CHAPTER 31

THE DYNAMICS OF A COAST WITH A GROYNE SYSTEM

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ABSTRACT

A mathematical theory will be given about the phenomena, which occur if on a coastal area groynee are constructed.

In former similar theories ([1] , [2], [3] , [4], [5]) the coast was schematised by one coastline. In the following theory it is presented by two lines, one line repreeenting the beach and the other one the inshore.

The theory is based upon the following assumptions·

1° the littoral drift along beach and inshore is linear dependent of the angle of wave incidence and therefore of the direction of the line of beach and inshore respectively;

2° the transport perpendicular to the coast depends on the eteepness of the profile.
If the distance between the line of the beach and the line of the inehore is less than a certain equilibrium distance, the profile is too steep and there is an offshore transport. In the opposite case there is an onshore transport. The relation between offshore transport and dietance between the mentioned lines is linearised.

Some results are shown in fig. 8, 9 and 13.

It is found, dat the influence of a groyne system is threefold: they reflect short-period beach processes on the adjacent areae, they retard erosion and they give a lee-side scour.
But the theory only gives one aspect: influence of diffraction and of currents is not yet taken into account.

INTRODUCTION

In order to know what will happen with a coast after the building of coastal structures, one can make use of several approaches.

If one should know the wave spectra during a long time at the site, if a reliable sand transport formula wae available, if one would know the interaction between waves and currente on the eand transport and if the rulee for the onshore- and offshore transport were known, one would probably be able to predict the changemente. Unfortunately, the state of knowledge is not so far just now.

Another approach, which we will use here ie more or less morphological. With the aid of the continuity equation and a simplified dynamical equation with some unknown constants one can find formulae for the coastline in couree of time. With the aid of curve fitting one can find the unknown conetants, which are only valid for the considered area. Furthermore one can find the constants from theories, following the first-mentioned approach, which gives a quick check of these theories.

The firet one, who published a paper about the second approach was PELNARD-CONSIDÈRE [1]. The original idea was of BOSSEN.

PELNARD-CONSIDÈRE assumes, that the profile of the coast always remains the equilibrium profile, so that he only needs to consider one coastline, being one of the contourlines. He assumee no currents, constant wave direction, small angle of wave incidence and a linear relation between angle of wavee incidence and littoral drift. Ae the angle of wave incidence at A is larger than at B, the littoral drift at A ie larger than at B; this means that there is accretion between. A convex coast erodes, a concave coast accretes.

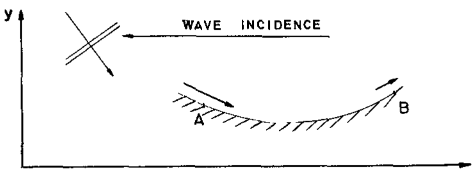

Fig 1 Accretion on a concave coast

He finds (cf "Appendix"), that the accretion is linear dependent of the curvature of the coast and inveree proportional with the depth D, up to where accretion takes place:

$$\frac{\partial y}{\partial t} = \frac{q}{D} \frac{\partial^2 y}{\partial x^2} \qquad \ldots\ldots\ldots \quad (1)$$

in which the x-direction is the main coastal direction, the y-axis points in seaward direction and in which

$$q = \frac{dQ}{d\alpha} \quad ,$$

the derivative of the littoral drift Q to the angle of wave incidence α.

From thie differential equation the coastline y as a function of x and t can be found for many boundary conditions. PELNARD-CONSIDÈRE finds solutions for the coastline of river deltas, the coastline in the vicinity of harbour moles and so on. His experiments confirm the theory.

GRIJM [2], [3] extends the theory by using a better formula for the littoral drift:

$$Q = \tilde{Q} \sin 2\alpha \, ,$$

in which α is the wave direction. He computee the shape of river deltas and finde fundamentally two possible solutione for these deltae: one in which the angle of wave incidence is everywhere less than 45° (fig. 2a) and the other one, in which this angle is everywhere more than 45°(fig. 2b). Aleo combinations are possible (fig. 2c).

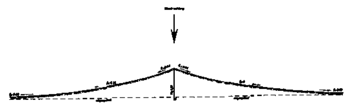

FIG 2a

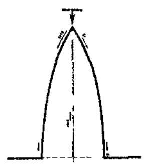

FIG 2b

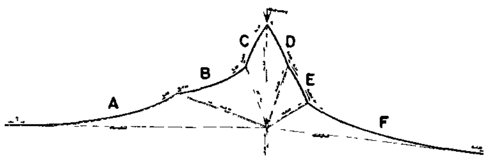

FIG 2c

In figure 2c the angle of wave incidence is less than 45° at the parts A, B, F and more than 45° at the parts C. D, E. As one never knows which combination one has to choose, the problem seems to be indefinite. BAKKER and EDELMAN [4] treat the same problem with the linear Pelnard-Considère approach. They investigate also the case of negative $q (= \frac{dQ}{d\alpha})$, which occurs if the angle of wave incidence is large. Their solutions are more or less similar to GRIJM, but opposite to GRIJM, they also find a periodical solution:

$$y = e^{-\frac{q}{D}K^2 t} \cos \dot{K} x \quad . . (2)$$

This is a sinusoidal shaped coastline of which the amplitude decreases in course of time if q is positive (small angle of wave incidence), but increases if q is negative (large angle of wave incidence). Therefore, solutions of the shape of fig. 2b are unstable and will be destroyed, because slight deviations trigger large deviations according to (2).

