \tag{37}$$

$$w_1 = \left\{\frac{i}{V_1}(Ae^{m_1 z}-Be^{-m_1 z}) + (Ce^{kz}+De^{-kz})\right\} e^{im_1(x-V_1 t)} + k_3 , \tag{38}$$

$$\frac{p_1}{\rho} = -(Ae^{m_1 z}+Be^{-m_1 z})e^{im_1(x-V_1 t)} - (g\cos\theta\cdot z + k_4) , \tag{39}$$

in which the constants A, B, C, D, k_1, k_2, k_3 and k_4 are given by the equations (35) and (36).

(2) CHARACTERISTICS OF SOLUTIONS

In the next paragraph, the second order approximate solutions will be induced by the using of the first order approximate solutions. In this paragraph, previous to this analyses, the some considerations relating to the hydraulic properties of the first order approximate solutions are given.

Now, let consider the change of the wave velocity and the wave height resulting from overlapping the surface waves upon the open channel flow. The boundary condition at the water surface is

$$\left(\frac{\partial\eta}{\partial t} + u_1\frac{\partial\eta}{\partial x}\right)_{z=h} = (w_1)_{z=h} .$$

Adopting u_{12} as u_1 in the equation (40) on the basis of the relation $|u_1| \ll |u_{12}|$ yields

$$V^2 - \frac{g \sin\theta}{\nu}\frac{h^2}{2}V - \frac{g}{2m\cosh mh}\Big\{2\sinh mh - \frac{m}{\beta}\cosh mh + \frac{m}{\beta}e^{-\beta h}\cdot$$
$$(\cos\beta h - \sin\beta h)\Big\} + \frac{gi}{2m\cosh mh}\Big\{\frac{m}{\beta}\cosh mh - \frac{m}{\beta}e^{-\beta h}(\cos\beta h + \sin\beta h)\Big\} = 0\,, \qquad (41)$$

in which the suffixes of m and V are omitted. Assuming that the terms including $e^{-\beta h}$ are negligible in comparison with the other terms, the equation (40) is written as

$$V^2 - \frac{g\sin\theta}{\nu}\frac{h^2}{2}V - \frac{g}{m}\left(\tanh mh - \frac{m}{2\beta}\right) + i\frac{g}{2\beta} = 0\ . \qquad (42)$$

First, solving the equation (42) for the case when there is only the wave motion of perfect fluid without flow, as $\nu=0$, and $\sin\theta=0$, the wave velocity is obtained as follows:

$$V_I = \sqrt{\frac{g}{m}\tanh mh}\ . \qquad (43)$$

This is the well-known wave velocity equation for frictionless liquid, and the first order approximate solution. Next, for the case when there is only the wave motion of viscous fluid without flow, as $\sin\theta=0$, the equation (42) is written as

$$V^2 - \frac{g}{m}\left(\tanh mh - \frac{m}{2}\sqrt{\frac{2\nu}{mV_I}}\right) + i\frac{g}{2}\sqrt{\frac{2\nu}{mV_I}} = 0\ , \qquad (44)$$

in which $\beta = \sqrt{mV/2\nu} \doteqdot \sqrt{mV_I/2\nu}$. Putting the solution of the equation (44), which is the second order approximate solution of the equation (42), as follows:

$$V_{II} = V_2 - i\frac{1}{m\tau_2}\ , \qquad (45)$$

the values V_2 and $1/\tau_2$ are written as

$$V_2 = V_I\left\{1 - \frac{\sqrt{m\nu}}{2\sqrt{2V_I}\tanh mh}\right\}\,, \qquad (46)$$
$$\frac{1}{\tau_2} = \frac{\sqrt{m^3 V_I \nu}}{2\sqrt{2}\tanh mh} + \frac{\nu m^2}{8\tanh^2 mh}\ . \qquad (47)$$

On the other hand, substituting the equation (45) into the equation of the surface profile η yields

$$\eta = a e^{im(x-Vt)} = a e^{-t/\tau_2}\cdot e^{im(x-V_2 t)}\,. \qquad (48)$$

By the equation (48), it is apparent that the equations (46) and (47) represent the change of the wave velocity and the damping of the wave height respectively. In other words, according to the equations (46) and (47), the wave velocity decreases for viscosity of fluid, and the damping of the wave height is taken place owing to the same reason. On the other hand, the relationship corresponding with these equations, which was obtained by S. S. Hough, is written as

$$V_2 = V_I\left\{1 - \frac{\sqrt{m\nu}}{\sqrt{2V_I}\tanh 2mh}\right\},$$
$$\frac{1}{\tau_2} = 2m^2\nu + \frac{\sqrt{m^3 V_I \nu}}{\sqrt{2}\sinh 2mh}.$$

Refering to these results, it may be considered that the first term and the second one of the right hand of the equation (47) represent the effects of the internal-viscosity of fluid and that of the bottom friction respectively.

