CHAPTER 57

A SIMPLE MATHEMATICAL MODEL OF WAVE MOTION ON A RUBBLE MOUND BREAKWATER FRONT

Anton Brandtzaeg, Dr. techn.
Professor of Construction and Harbour Engineering
Technical University of Norway
Trondheim, Norway

and

Alf Torum
Laboratory Engineer, Institute of Construction and Harbour Engineering, Technical University of Norway, now on leave, working with Science Engineering Associates, San Marino, California, U.S.A.

A. ABSTRACT

This paper is a continuation of a paper under the same title, presented at the VIIIth Conference in Mexico City, 1962, where a mathematical model was proposed, intended to give a rough idea of the order of magnitude of velocities and accelerations in the downrushing wave on a rubble mound breakwater front. Here observations of 85 individual waves of various dimensions are presented and compared with the formulae derived from the model. Considerable scatter is evident, but it is concluded that the model does correspond roughly to the actual displacements of the water surface during downrush, and therefore may be expected to give useful indications also of velocities and accelerations. The importance of the slope of the surface is emphasized, and, within the scope of the tests, this slope seems to stand in linear relation to the wave steepness.

B. INTRODUCTION

In spite of the many novel types of breakwaters that have appeared during later years, the traditional rubble mound type probably to a great extent still holds the position as the most economical one in locations where suitable rock material is easily available. The stability problems pertaining to such breakwaters therefore still are of particular importance.

Most stability formulae at present in use for such structures are partly empirically based, but partly also based on relations of hydraulics, applicable to uniform and steady flow, while the flow of water up and down a sloping breakwater front certainly is neither.

A complete hydrodynamic solution of such fluid motion is known to be quite complicated, even with an ideal fluid on an inclined plane with no friction, and considerably more so with a viscuous fluid on a breakwater front covered with heavy rocke. At the same time it is, in several relations, desirable to have at least a rough idea of what velocities and accelerations to expect in the downrushing water from a wave of known dimensions on such a breakwater front.

Based on a study of the obeervable water surface during downrush, it has been attempted to develop a simple mathematical model which could give indications as to the order of magnitude of the velocities and accelerations involved, and at the same time as to the pressures in the fluid under the sloping surface.

At the Eighth Conference on Coaetal Engineering in Mexico City, 1962, a model for that purpose was proposed in a paper with the same title as the present one (1). The experimental basis was very scant, consisting of the observation of only three individual waves. Since then a total of 85 more waves have been observed, the procedure being the same as described in detail in the 1962 paper. The studies have gone over a period of about three years, carried out partly by the staff of the River and Harbour Research Laboratory of the Technical Univereity of Norway *), partly by groups of graduate students.

The purpose of the present paper is to present the resulte of these studies, and at the eame time to show how the model can be applied to an actual case with waves of known dimensions and how the result of such application agrees with the observations made.

C. THE MODEL

To restate briefly the basic concepts of the model, reference is made to Fig. 1:

1) The body of downruehing water is considered as a triangle. That is, the surface profile is assumed to be a straight line forming an angle, β, with the breakwater front and an angle, δ, with the horizontal.

2) The triangular body is divided into individual slices, "s_u". Each slice is defined by its original distance, u, from the top, O, of the triangle. The height of each slice is $z = u \tan\beta$, and the width, Δu.

3) Each individual slice is taken to move integrally and independently, without regard to continuity of the fluid, but otherwise in accordance with the gravity, the pressures and the boundary resistance, frictional and inertial, acting in the fluid.

Waving the requirement of continuity is, of course, most unusual. It may not, however, in this particular context, lead to any very great error, as may be seen from Fig. 2. The upper triangle has been divided into five parts, and each part is assumed to move as its middle slice will move according to the model. At some later time the five parts will have separated. Actually, of course, the water remains continuous, and the surface profile will therefore assume a shape somewhat like the line A - B, which is very like what is actually seen to happen.