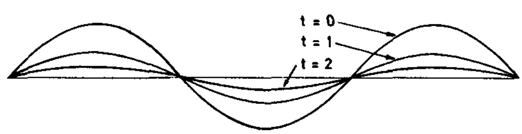

Fig 3 Decay of sinussoidal shaped coastline

This solves the problem of the indefiniteness: nature will prefer solutions of category I. GRIJM did not find this solutions, because he confined himself to solutions growing with $\sqrt{t}$ in all directions.

PELNARD-CONSIDÈRE [1] considered river deltas and coastal structures, GRIJM [2], [3] and BAKKER and EDELMAN [4] merely treat river deltas. BAKKER [5] investigates the periodical solutions of eq.(1). Besides the "standing" and attenuating wave of fig. 3, also propagating and attenuating waves are possible. The propagating sandwave found in the prototype on Vlieland could be explained with theory. BAKKER also examines the influence of coastal structures on these sandwaves. The sandwave appears to be reflected by the structure: the amplitude at the site of the structure is enlarged. One can sense this, because there is an analogy between these moving sandwaves and tidal waves. The coastline is analogous to the vertical tide and the littoral drift to the horizontal tide. If one stops the littoral drift (current) by a dam, one increases the variations of the coastline (vertical tide).

One of the beauty failures of the solutions of PELNARD-CONSIDÈRE [1] and BAKKER [5] is the assumption of parallel depth contours. Near coastal structures the deviations of the prototype can be considerable. For instance, the solution near a breakwater is sketched in fig. 4a. PELNARD-CONSIDÈRE finds, that the coast on the left-hand side builds up to the head of the breakwater and that the coastline on the right-hand side erodes the same amount. This might be true for constructions, extending to large depths. But in the case of groynes only the littoral drift on the beach is prevented: at the beach there will be sedimentation of material on one side of the groyne and erosion on the other side. But in deeper regions this disturbance does not take place, so on the left-hand side the profile becomes steeper than the equilibrium profile and the sand drops down, and on the right-hand side the profile is flatter than the equilibrium profile and the sand is pushed by the waves in upward direction.

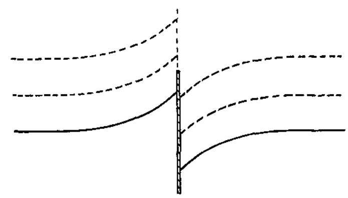

Fig 4^a Coastline with parallel contour - lines

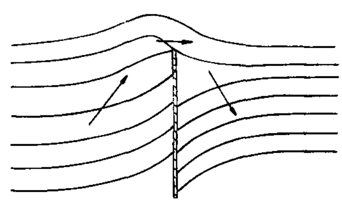

Fig 4^b More reasonable approximation
The arrows give the direction of the sanddrift

In order to reproduce this feature in a mathematical model it is necessary to schematise the coast by two lines instead of one. This will be done in this article. The difference with former theories is, that thus off- and onshore transport are taken into account.

Definitions and assumptions.

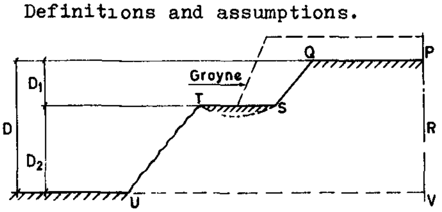

Fig 5^a Schematised profile

Fig. 5a denotes a schematized profile. The profile is under-divided into two parts, one part consisting of the profile between 0 and D_1 below sea level (beach), the second one between D_1 and $D_1 + D_2$ (inshore).

Between beach and inshore is a horizontal shelf at depth D_1, the total depth D being $D_1 + D_2$.

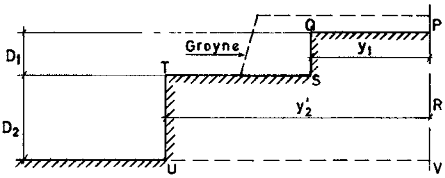

Fig 5^b Two-line theory

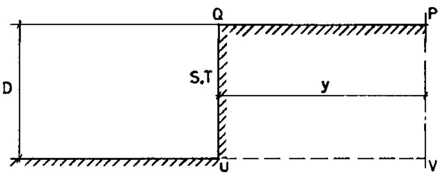

Fig 5^c One-line theory
the area PQSTUV
is always the same

The depth D is aesumed to be so large, that no littoral drift takes place here.

In reality one can imagine, that a breaker ridge occurs at depth D_1 and that a trough links the two parts of the coast (dotted line), fig. 5a.

It is assumed, that a groyne reachee up to the horizontal ehelf at depth D_1 and prevente all littoral drift along the beach, but of course not along the inshore.

In the theory the profile is still more schematized, according to fig. 5b. A stepwiee profile remains. The areas PQSR and RTUV in fig. 5a are equal to the corresponding areas in fig. 5b.

In top view one sees two lines at a dietance y_1 and y_2' from the x-axis, which will be called "the line of the beach" and "the line of the inshore" respectively.

The "equilibrium distance" W is the dietance $y_2' - y_1$ between beach and inshore, when the profile is an equilibrium profile.

The following dynamic equations are aesumed.
If the distance $y_2' - y_1$ is equal to the equilibrium distance W, no interaction is assumed. If the distance $y_2' - y_1$ is less than W, the profile is too steep and an offshore transport will be the result. An onshore traneport will occur in the oppoeite case.