Next, supposing the approximation $\beta = \sqrt{mV/2\nu} \fallingdotseq \sqrt{mV_I/2\nu}$, and substituting the value V_{II} of the equation (45) into the value V including in the second term of the left hand of the equation (42) yields

$$V^2 - \frac{g\sin\theta}{\nu}\frac{h^2}{2}V_{II} - \frac{g}{m}\left(\tanh mh - \frac{m}{2}\sqrt{\frac{2\nu}{mV_I}}\right) + \frac{gi}{2}\sqrt{\frac{2\nu}{mV_I}} = 0. \qquad (49)$$

Putting the solution of the equation (49), which is the third order approximate solution, as follows:

$$V_{III} = V_3 - \frac{i}{m\tau_3},$$

the quantities V_3 and $1/\tau_3$ are written as

$$V_3 = V_I(1-\alpha) + \left\{\frac{V_I}{2} - \frac{1}{8}\frac{\sqrt{2\nu m V_I}}{\tanh mh}(1-2\alpha)\right\}\left(\frac{V_S}{V_I}\right) + \frac{(1-\alpha)V_I}{4}\left(\frac{V_S}{V_I}\right)^2, \qquad (50)$$
$$\frac{1}{\tau_3} = \frac{1}{\tau_2} + \frac{1}{2}\frac{\nu m^2}{8\tanh^2 mh}\left(\frac{V_S}{V_I}\right) - \frac{1}{4}\frac{1}{\tau_2}\left(\frac{V_S}{V_I}\right)^2, \qquad (51)$$

in which $\alpha = \sqrt{m\nu}/\{2\sqrt{2V_I}\tanh mh\}$, and $V_S = [u_{12}]_{z=h} = gh^2\sin\theta/2\nu$. The second and the third terms of the right-hand of the equation (50) represent the effects of the coexistence of flow for the wave velocity, and the second and the third terms of the right hand of the equation (51) do the its effects for the wave damping.

Let estimate quantitatively the effects of the coexistence of flow, by using of the following data, the water mean depth h=12.0 cm, the period of the surface waves T=1.53 sec, the wave length L=160.6 cm and the coefficient of kinematic viscosity $\nu = 1.346 \times 10^{-2}$ cm / sec, the equations (50) and (51) are respectively written as

$$V_3 = 104.6 + 52.3\left(\frac{V_S}{V_I}\right) + 26.1\left(\frac{V_S}{V_I}\right)^2, \qquad (50)'$$
$$\frac{1}{\tau_3} = 7.417\times 10^{-3} + 0.007\times 10^{-3}\left(\frac{V_S}{V_I}\right) - 1.854\times 10^{-3}\left(\frac{V_S}{V_I}\right)^2. \qquad (51)'$$

Fig.2 indicates the relationship between the wave velocity V_3 and the relative flow velocity V_s/V_I. The value V_3 increases with the increase in the value V_s/V_I. Fig.3 indicates the relationship between the coefficient of the wave damping $1/\tau_3$ and the value V_s/V_I. It is found from this results that the damping of the wave height becomes slow against the effects of the fluid viscosity, and the tendency becomes remarkable with the increase in the value V_s/V_I.

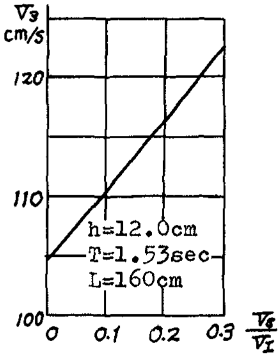

Fig. 2. Relation between V_3 and V_s/V_I

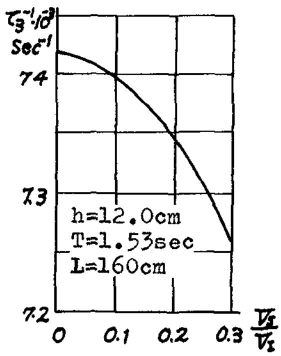

Fig. 3. Relation between $1/\tau_s$ and V_s/V_I.