From these basic concepts the distance, x, travelled by any one "slice", its velocity, v, and its acceleration, a, at a time, t, since it started downwards, was calculated:

---

*) Referred to later as the RHRL.

$$x = B^2 \ln\left(\mathrm{Cosh}\left(\frac{A}{B} t\right)\right) \qquad (1)$$

$$v = AB\ \mathrm{Tanh}\left(\frac{A}{B} t\right) \qquad (2)$$

$$a = \frac{A^2}{\mathrm{Cosh}^2\left(\frac{A}{B} t\right)} \qquad (3)$$

$$A^2 = \frac{g(\sin\alpha - \tan\beta\ \cos\alpha)}{1 + 0{,}5\ C_{MP} \frac{k}{z}} \qquad (4)$$

$$B^2 = \left(1 + 0{,}5\ C_{MP} \frac{k}{z}\right) 32\ z \left(\log_{10} \frac{5z}{k}\right)^2 \qquad (5)$$

For any particular slice, A and B are invariant with respect to t. The volume of an armour block is assumed to be $V = 0{,}5\ k^3$, k being a characteristic, which means approximately a mean linear dimension of the block. This assumption agrees fairly well with the actual shape of blocks. C_{MP} is the inertial coefficient, in 1962 taken to be 0,4. In view of later information from several sources, (2) and (3) and others, $C_{MP} = 1{,}0$ and 1,5 has now been used, and the figure 14.8 in the last parenthesis of Eq. (5) has been changed to 5. (4).

D. THE OBSERVATIONS

The essential features of the test procedure were as follows:

Waves were run against a rubble-covered board with slopes of 1:1,25, 1:1,5 and 1:2, and motion pictures were taken of the wave profiles during up- and downrush. For sample, see Fig. 5 of Reference (1). On each picture also appeared a "clock", making two full revolutions per second. One hundreth of a revolution could be read quite easily, and was used as the unit of time, equal to 1/200 s.

Projections of the pictures, to about one-half of natural size, were made on sheets of paper, and the surface profile of each wave was traced off, together with the grid of lines on the glass panel of the wave channel. The time reading for each picture is shown on the diagrams, as seen in Fig. 3.

To each wave profile a tangent was drawn in the region around its point of intersection with the normal, M - N, to the breakwater slope at the SWL. This tangent was taken to represent the rectlinear surface profile, corresponding to lines O - N, resp. O'- N' in Fig. 1. The distance from the point of intersection, O, of this tangent with the breakwater slope at time t = 0, to the corresponding point at time t = t, was taken as the distance, x, travelled during the time, t, by the slice $u = l_{u_0} - x$, which passes the SWL just at time, t. This, of course, is strictly so only if the successive tangents are parallel to each other, if the angle β is all the time the same.

The reasons for referring all observations to the SWL were, first that failure of the slope generally occurs in this region (see (5), Fig.10) and, second, that down to this region the motion of downrush should hardly be much influenced by the oncoming new wave.

The wave profiles naturally were rather irregular, as may be seen from the profiles of one wave presented in Fig.4 as an example. Here six successive profiles of the same wave have been traced as photographed at six successive times, t. The principle of drawing the tangents at the SWL-normal obviously could only be applied in a general way. Account had to be taken of the general trend of the profiles throughout a wide region around the SWL.

The drawing of the tangents therefore involved a certain amount of personal judgement. The tangents were drawn by several persons, and no subsequent ajustment has been made. From these tangents the experimental values of l_{uo}, x, u, z, and $\tan\beta = z/u$ corresponding to each value of t was taken off.

Naturally, great scattering in the values, in particular of x and u are to be expected, due to the causes mentioned above and also due to the acuteness of the angle β.

As seen from Figures 3, upper left diagram, and 4, there is a difference between the actual length of uprush, l_u, and the "idealized" length, l_{uo}, corresponding to the tangent drawn at the SWL. In some cases the former, in others the latter, is the larger.

In Reference (1), Tables I, II and III, calculations were made with both l_u and l_{uo}, in the former case with the angle β as observed at each time, t, in the latter with an average value of β. Relatively little difference was found.

When the formulae presented are used for the purpose of estimating probable values of velocities and accelerations in a particular case, of course l_u must be substituted for l_{uo}, since only l_u can be estimated from published uprush data.

E. CALCULATIONS OF x, v AND a FOR A PARTICULAR CASE

If the model is to serve as a rough guide in estimating the velocities and accelerations near the SWL in the downrushing stream of water on any particular rubble mound breakwater slope it must be possible to calculate these quantities from Eq.(1) through (5). This requires, besides application of hydraulic coefficients and geometrical relations of the structure, introduction of the quantities z and $\tan\beta$, without recourse to specific test data.

If $\tan\beta$ is known, successive values of z can be calculated, starting with the length, l_u, of uprush along the slope. About this quantity current literature yields a great deal of information. In our tests l_u was read off from each wave profile, and the average value

$$l_u = \frac{1.23\ H}{\sin\alpha} \qquad (6)$$

was arrived at, with a standard deviation of 13 %. The height of uprush accordingly was 1,23 H, as an average for all values of H, T and α in our case. In applying the model to actual cases, one should use values of l_u corresponding to each case, which may differ from those of these tests.