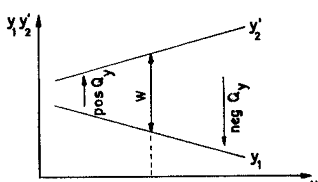

Fig 6 On- and offshore transport

We linearize this relation and take for the offshore transport Q_y per unit length:

$$Q_y = q_y \left\{ y_1 - (y_2' - W) \right\} \quad . . . . (3a)$$

in which q_y is a proportionality constant. The dimeneion of q_y ie $[1/t]$. For a simpler notation, we denote:

$$y_2 = y_2' - W \quad . . . . (4)$$

Then (3a) becomee

$$Q_y = q_y \ (y_1 - y_2) \quad . . . . (3)$$

With respect to the littoral drift, the assumption of PELNARD-CONSIDÈRE [1] is applied, bothfor beach and inshore: the transport in linearized:

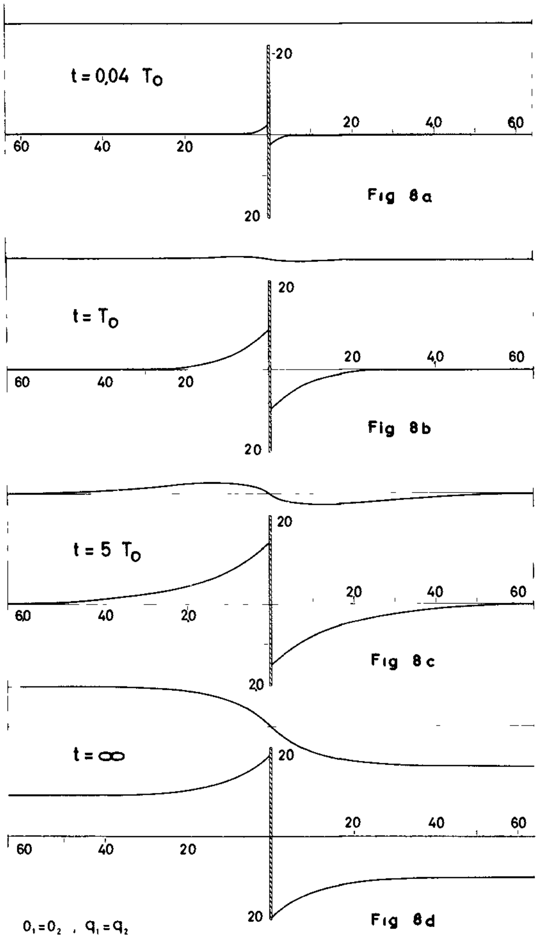

Fig 8a

Fig 8b

Fig 8c

Fig 8d

$Q_1 = Q_2$, $q_1 = q_2$

Scales

Horizontal $1 \hat{=} L_0 = \sqrt{\frac{1}{q_y\left(\frac{1}{q_1}+\frac{1}{q_2}\right)}}$

Vertical $1 \hat{=} L_0 \tan\alpha = \frac{Q_1}{q_1}\sqrt{\frac{1}{q_y\left(\frac{1}{q_1}+\frac{1}{q_2}\right)}}$

Time $T_0 = L_0^2 \frac{D_1}{q_1} = \frac{1}{2}\frac{D_1}{q_y}$

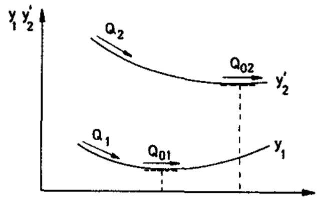

Fig 7 Littoral drift along beach and inshore

$$\left.\begin{aligned} Q_1 &= Q_{o1} - q_1 \frac{\partial y_1}{\partial x} \\ Q_2 &= Q_{o2} - q_2 \frac{\partial y_2}{\partial x} \end{aligned}\right\} \quad \ldots\ldots\ldots (5)$$

in which Q_{o1} and Q_{o2} are respectively the "stationary transport" (littoral drift where $\frac{\partial y_1}{\partial x} = 0$, resp. $\frac{\partial y_2}{\partial x} = 0$, fig. 7) on beach and inshore and in which q_1 and q_2 are proportionality constants. The dimension of q_1 and q_2 is $[l^3/t]$.

RESULTS

By making use of the continuity equation and the above-mentioned dynamical equations one can compute many stationary and instationary cases (cf. "Appendix").
Of importance appears to be a reference length:

$$L_o = \sqrt{\frac{1}{q_y\ (\frac{1}{q_1} + \frac{1}{q_2})}} \quad \ldots\ldots\ldots\ldots (6)$$

Fig. 8 shows the result in the case of accretion and erosion near one groyne when $q_1 = q_2$ and $D_1 = D_2$. (cf Appendix, 3).

In the initial situation the lines of beach and inshore are parallel. Fig. 8a shows the situation immediately after the construction of the groyne. Only the beach shows some build-up on the right-hand side and erosion on the left-hand side. It must be stressed, that the influence of diffraction is not taken into account.

In fig. 8b and 8c the profile on the left-hand side becomes too steep and sand drops down to the inshore. Here the littoral drift was originally everywhere the same. The supply of sand from the beach overcharges the transport capacibility of the inshore and therefore sand sedimentates here.