(3) THE SECOND ORDER APPROXIMATE SOLUTION [2)]

(a) FUNDAMENTAL EQUATION

Substituting the first order approximate solutions into the non-linear terms of the Navier-Stokes equations , the equations are linearized as follows:

$$\frac{\partial u_2}{\partial t}+u_1\frac{\partial u_2}{\partial x}+w_1\frac{\partial u_2}{\partial z}=g\sin\theta-\frac{1}{\rho}\frac{\partial p_2}{\partial x}+\nu\left(\frac{\partial^2 u_2}{\partial x^2}+\frac{\partial^2 u_2}{\partial z^2}\right), \qquad (52)$$

$$\frac{\partial w_2}{\partial x}+u_1\frac{\partial w_2}{\partial x}+w_1\frac{\partial w_2}{\partial z}=-g\cos\theta-\frac{1}{\rho}\frac{\partial p_2}{\partial z}+\nu\left(\frac{\partial^2 w_2}{\partial x^2}+\frac{\partial^2 w_2}{\partial z^2}\right), \qquad (53)$$

and the continuous equation is written as

$$\frac{\partial u_2}{\partial x}+\frac{\partial w_2}{\partial z}=0, \qquad (54)$$

in which suffixes 1 and 2 denote the first and the second order approximate solution respectively. According to the properties of the first order approximate solution, the hydraulic effects, which may be resulted in the existence of the fluid viscosity, are negligible in general. In this analysis, it is assumed for the sake of simplicity that the third terms of the right side of the equations (52) and (53) are omitted as the very small quantities.

Introducing a stream function ψ into these equations, it may be considered that the function ψ is given by the sum of a periodic solution and a non-periodic one as follows:

$$\psi=f_{21}(z)e^{im_2(x-V_2t)}+f_{22}(z). \qquad (54)'$$

Coinciding a stream line $\psi=0$ with the bottom $z=0$, the boundary conditions are written as [5)]

$$\varphi_{21}(0)=0\ ,\quad \varphi_{22}(0)=0\,. \tag{55}$$

And in the case, u_2 and w_2 are respectively represented as

$$u_2=\varphi_{21}'(z)e^{im_2(x-V_2t)}+\varphi_{22}'(z)\ , \tag{56}$$

$$w_2=-\varphi_{21}(z)im_2\,e^{im_2(x-V_2t)}\,. \tag{57}$$

Eliminating the prresure p_2 from the two equations obtained by differentiating the equations (52) and (53) with respect to z and x respectively yields

$$\frac{\partial}{\partial t}\left(\frac{\partial u_2}{\partial z}-\frac{\partial w_2}{\partial x}\right)+\frac{\partial u_1}{\partial z}\frac{\partial u_2}{\partial x}-\frac{\partial u_1}{\partial x}\frac{\partial w_2}{\partial x}+u_1\frac{\partial}{\partial x}\left(\frac{\partial u_2}{\partial z}-\frac{\partial w_2}{\partial x}\right)$$
$$+\frac{\partial w_1}{\partial z}\frac{\partial u_2}{\partial z}-\frac{\partial w_1}{\partial x}\frac{\partial w_2}{\partial z}+w_2\frac{\partial^2 u_1}{\partial z^2}-w_1\frac{\partial^2 w_2}{\partial x\partial z}=0\ , \tag{58}$$

in which the following approximation,

$$w_1\frac{\partial^2 u_2}{\partial z^2}\doteqdot w_2\frac{\partial^2 u_1}{\partial z^2}\ , \tag{59}$$

is assumed. Futhermore, substituting the quantities, u_1 , w_1 , u_2 and w_2 given by the equations (4), (5), (56) and (57) respectively, into the equation (58), the equation is written as

$$e^{im_1(x-V_1t)}\{\varphi_{21}''(im_2u_{11}+w_{11}')+\varphi_{21}'(im_2u_{11}'-m_1m_2w_{11}-m_2^2w_{11})+\varphi_{21}(-im_1m_2^2u_{11}$$
$$-im_2^3u_{11}-im_2u_{11}'')\}+\left[\varphi_{22}''(-im_2\varphi_{21}')+\{\varphi_{21}''(im_2u_{12}-im_2V_2)+\varphi_{21}'(im_2u_{12}')+\right.$$
$$\left.\varphi_{21}(im_2^3V_2-im_2^3u_{12}-im_2u_{12}'')\}\right]=0\ , \tag{60}$$

in which the following approximation,

$$\frac{\partial w_1}{\partial z}\{\varphi_{21}''e^{im_2(x-V_2t)}+\varphi_{22}''\}\doteqdot\frac{\partial w_1}{\partial z}\varphi_{21}''e^{im_2(x-V_2t)}+\frac{\partial w_2}{\partial z}\varphi_{22}''\ , \tag{61}$$