Besides, l_u, $\tan\beta$ must be known. The wave profiles and the corresponding tangents, like those in Fig. 4, consistently show that the angle β decreases in the course of downrush, which also follows from the model itself (Fig. 2). This is of considerable theoretical interest, especially as it influences the pressure in the fluid and thereby also the buoyancy of the cover blocks. In application of the model for estimation of velocities and accelerations in a particular case, however, an average value of β for each wave must be used.

This average value of β varies with the wave characteristics. It was found that the angle $\delta = \alpha - \beta$ (the angle of the wave surface with the horizontal) varied, with reasonable scatter, linearly with the wave steepness, H/L. This is shown in Fig. 5, where the observed values of δ have been plotted against H/L. Some of the points represent one wave, others the average of a group of waves, the number of which is indicated beside each point. The weighted average is represented by the line

$$\delta = \alpha - \beta = 6{,}56\ H/L \tag{7}$$

Eq. (7) together with Eq. (6) or some other relation defining the uprush, form the basis on which the equations (1) to (5) can be applied to actual cases. The calculation itself must be done by trial. One may start with an assumed value of $x = x_1$ at the time, t, considered. From this, $z = (l_u - x_1)\tan\beta$ can be entered in Eq. (4) and (5). By entering A, B and t in Eq. (1) another value, $x = x_2$, is found, which generally does not satisfy the requirement that $x + u = l_u$. Another value of x, $x = x_3$, preferably between x_1 and x_2 and closer to the latter, is chosen, and the calculation is repeated. Usually two or three repetitions lead to a satisfactory value of x. With the corresponding values of A and B, v and a can be calculated from Eq. (2) and (3). The calculation is fairly simple, and, in the cases where many different wave data must be considered, it is easily adaptable for a digital computer.

F. COMPARISON OF CALCULATION WITH TEST DATA

It remains to see how the results of a computation as described compare with test data. Unfortunately we have so far not been able to get reliable measurements of velocities in the downrushing stream at the SWL. The only measured quantity with which to compare therefore is the distance x, the distance travelled down along the slope during the time, t, of the point of intersection with the breakwater front of the tangent to the wave profile near the SWL.

As explained before, great scatter of this quantity must be expected, mainly because of irregularities of wave surfaces and of the acuteness of the angles β . In fact, in some cases the irregularity of the wave profile was such that the test had to be discarded, - for instance when the point of intersection moved upwards, towards negative x, during the first part of downrush, because β initially diminished extra quickly. Out of the total number of 85 waves included in the investigation, 14 were discarded for such reasons.

Since the model is intended as a means to predict probable velocities and accelerations in actual cases, the main question is how computed and experimental data compare with regard to that part of the wave syclus where damage mostly occurs, which is rather late in the stage of downrush. The time, t, elapsing between the start of downrush and this critical stage is shorter for the lower than for the higher waves. Therefore, a comparison made at one definite time, t, will not coincide with the critical stage for all wave heights. Consequently, a compromise is necessary, whereby some of the smaller wave heights fall out of the comparison. The time, t = 0,43 s was chosen, and 21 waves thereby fell out.

The remaining 50 of the 85 waves are represented in Fig. 6, where measured values of x are plotted against those calculated as described in Section E. There is a considerable scatter, but still the individual points group themselves fairly evenly around the line $x_{cal} = x_m$ and 43 points are within the ± 30 %-lines. *) It is noted that the five points representing a 1:1,5 slope all are close to the -30 %-line. It is, however, difficult to draw any conclusions from that fact, since just for this slope, 10 out of 15 waves fell out due to short time of downrush.

Another comparison between experiment and calculation is shown in Fig. 7, by curves representing averages of the values of x at various times, t, observed and calculated as described. The observed values represent 71 waves, as the 14 waves mentioned before are left out. The "calculated values" are averages of x_{cal} for all wave characteristics and all slopes at successive times, t.

Taking into account the unavoidable scatter discussed above, and the approximations and simplifications necessary to make the model applicable to actual cases, there seems to be sufficient agreement between tests and calculation to indicate that the model presented may be useful as a rough indication of what velocities and accelerations may be expected in the downrushing wave.

G. CONCLUSIONS

1. The mathematical model presented and the method of studying the motion of a downrushing wave on a breakwater front by observing the sloping water surface at known time intervals, may make possible a rough over all estimate of the displacements, velocities and accelerations in the fluid as functions of time.

2. There is considerable scatter of the experimental data, but still it is believed that the model may be used as stated.

3. The steepness of the wave surface is important in influencing pressures and accelerations in the fluid.

4. The tests indicate a linear relation between the wave steepness and the average value of the angle, δ, between the water surface and the horizontal.