Now the littoral drift Q_2 along the groyne at the inshore becomes larger ($\frac{\partial y_2}{\partial x}$ becomes negative, cf (5)). In the final stage (fig. 8d) beach and inshore on the left-hand side and on the right-hand side are shifted with respect to each other. This is in correspondence with the results of PELNARD-CONSIDÈRE [1], but he finds, that the coast builds up to the top of the groyne, and here it is found, that it builds up to a distance, only dependent of q_1, q_2, q_y and the angle α of wave incidence, where $\tan\alpha = \frac{Q_{o1}}{q_1}$

Fig. 9 shows several stationary cases.
Fig. 9d gives again the final state of fig. 8d. The transport is the same as without a groyne, because the transport at a long distance of the groyne does not change. If more groynes are constructed, the littoral drift along the beach is stopped more and more, because the beach turns in the direction

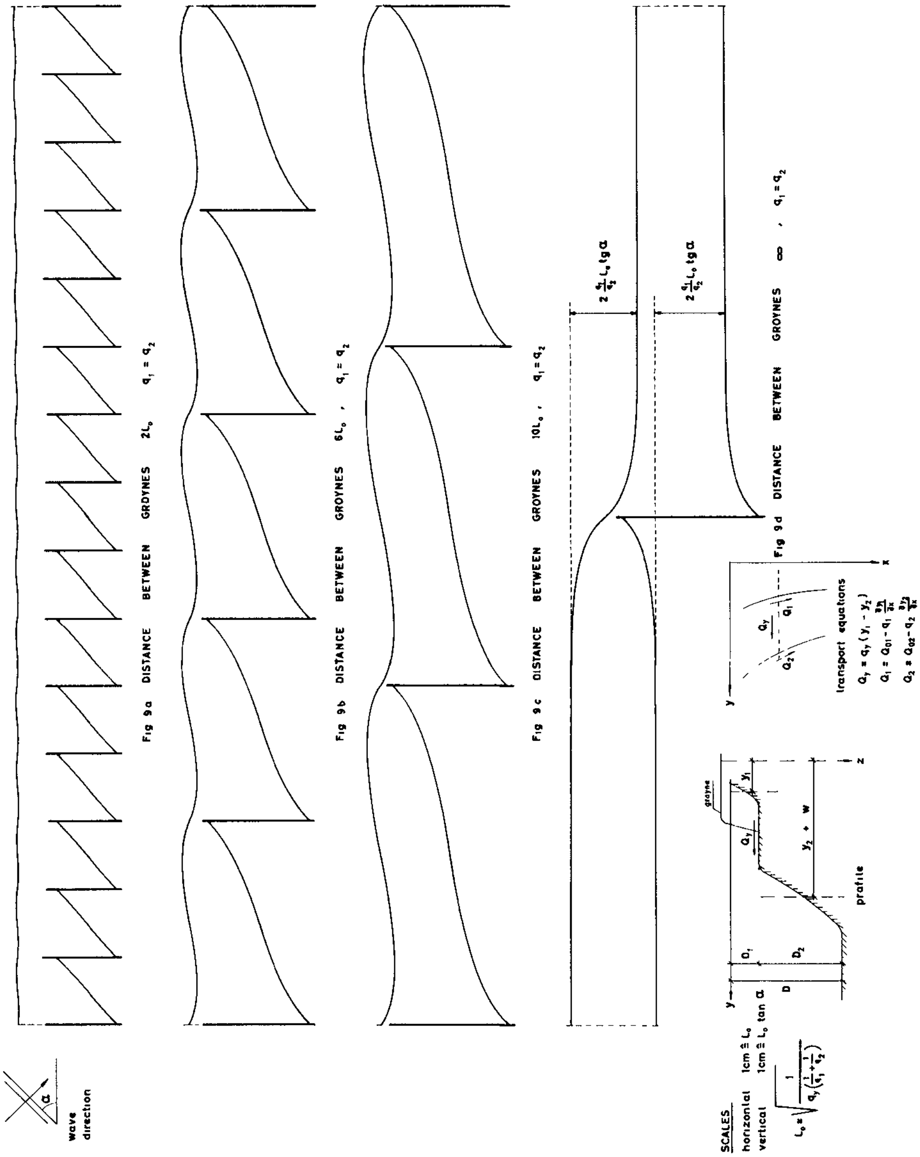

Fig 9a DISTANCE BETWEEN GROYNES $2L_0$ $q_1 = q_2$

Fig 9b DISTANCE BETWEEN GROYNES $6L_0$, $q_1 = q_2$

Fig 9c DISTANCE BETWEEN GROYNES $10L_0$, $q_1 = q_2$

Fig 9d DISTANCE BETWEEN GROYNES ∞, $q_1 = q_2$

of the wave crest (fig. 9c,b,a).

We now consider the case, that the groynes are so near to each other, that they prevent all the transport along the beach (fig. 9a). In this case the total littoral drift is the drift along the inshore:

$$Q = Q_2 = Q_{o2} - q_2 \frac{\partial y_2}{\partial x}$$

Before the construction of the groynes this transport was:

$$Q = Q_1 + Q_2 = Q_{o1} + Q_{o2} - q_1 \frac{\partial y_1}{\partial x} - q_2 \frac{\partial y_2}{\partial x}$$

Following the conception of PELNARD-CONSIDÈRE (cf Appendix, 1), and assuming that the sedimentation takes place equable on beach and inshore (q_y sufficiently large), the coastal equation for a protected area would be:

$$\frac{\partial y}{\partial t} = \frac{q_2}{D} \frac{\partial^2 y}{\partial x^2}$$

The coastal constant q/D is changed in q_2/D. The assumption, that the sedimentation takes place equable on beach and inshore, is about correct for long-term coastal processes (long with respect to T_o, cf Appendix 2,(19),7.

We considered the case, that the groynes were so near to each other, that they prevent all the transport along the beach. If the distance between the groynes is larger, the coastal constant will not diminish with a factor q_2/q, but less:

$$\frac{\partial y}{\partial t} = \frac{q'}{D} \frac{\partial^2 y}{\partial x^2}, \text{ in which } q_2 < q' < q_1 + q_2 \quad \ldots (7)$$

This factor can be computed (cf Appendix, 4) and will be called $\frac{1}{p^2}$:

$$\frac{1}{p^2} = \frac{q'}{q_1 + q_2} \longrightarrow p = \sqrt{\frac{q_1 + q_2}{q'}} \quad \ldots\ldots\ldots (8)$$

We now have returned to the one-line theory of PELNARD-CONSIDÈRE a protected area can be considered as an area with another coastal constant ($\frac{q'}{D}$) than the neighbouring unprotected area, and this constant can be computed with the two-line theory.