is assumed. By the reason of that the equation (60) must be always satisfied without respect to time and space, the following two conditions,

$$\frac{d^2\varphi_{21}}{dz^2}(im_2u_{11}+u_{11}')+\frac{d\varphi_{21}}{dz}(im_2u_{11}'-m_1m_2w_{11}-m_2^2w_{11})+\varphi_{21}(-im_1m_2^2u_{11}-im_2^3u_{11}-im_2u_{11}'')=0, \tag{62}$$

$$\frac{d^2\varphi_{22}}{dz^2}(-im_2\varphi_{21}')+\{\varphi_{21}''(im_2u_{12}-im_2V_2)+\varphi_{21}'(im_2u_{12}')+\varphi_{21}(im_2^3V_2-im_2^3u_{12}-im_2u_{12}'')\}=0, \tag{63}$$

are obtained. Studying mathematically the types of these equations, it is apparent that the periodic solution φ_{21} is expected to obtaine as a solution of the equation (62) and by using of the solution the non-periodic solution φ_{22} is to be obtained as a solution of the equation (63). In this paragraph, the equations (62) and (63) are defined as the fundamental equations for the following analyses.

b) BOUNDARY CONDITIONS

It is desirable that the boundary conditions to be adopted to the second order approximate solutions define the same physical meaning as those of the conditions, (21), (22), (25)' and (26)', applied to the first order approximate solutions. But, for the simplicity, the conditions including the some different contents in comparison with them are set as follows:

1) THE SURFACE CONDITIONS

$$\left(\frac{\partial \eta}{\partial t} + U_{22}\frac{\partial \eta}{\partial x} - w_2\right)_{z=h} = 0 \,, \qquad (64)$$

$$\left(\sigma = -p_2 + 2\rho\nu\frac{\partial w_2}{\partial z}\right)_{z=h} = -\rho g \eta \,. \qquad (65)$$

2) THE BOTTOM CONDITIONS

$$[u_2]_{z=0} = 0 \,, \quad [w_2]_{z=0} = 0 \,. \qquad (66),(67)$$

Assuming that η, the height of the free surface above the plane z=h, is expressible in the form $\eta = a e^{im_2(x-V_2t)}$, the conditions (64) and (65) are written respectively as

$$a[\varphi_{22}']_{z=h} + [\varphi_{21}]_{z=h} - V_2 a = 0 \,, \qquad (68)$$

$$\int_0^h (-m_2^2 V_2 \varphi_{21} + m_2^2 U_{12}\varphi_{21})dz - 2im_2\nu[\varphi_{21}'']_{z=h} + ga = 0 \,. \qquad (69)$$

Furthermore, the conditions (66) and (67) are written respectively as

$$[\varphi_{21}']_{z=0} = 0 \,, \quad [\varphi_{22}']_{z=0} = 0 \qquad (70)$$

and,

$$[\varphi_{21}]_{z=0} = 0 \,. \qquad (71)$$

Now, transforming the variable z including in the equations (68)~(71) to ξ by the equation $\xi = m_1 z$ yields

$$a m_1\left[\frac{d\varphi_{22}}{d\xi}\right]_{\xi=m_1h} + [\varphi_{21}]_{\xi=m_1h} - V_2 a = 0 \,, \qquad (68)'$$

$$-\frac{m_2^2}{m_1}V_2\int_0^{m_1h}\varphi_{21}d\xi + \frac{g\sin\theta}{2\nu}\frac{m_2^2}{m_1^3}\int_0^{m_1h}(2hm_1\xi - \xi^2)\varphi_{21}d\xi - 2i\nu m_1 m_2\left[\frac{d\varphi_{21}}{d\xi}\right]_{\xi=m_1h} + ga = 0, \qquad (69)'$$

$$\left[\frac{d\varphi_{21}}{d\xi}\right]_{\xi=0} = 0 \,, \quad \left[\frac{d\varphi_{22}}{d\xi}\right]_{\xi=0} = 0 \,, \qquad (70)'$$

$$\left[\varphi_{21}\right]_{\xi=0} = 0 \quad . \tag{71'}$$

Let solve the fundamental equations (62) and (63) under the boundary conditions (68)'~(71)' and the incidental condition (55) relating to the stream function.