---

*) The x-values plotted here have been calculated by the trial method described, and satisfy the condition that $x = l_u - u$ at all values of t. In the "detailed summary", printed before the Conference, Fig.3 was plotted from the same data, but, due to a misunderstanding the calculation was different. It started with $x_1 = x_{measured}$ and arrived at x_2, which did not make $x_2 + u = l_u$, but still was plotted as x_{cal} without further trial.

5. The angle, δ, as measured at the SWL, increases (β decreases) throughout the downrush.

ACKNOWLEDGEMENT

Mr. Ole R. Ostby, assistant at this Institute, has given valuable suggestions and helped processing data and preparing their presentation.

H. REFERENCES

(1) Anton Brandtzæg: "A Simple Mathematical Model of Wave Motion on a Rubble Mound Breakwater Slope", Proc. Eighth Conference on Coastal Engineering, 1963.

(2) Robert L. Wiegel: "Oceanographical Engineering", Prentice-Hall International Inc., London, 1965, pp. 269-270.

(3) Borje Johansson: "Vågkrafter mot en på havsbottnen liggande, cirkulär rörledning", Institutionen for Vattenbyggnad, Kungliga Tekniska Hogskolan, Stockholm, Sweden, 1965.

(4) Soren Andersson: "Stabilitet hos skyddslager av ensartad sten i rinnande vatten", Institutionen for Vattenbyggnad, Chalmers Tekniska Hogskola, Goteborg, Sweden, 1963, p. 13.

(5) Anton Brandtzæg: "The Effect of Unit Weights of Rock and Fluid on the Stability of Rubble Mound Breakwaters", Paper 3-14, Tenth Conference of Coastal Engineering, 1966.

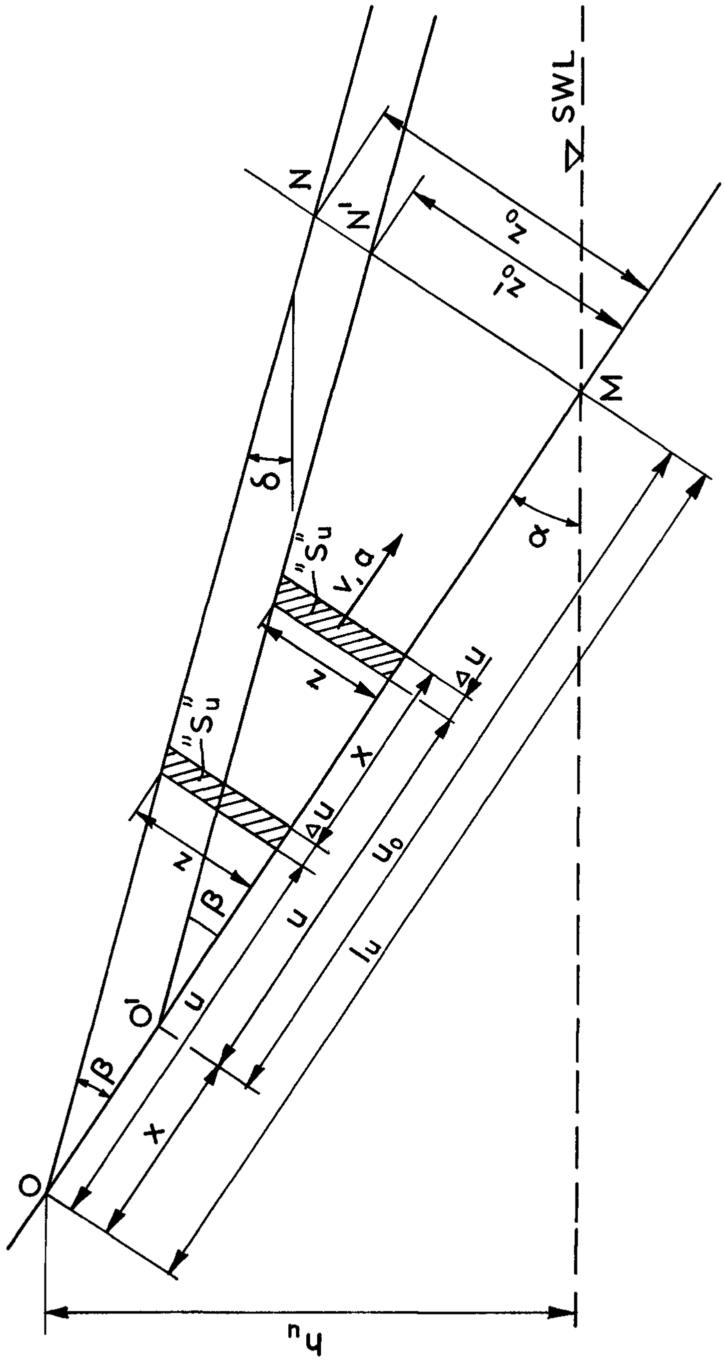

Fig. 1. Concept of motion used in the model.