In the following section we shall give first some rough statements, making use of the one-line theory, considering a protected area as an area with another coastal constant, and afterwards we shall illustrate it with more accurate computations with the two-line theory. The advantage of the one-line theory is, that it gives a quick insight in the essence of the matter.

LAWS OF SCALE

One can make the coastal equation (1) dimensionlees by substituting: $x = n_x . \chi$, $y = n_y . \eta$, $t = n_t . \tau$, $q/D = n_{cc} . C$, in which χ , η ,τ

and C are dimensionless and n_x, n_y, n_t, n_{cc} give the scale factor of x, y, t and the coastal constant q/D.

One finds the following relation:

$$n_{cc} \frac{n_t}{n_x^2} = \text{constant} \quad \ldots\ldots\ldots\ldots \quad (9)$$

The scale n_y of y can be chosen arbitrary, because y occurs on both sides of eq. (1). We shall give some examples:

Consider two half-infinite coastal areas with different coastal constants, which are in rest at infinity. Suppose that the ends of both areas carry out the same movements in course of time. The shape of the coastline will be the same in both cases, but the x-scale will be equal to the square root of the coastal constants (fig. 10a).

Consider now two areas with different coastal constants, which are identical at time t = 0 and of which the ends of the areas are in rest. Now the x-scale is the same and therefore the timescale of the changements will be inverse proportional to the coastal constants (fig. 10b).

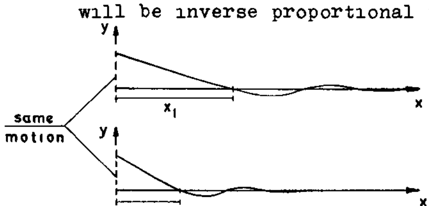

Fig 10a

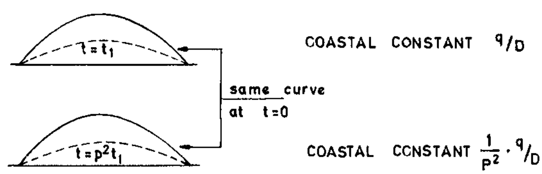

Fig 10b

INFLUENCE OF A ROW OF GROYNES

We consider a coast, where over a long stretch groynes are constructed at time t = 0.

What is the influence of the groynes on the coast?

We will consider three influences, which can be superponed, because all equations are linear:

1° the influence of external causes (boundary conditions);

2° the influence of the oblique incidence of the waves (stationary transport);

3° the influence of the shape of the coast.

1° Influence of external causes.

Assume a wave direction, perpendicular to the coast and a straight coastline at time t = 0.

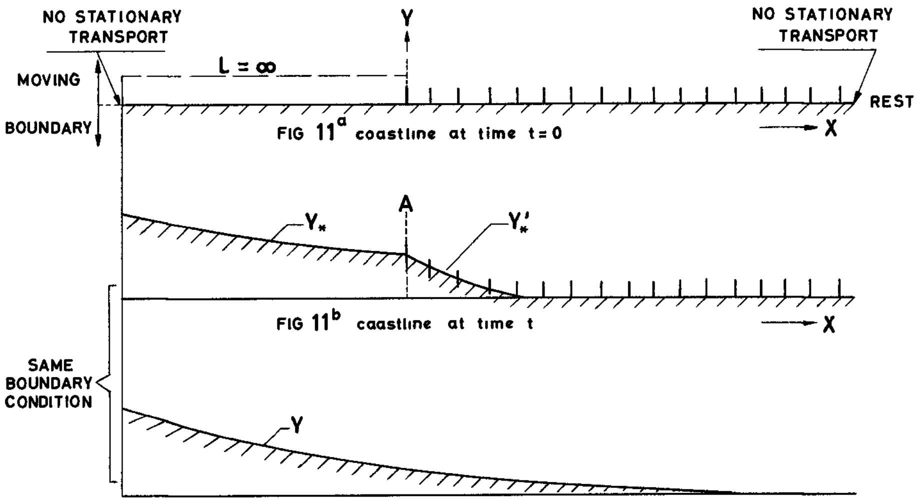

FIG 11[a] coastline at time t = 0

FIG 11[b] coastline at time t

FIG 11[c] coastline at time t, if no groynes would have been constructed

Assume, that by one or another external cause (for instance, a river or tidal canal, which embouches left of the left boundary, fig. 11) the coastline at time t, would have been according to fig. 11c, if no groynes would have been constructed.

What is the coastline if groynes are constructed at time t = 0 ?

This coastline can be constructed with the following method.

The coastline y_*' of the protected area can be found by reducing the x-scale of y by a factor 1/p (p defined by (8)) and by multiplying the y-scale with a factor 1 + r, in which:

$$r = \frac{p - 1}{p + 1}, \quad \text{so} \quad 1 + r = \frac{2p}{p + 1} \qquad \ldots\ldots\ldots (10)$$

In formula:

$$y_*'(x) = (1 + r)\, y\, (px) \qquad \ldots\ldots\ldots (11a)$$

The coastline of the unprotected part y_* can be found as the sum of the original y plus a "reflected y"; the latter one being the reflection of the original y (for $x > 0$) with respect to the y-axis and multiplied with the reflection factor r, given in eq. (10):