c) THE PERIODIC SOLUTION

Transforming the variable z including in the equation (62) to ξ by the equation $\xi = m_1 z$, and representing the coefficients of the equation obtained by an exponential expansions, the equation is written as

$$\xi f_1(\xi)\frac{d^2\varphi_{21}}{d\xi^2} + F_2(\xi)\frac{d\varphi_{21}}{d\xi} + F_3(\xi)\varphi_{21} = 0 \ , \tag{72}$$

in which

$$\left.\begin{aligned} f_1(\xi) &= \sum_{n=1}^{\infty} A_n \xi^{n-1} = A_1 + A_2\xi + A_3\xi^2 + \cdots \ , \\ F_2(\xi) &= \sum_{n=0}^{\infty} B_n \xi^n = B_0 + B_1\xi + B_2\xi^2 + \cdots \ , \\ F_3(\xi) &= \sum_{n=0}^{\infty} C_n \xi^n = C_0 + C_1\xi + C_2\xi^2 + \cdots \ , \end{aligned}\right\} , \tag{73}$$

and,

$$\left.\begin{aligned} A_1 &= \frac{g a m_1 (k^2 - m_1^2)(m_1 - m_2)}{2\beta V_1 \cosh m_1 h}(1-i) \ , \\ A_2 &= \frac{-g a (m_1 - m_2)}{4\beta V_1 \cosh m_1 h}\left\{k^3 - i(k^3 - 2\beta m_1^2)\right\} \ , \\ A_3 &= \frac{g a (k^4 - m_1^4)(m_1 - m_2)}{12 m_1 \beta V_1 \cosh m_1 h}(1-i) \\ A_4 &= \frac{-g a (m_1 - m_2)}{48 m_1^2 \beta V_1 \cosh m_1 h}\left\{k^5 - i(k^5 - 2\beta m_1^4)\right\} . \\ & \cdots\cdots\cdots\cdots\cdots\cdots , \end{aligned}\right\} , \tag{74}$$

and,

$$\left.\begin{aligned} B_0 &= -\frac{g a m_2}{2\beta V_1 \cosh m_1 h}\left\{m_1(k^2 + m_1^2 + m_1 m_2) - m_1^2(2m_1 + m_2)\right\}(1-i) \ , \\ B_1 &= \frac{g a m_2}{2\beta V_1 \cosh m_1 h}\left[k(k^2 + m_1^2 + m_1 m_2) - i\left\{k(k^2 + m_1^2 + m_1 m_2) - 2m_1\beta(2m_1 + m_2)\right\}\right] \ , \\ B_2 &= \frac{-g a m_2}{4 m_1 \beta V_1 \cosh m_1 h}\left\{k^2(k^2 + m_1^2 + m_1 m_2) - m_1^3(2m_1 + m_2)\right\}(1-i) \ , \\ B_3 &= \frac{g a m_2}{12 m_1^2 \beta V_1 \cosh m_1 h}\left[k^3(k^2 + m_1^2 + m_1 m_2) - i\left\{k^3(k^2 + m_1^2 + m_1 m_2) - 2m_1^3\beta(2m_1 + m_2)\right\}\right] , \\ B_4 &= \frac{-g a m_2}{48 m_1^3 \beta V_1 \cosh m_1 h}\left\{k^4(k^2 + m_1^2 + m_1 m_2) - m_1^5(2m_1 + m_2)\right\}(1-i) \ , \\ & \cdots\cdots\cdots\cdots\cdots\cdots , \end{aligned}\right\} , \tag{75}$$

and,

$$C_0 = \frac{-g a m_2}{2\beta V_1 \cosh m_1 h}\left[k(k^2 + m_1 m_2 + m_2^2) - i\left\{k(k^2 + m_1 m_2 + m_2^2) - 2\beta(m_1^2 + m_1 m_2 + m_2^2)\right\}\right] ,$$

$$\left.\begin{aligned}
C_1 &= \frac{gam_2}{2m_1\beta V_1\cosh m_1h}\left\{k^2(k^2+m_1m_2+m_2^2)-m_1^2(m_1^2+m_1m_2+m_2^2)\right\}(1-i)\,,\\
C_2 &= \frac{-gam_2}{4m_1^2\beta V_1\cosh m_1h}\left[k^3(k^2+m_1m_2+m_2^2)-i\left\{k^3(k^2+m_1m_2+m_2^2)-2m_1^2\beta(m_1^2+m_1m_2+m_2^2)\right\}\right],\\
C_3 &= \frac{gam_2}{12m_1^3\beta V_1\cosh m_1h}\left\{k^4(k^2+m_1m_2+m_2^2)-m_1^4(m_1^2+m_1m_2+m_2^2)\right\}(1-i)\,,\\
&\text{-- -- -- -- -- -- -- -- -- -- -- -- -- -- -- -- -- -- --},
\end{aligned}\right\}. \tag{76}$$