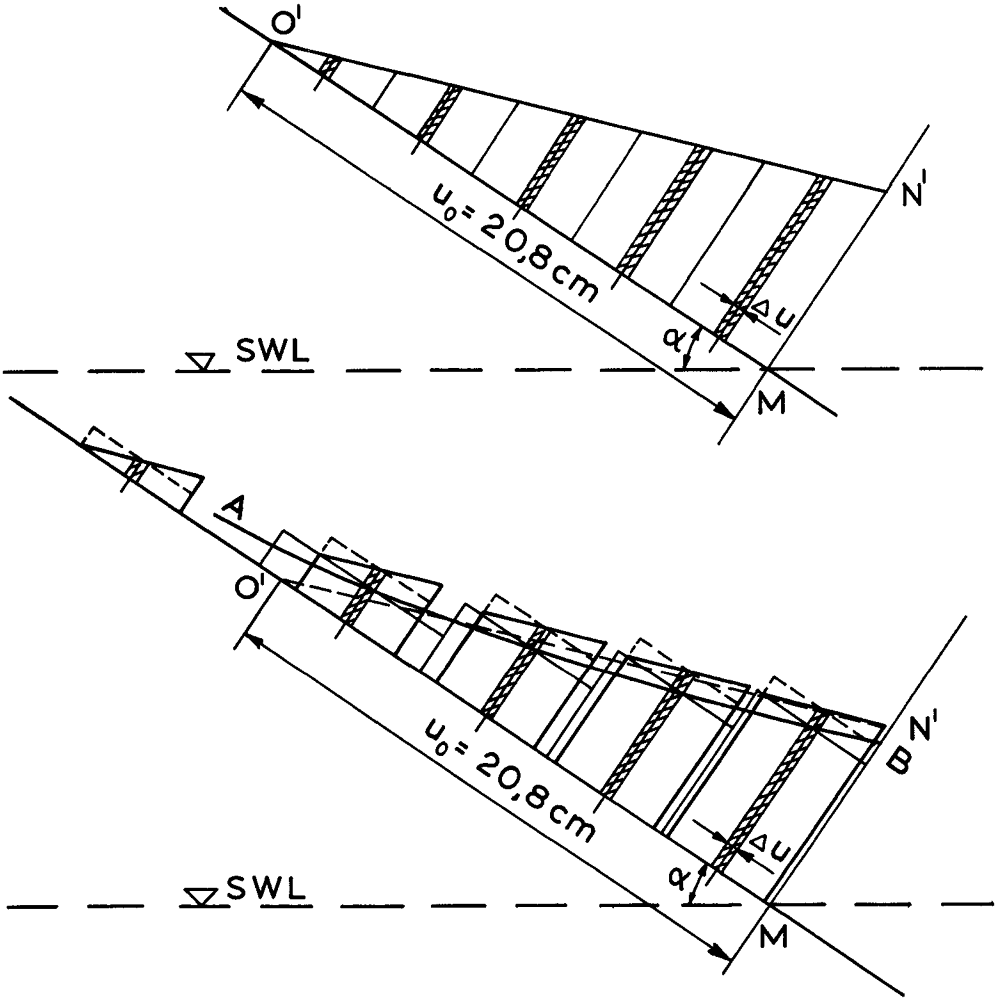

Fig. 2. Restoration of continuity.