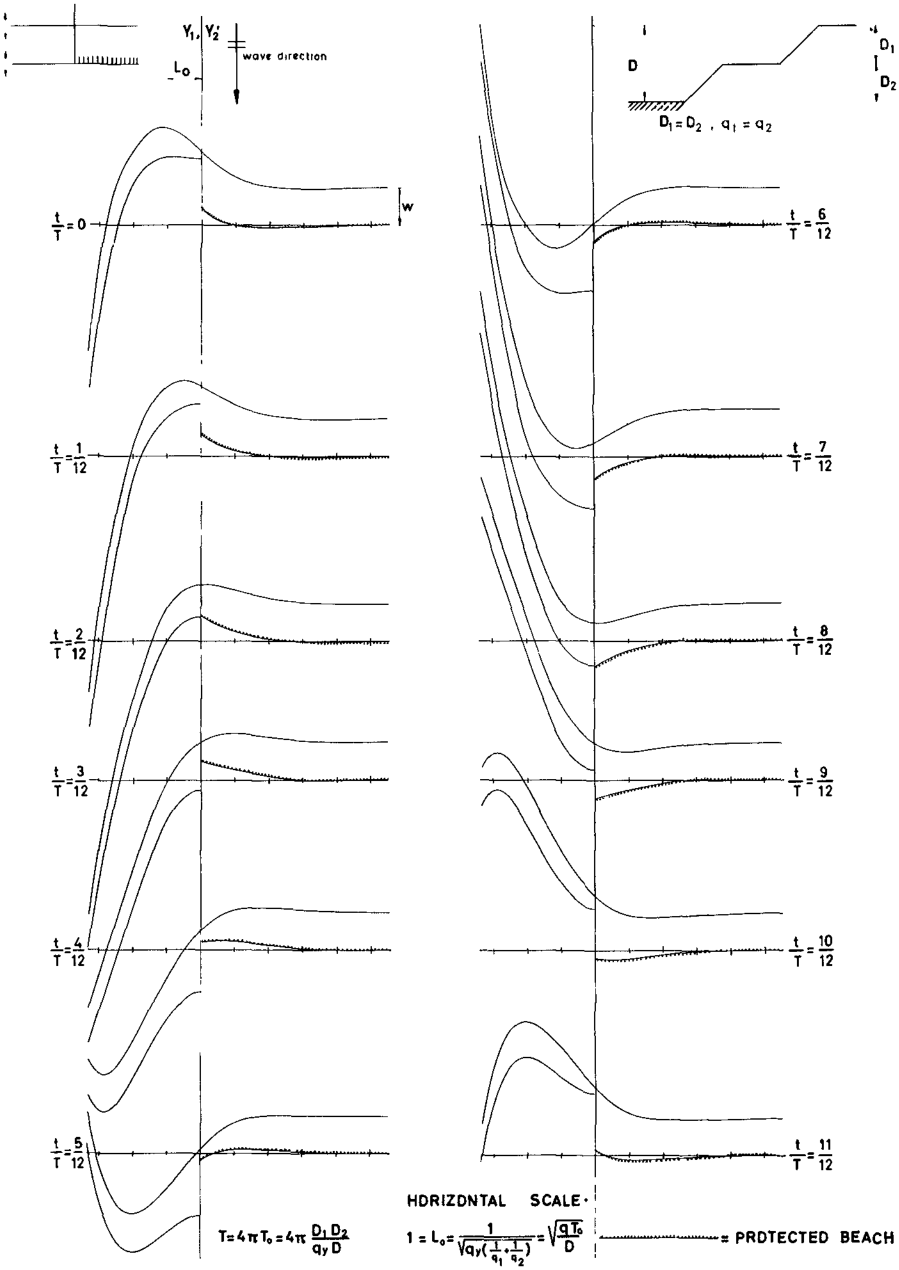

FIG 13 PROPAGATING SANDWAVE ACCORDING TO TWO-LINE SYSTEM

if $x < 0$, then $y_*(x) = y(x) + r \cdot y(-x)$ . . . . (11b)

In fig. 12 the method of construction is visualised.

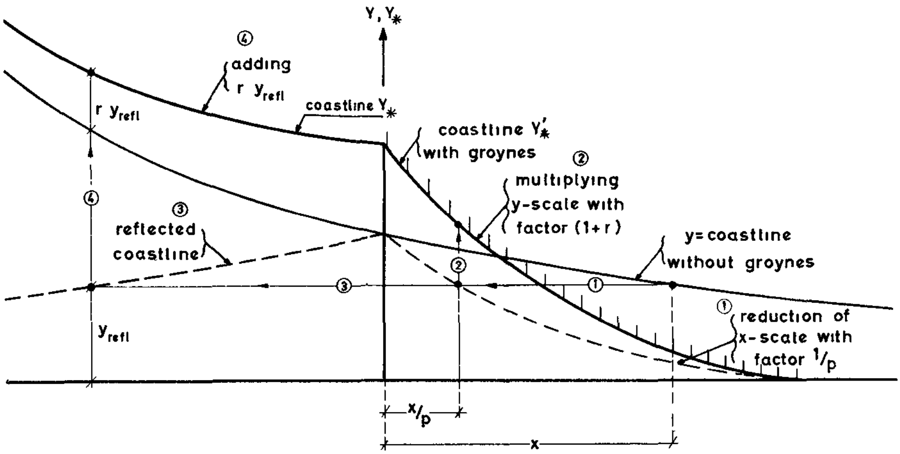

FIG 12

One can proof easily, that:

a. the new coastline of the unprotected part y_* suffices the coastal equation (1) for an unprotected area: it consists of the sum of two functions, both obeying this linear equation.

b. the new coastline of the protected part y_* suffices the coastal equation (7) for a protected area. For the timescale of the changements at A is the same in fig. 11b and 11c and the x-scale is proportional to the square root of the coastal constants (fig. 10a).

c. the transport a little bit on the left of A in fig. 11b is equal to the transport a little bit on the right of A.

d. the y-coordinates at A of the protected and unprotected area are the same.