By using of the so-called Frobenius's Method, the exponential solution of the equation (72) is obtained as follows:

$$\begin{aligned}
\varphi_{21} &= M\sum_{n=0}^{\infty}h_n\xi^n + N\xi^{\delta}\sum_{n=0}^{\infty}g_n\xi^n\\
&= M(h_0+h_1\xi+h_2\xi^2+\cdots) + N\xi^{\delta}(g_0+g_1\xi+g_2\xi^2+\cdots)\,,
\end{aligned} \tag{77}$$

in which the quantities M and N are integral constants, and the quantities h_n and g_n are coefficients of the exponential solution. By the unequal equation, $m_2 < m_1$, which is considered to be satisfied in general, a relation about the exponent δ is obtained as follows:

$$\delta = 1-\frac{B_0}{A_1} = 1+\frac{m_2}{m_1-m_2} > 1\ . \tag{78}$$

The equation (77) represents the general solution of the fundamental equation (62). Let determine the constants M and N under the boundary conditions (69)' and (71)'. Under the condition (71)', which is the same condition as the first one of the equation (55), the relation $M = 0$ is obtained, and furthermore the following relation is derived under the condition (69):

$$N = \frac{-ga}{X_0+Y_0+Z_0}\ , \tag{79}$$

in which

$$\left.\begin{aligned}
X_0 &= \frac{-m_2^2V_2}{m_1}\left\{\frac{g_0}{\delta+1}(m_1h)^{\delta+1}+\frac{g_1}{\delta+2}(m_1h)^{\delta+2}+\frac{g_2}{\delta+3}(m_1h)^{\delta+3}+\cdots\right\},\\
Y_0 &= \frac{g\sin\theta}{2\nu}\frac{m_2^2}{m_1^3}\left\{2hm_1\frac{g_0}{\delta+2}(m_1h)^{\delta+2}+\frac{(2hm_1g_1-g_0)}{\delta+3}(m_1h)^{\delta+3}+\frac{(2hm_1g_2-g_1)}{\delta+4}(m_1h)^{\delta+4}+\cdots\right\},\\
Z_0 &= -2i\nu m_1m_2(m_1h)^{\delta-1}\left\{\delta g_0+g_1(\delta+1)m_1h+g_2(\delta+2)(m_1h)^2+g_3(\delta+3)(m_1h)^3+\cdots\right\}
\end{aligned}\right\}. \tag{80}$$

Substituting these relations obtained into the equation (77) yields

$$\varphi_{21}(\xi) = \frac{-ga}{X_0+Y_0+Z_0}\xi^{\delta}(g_0+g_1\xi+g_2\xi^2+\cdots\cdots)\,, \tag{81}$$

in which

$$g_0 = g_0\ ,\quad g_1 = \frac{g_0km_2(m_1-2m_2)}{2m_1(m_1-m_2)(2m_1-m_2)}\ ,$$

$$\left.\begin{aligned}
g_2 &= \frac{-g_0 k^2 m_2 (2m_1-3m_2)(5m_1^2-6m_1m_2+4m_2^2)}{24m_1^3(m_1-m_2)^2(2m_1-m_2)(3m_1-2m_2)}\\
g_3 &= \frac{g_0 k^3 m_2 (24m_1^5-114m_1^4m_2+255m_1^3m_2-312m_1^2m_2^3+192m_1m_2^4-48m_2^5)}{144m_1^3(m_1-m_2)^3(2m_1-m_2)(3m_1-2m_2)(4m_1-3m_2)}\\
g_4 &= \frac{-g_0 k^4 m_2 (186m_1^4-77m_2m_1^3+371m_2^2m_1^2-413m_2^3m_1+6m_2^4)}{1440m_1^4(m_1-m_2)^3(2m_1-m_2)(3m_1-2m_2)(5m_1-4m_2)}\\
&\cdots\cdots\cdots\cdots
\end{aligned}\right\} \qquad (82)$$

Furthermore, the first one of the boundary condition (71)' is satisfied by the solution (81) itself.

d) NON-PERIODIC SOLUTION

Substituting the 1st order approximate solutions, u_{12} , u_{12}' and u_{12}'' , into the fundamental equation (63), and transforming the variable z of the equation obtained to ξ by the equation $\xi = m_1 z$ yields

$$m_1^2\frac{d^2\varphi_{22}}{d\xi^2}\left(-im_2m_1\frac{d\varphi_{21}}{d\xi}\right)+m_1^2\frac{d^2\varphi_{21}}{d\xi^2}\left\{im_2\frac{g\sin\theta}{\nu}\left(h\frac{\xi}{m_1}-\frac{1}{2}\frac{\xi^2}{m_1^2}\right)-im_2V_2\right\}+m_1\frac{d\varphi_{21}}{d\xi}\left\{im_2\frac{g\sin\theta}{\nu}\left(h-\frac{\xi}{m_1}\right)\right\}+\varphi_{21}\left\{im_2^3V_2-im_2^3\frac{g\sin\theta}{\nu}\left(h\frac{\xi}{m_1}-\frac{1}{2}\frac{\xi^2}{m_1^2}\right)+im_2\frac{g\sin\theta}{\nu}\right\}=0. \qquad (83)$$