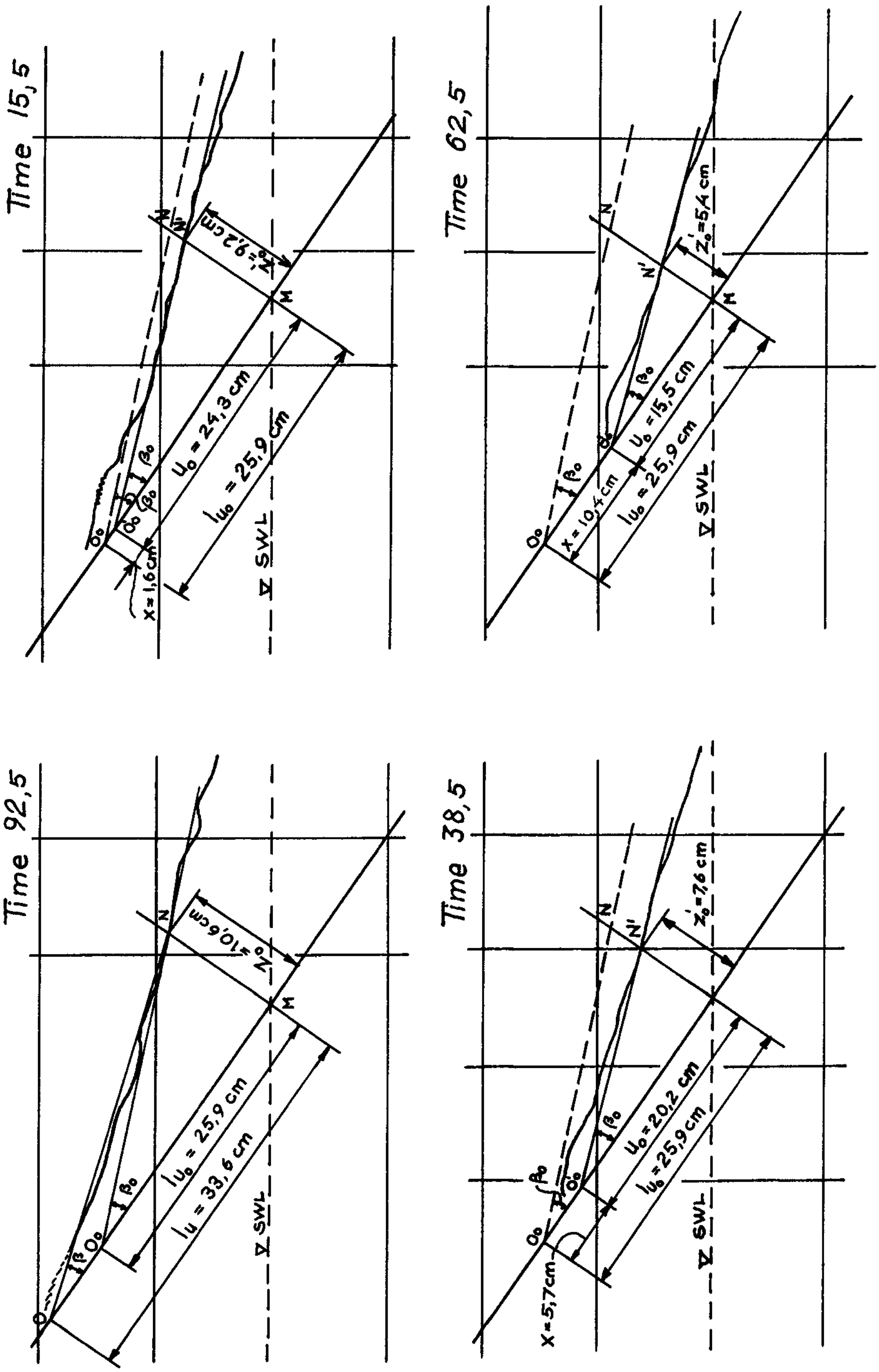

Fig. 3. Surface profiles from Reference (1).