As an example of the influence of external causes in the case of the two-line system, fig. 13 shows the influence of groynes on a harmonical, propagating sandwave. These sandwaves occur, if the left-hand boundary erodes and accretes harmonically (by an external cause), if the right-hand boundary (protected coast) is in rest, and if the profile at the boundaries is an equilibrium profile [4], [5]. In the case of fig. 13, it is assumed, that the distance between the groynes is so small, that no littoral drift takes place on the beach. The protected beach just reacts as a store. The formulae are given in the Appendix, 7.

As fig. 13 shows, for short-period processes[1)], the motion of the

1) defined in the appendix

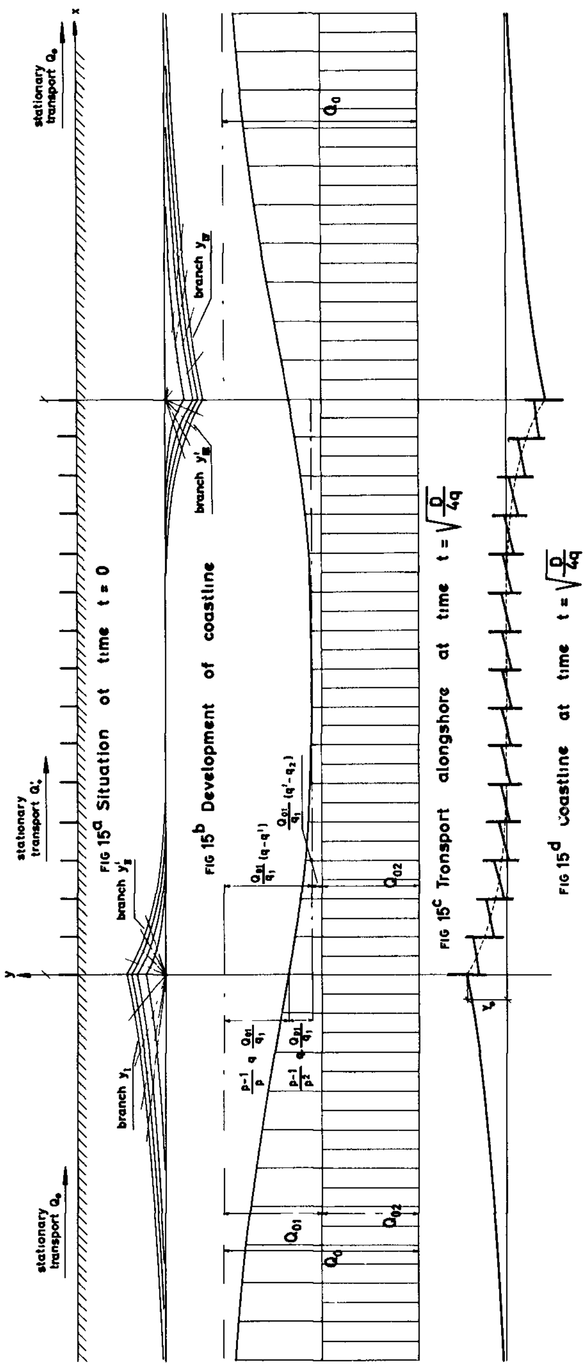

FIG 15a Situation at time $t = 0$

FIG 15b Development of coastline

FIG 15c Transport alongshore at time $t = \sqrt{\frac{D}{4q}}$

FIG 15d Coastline at time $t = \sqrt{\frac{D}{4q}}$

inshore is practically the same either if groynes are constructed, either if they are not. Then the motion of the protected beach is small, but there is a large edge effect on the unprotected beach, near the beginning of the groyne system.

For long-period processes[1] the results of the one-line theory are confirmed: the wave-length along the protected beach and inshore is a factor $1/p$ times the wave-length along the unprotected beach and the amplitude is enlarged by a factor $1 + r$, according to (10).

2^o Influence of oblique incidence of waves

According to the one-line system, the transport along an unprotected coast y and along a protected coast y' are respectively:

$$Q = Q_o - q\,\frac{\partial y}{\partial x}$$

$$Q' = Q_o' - q'\,\frac{\partial y'}{\partial x}$$

In this formula Q_o' and q' are smaller than Q_o and q. If the transport along the beach is prevented totally, Q_o' and q' are respectively Q_{o2} and q_2, the constants of the inshore.

Consider an area, partly protected with groynes, which is at $t = 0$ parallel to the x-axis (fig. 15a). On the unprotected beach the transport will be Q_o and on the protected beach Q_o' and therefore the sedimentation per unit time will be $Q_o - Q_o'$. With the same considerations as in the chapter "influence of external causes" it can be shown, that a kind of delta will be formed, which will increase with a velocity proportional to $\sqrt{t}$ (fig. 15b). This delta is not symmetrical: the same y-coordinate at the point $(-x)$ of the unprotected coast occurs in the point x/p of the protected coast (fig. 14). The inverse will occur on the lee-side of the groyne system. Here a similar shape scour-hole will be formed. Fig. 15 shows the shape and the transport, the formulae are given in appendix, 8.

The corresponding two-line system is rather intricate and still in study.