Furthermore, substituting the periodic solution φ_{21} into the equation (83), and representing the equation obtained in the form of an exponential expansions, the equation is written as

$$\xi\frac{d^2\varphi_{22}}{d\xi^2} = L_0 + L_1\xi + L_2\xi^2 + L_3\xi^3 + L_4\xi^4 + \cdots, \qquad (84)$$

in which

$$\left.\begin{aligned}
L_0 &= -\frac{1}{m_1}(\delta-1)V_2,\\
L_1 &= -\frac{1}{m_1}\frac{\delta+1}{\delta}\frac{g_1}{g_0}V_2+\frac{1}{m_1^2}\delta h\frac{g\sin\theta}{\nu},\\
L_2 &= \frac{-1}{m_1}\left\{\frac{2(\delta+2)g_2}{\delta g_0}-\frac{(\delta+1)^2g_1^2}{\delta^2g_0^2}-\frac{1}{\delta}\frac{m_2^2}{m_1^2}\right\}V_2+\frac{1}{m_1^2}\left\{\frac{(\delta+1)h}{\delta}\frac{g_1}{g_0}-\frac{1}{2m_1}\frac{(\delta+2)(\delta-1)}{\delta}\right\}\frac{g\sin\theta}{\nu},\\
L_3 &= \frac{-1}{m_1}\left\{\frac{m_2^2}{\delta^2m_1^2}\frac{g_1}{g_0}-\frac{3(\delta+1)(\delta+2)}{\delta^2}\frac{g_1g_2}{g_0^2}+\frac{3(\delta+3)}{\delta}\frac{g_3}{g_0}+\frac{(\delta+1)^3}{\delta^3}\frac{g_1^3}{g_0^3}\right\}V_2\\
&\quad -\frac{1}{m_1^2}\left\{\frac{(\delta^2+\delta+2)}{2\delta^2m_1}\frac{g_1}{g_0}-\frac{2(\delta+2)h}{\delta}\frac{g_2}{g_0}+\frac{(\delta+1)^2h}{\delta^2}\frac{g_1^2}{g_0^2}+\frac{m_2^2h}{\delta m_1^2}\right\}\frac{g\sin\theta}{\nu},\\
&\cdots\cdots\cdots\cdots,
\end{aligned}\right\}. \qquad (85)$$

Integrating the equation (84) with respect to ξ yields

$$\frac{d\varphi_{22}}{d\xi} = L_0\log\xi + L_1\xi + \frac{1}{2}L_2\xi^2 + \frac{1}{3}L_3\xi^3 + \cdots + S_1, \qquad (86)$$

$$\varphi_{22} = L_0\xi(\log\xi - 1) + \frac{1}{2}L_1\xi^2 + \frac{1}{6}L_2\xi^3 + \cdots + S_1\xi + S_2. \qquad (87)$$

in which S_1 and S_2 are integral constants. Let determine the constants under the second one of the boundary conditions (55) and (70) respectively. Substituting the equation (87) into the former condition, $S_2=0$ is obtained, but the constant S_1 is not determined by means of doing the equation (86) into the latter condition. However, let advance this analysis under the assumption that the constant S_1 was determined by some condition.

e) THE SECOND ORDER APPROXIMATE SOLUTION

Substituting the periodic solution $\varphi_{21}(\xi)$ and the non-periodic one $\varphi_{22}(\xi)$ into the equation (54)', the stream function ψ is written as

$$\psi=\left\{\frac{-ga}{x_0+y_0+z_0}\xi^{\delta}(g_0+g_1\xi+g_2\xi^2+\cdots)\right\}e^{im_2(x-V_2t)}$$
$$+\left\{L_0\xi(\log\xi-1)+\frac{1}{2}L_1\xi^2+\frac{1}{6}L_2\xi^3+\cdot\quad+S_1\xi\right. . \qquad (88)$$