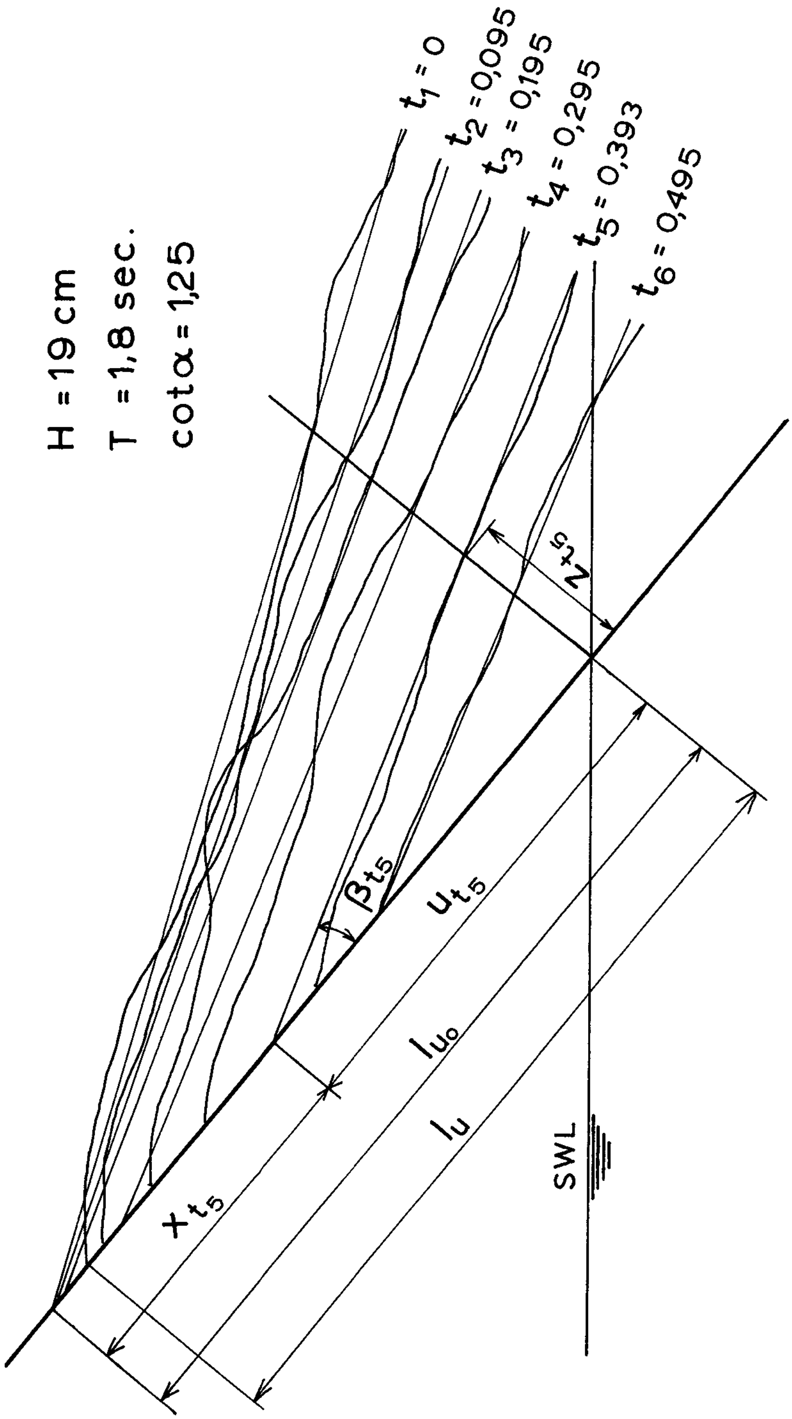

Fig. 4. Sample of wave profiles observed during downrush of one wave, with tangents drawn.

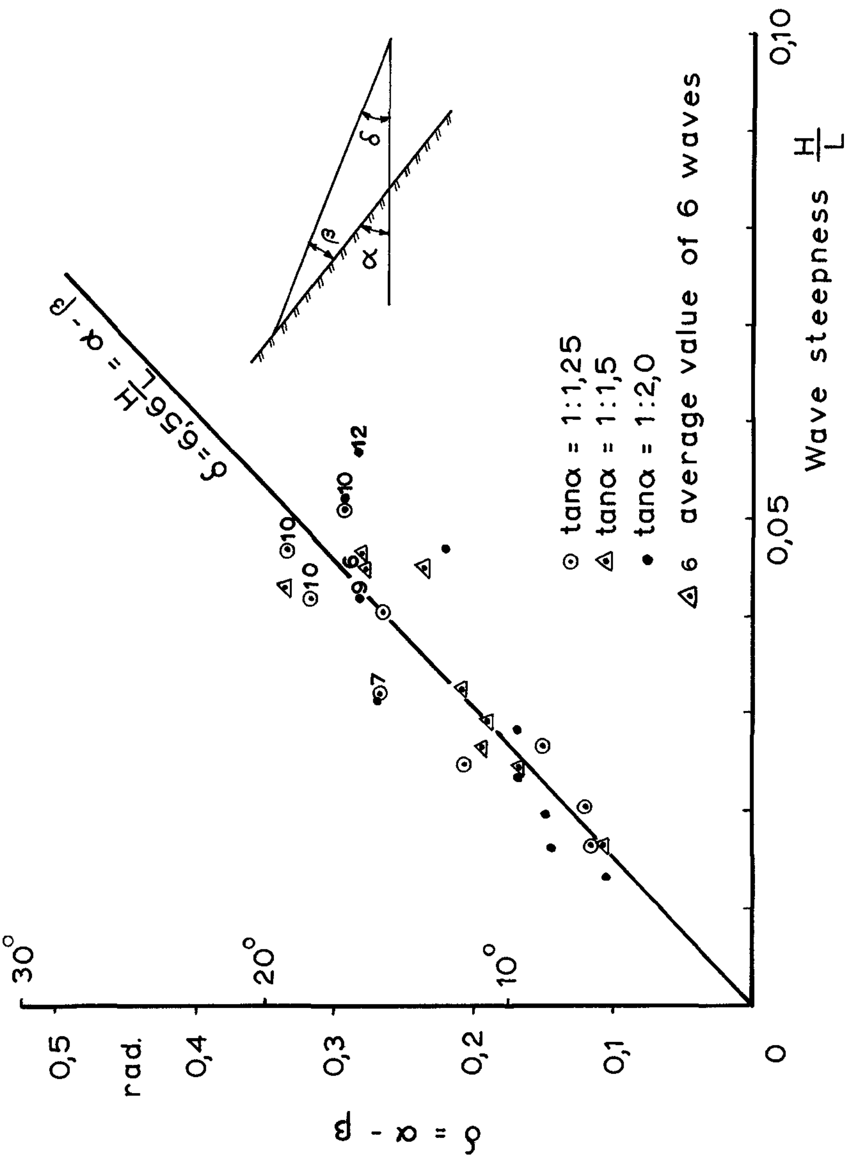

Fig. 5. Relation between wave steepness and average slope of the water surface at SWL during downrush.

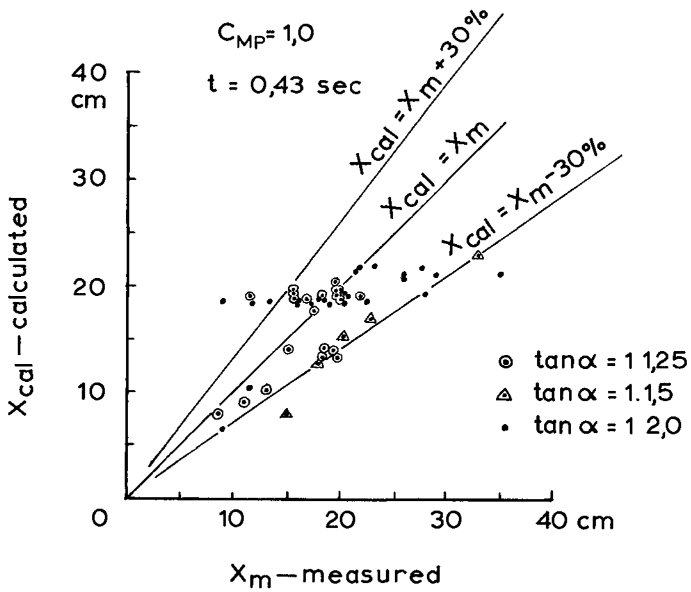

Fig. 6. Comparison between measured and calculated values of x.

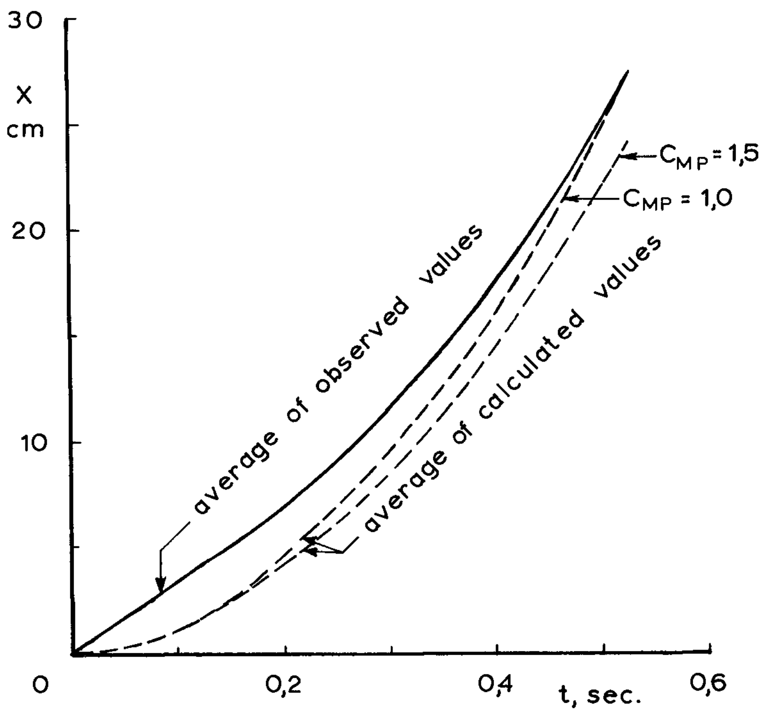

Fig. 7. Comparison of average values of x as observed and as calculated.