Furthermore, substituting the equation (88) into the equation (56) and (57), the velocity components u_2 and w_2 written as

$$u_2=\left[\frac{-gam_1}{x_0+y_0+z_0}\xi^{\delta-1}\{\delta g_0+(\delta+1)g_1\xi+(\delta+2)g_2\xi^2+(\delta+3)g_3\xi^3+\cdots\}\right]e^{im_2(x-V_2t)}$$
$$+\left[m_1\{L_0\log\xi+L_1\xi+\frac{1}{2}L_2\xi^2+\frac{1}{3}L_3\xi^3+\cdot\quad+S_1\}\right], \qquad (89)$$
$$w_2=\left\{\frac{igam_2}{x_0+y_0+z_0}\xi^{\delta}(g_0+g_1\xi+g_2\xi^2+g_3\xi^3+\cdots)\right\}e^{im_2(x-V_2t)}. \qquad (90)$$

On the other hand, the frictional stress τ_b acting on the bottom is given as follows:

$$\tau_b=\mu m_1\left(\frac{\partial u_2}{\partial\xi}\right)_{\xi=0}=\tau_w+\tau_f\quad, \qquad (91)$$

in which τ_w and τ_f denote the periodic frictional stress due to wave motion and the non-periodic one due to flow respectively, and it is assumed that the limiting values, $(\partial w_2/\partial x)_{z=0}$ and $(\partial u_2/\partial z)_{z=0}$, are able to determined and the relation,

$$\left|\left(\frac{\partial w_2}{\partial x}\right)_{z=0}\right|\ll\left|\left(\frac{\partial u_2}{\partial z}\right)_{z=0}\right| .$$

Now, measuring experimentally the values τ_b and putting these mean value to $\overline{\tau}_b$, the value $\overline{\tau}_b$ is expected to be identical with the theoretical value τ_f . However, the value τ_f are indefinite because of that a limiting value $(m_1L_0/\xi)_{\xi=0}$ is undetermined. Therefore, introducing here an infinitesimal quantity ε , these problems are treated as follows. First, assuming that the experimental value $\overline{\tau}_b$ is identical with the theoretical value τ_f at the level $z=\varepsilon/m_1$, the following relation is obtained from the equations (89) and (91):

$$\overline{\tau}_b=\mu m_1^2\left(\frac{L_0}{\xi}+L_1+L_2\xi+L_3\xi^2+\cdots-\right)_{\xi=\varepsilon} . \qquad (92)$$

Determining the quantity ε from the equation (92), the value τ_ω is written as follows, by using of the equation (89):

$$\tau_\omega = -\frac{\mu a g m_1^2}{x_0+y_0+z_0}\left[\xi^{\delta-2}\left\{(\delta-1)\delta g_0+\delta(\delta+1)g_1\xi+(\delta+1)(\delta+2)g_2\xi^2+\cdots\right\}\right]_{\xi=\varepsilon} e^{im_2(x-V_2t)}. \quad (93)$$

Furthermore, assuming that the mean value $\overline{u_2}$ with time is equal to zero at the level $z=\varepsilon/m_1$, the integral constant S_1 is determined as follows by using of the equation (89):

$$S_1 = -\left(L_0 \log\varepsilon + L_1\varepsilon + \frac{1}{2}L_2\varepsilon^2 + \frac{1}{3}L_3\varepsilon^3 + \cdots\right). \quad (94)$$

In the analyses mentioned above, the second order approximate solutions are given, but the values, m_1, V_1, m_2 and V_2, including in these solutions must be determined in order to that these solutions are established. For the given data, the wave-height $2a$, the wave period T, the bottom slope of channel $J=\sin\theta$ and the discharge per unit width, the value m_1 is expected to be determined by using of the equation (50), in which the value V_3 must be regarded as the value V_1. By using of the solution m_1, the value L_1 and V_1 are obtained as follows: $L_1=2\pi/m_1$ and $V_1=L_1/T$. Furthermore, substituting the quantities, the wave height $2a$, the wave period T, m_1 and V_1, into the surface condition (68)', the values m_2 and V_2 are expected to be calculated.

2 SUMMARY AND CONCLUSIONS

In this paper, analysing the coexistence system of wave and flow as a kind of motion of viscous fluid, the equations of the velocity components, u and w, and that of the frictional stress, , acting on the bottom are derived theoretically. It should be noticed that the quantities, u and τ_b, are given by the sum of the periodic solution relating to the effects of surface waves and the non-periodic one relating to the effects of flow. According to the first order approximate solution, the wave velocity increases with increase in the flow velocity and the decaying of the wave height becomes slow against the effect of the fluid viscosity with the increase. The results of numerical analysis of the second order approximate solution will be published in near future.

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