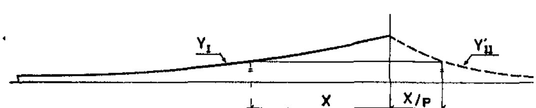

Fig 14

3^o The influence of shape

Even if the coastline would not change because of external causes and even if the influence of stationary transport is not taken into account, a convex coast would erode and a concave coast accrete according to eq. (1):

$$\frac{\partial y}{\partial t} = \frac{q}{D}\,\frac{\partial^2 y}{\partial x^2}$$

In the one-line system the difference between a protected coast and an unprotected coast is a difference in the constant q/D, which differs a factor p^2, according to (8). This means, that a protected coast accretes

---
[1] defined in the appendix

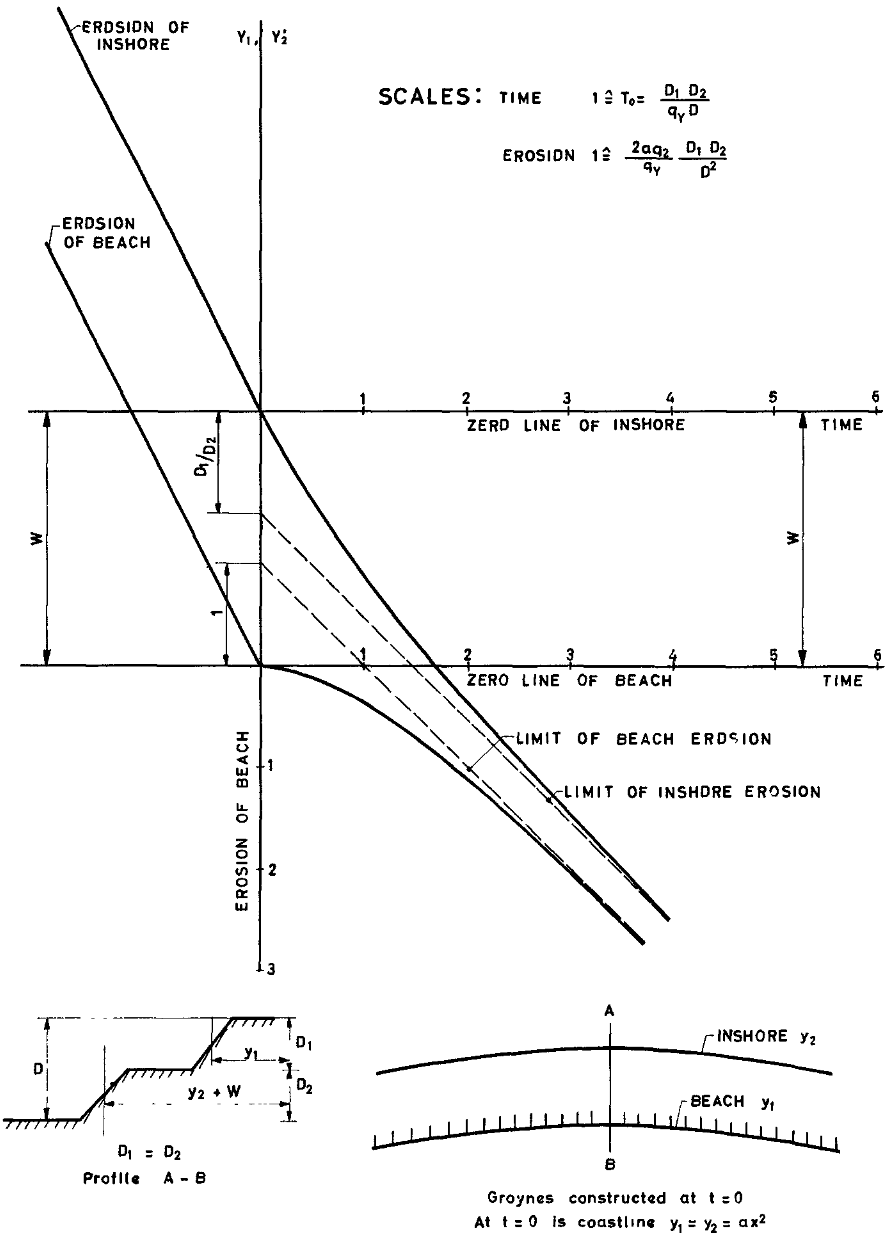

Fig 16 EROSION OF BEACH AND INSHORE ACCORDING TO THE TWO - LINE SYSTEM

(erodes) slower than an unprotected coast with the same curvature. The timescale is p^2 times as large, according to (9) and fig. 10b.

At the boundary between a protected part and an unprotected part edge effects arise, which can be computed graphically with the method of Schmidt [5].

Fig. 16 shows the erosion of a convex coast, according to the two-line system. The formulae are given in appendix, 9.

It is assumed, that before t = 0 beach and inshore have the same parabolic shape and an equilibrium profile. They erode with the same velocity. At time t = 0 groynes are constructed at the beach, so near to each other, that they prevent all transport along the beach. The erosion along the inshore is not stopped, however, the profile becomes too steep and sand moves from beach to inshore.

The erosion of the inshore diminishes and the erosion of the beach begins again. Finally, beach and inshore erode together again, but the profile remains steeper than the equilibrium profile, and the total rate of erosion is less than before, according to the one-line theory.

DISCUSSION

The theory only deals with one aspect.

Other aspects are:

1° the influence of rip-currents near the groynes.
This influence is two-fold: rip-currents transport material from beach to inshore and they cause stream-refraction. These two influences work against each other. The transport of material from the beach causes a scour-hole on the beach and the stream-refraction causes a spit on the beach.
At the moment experiments with dyed water in the prototype are carried out to get an impression of the order of magnitude of these rip-currents. Rip-currents flatten the profile and lower the rate of effect of a groyne-system. Therefore it is very important, that they give the correct transport in models, because otherwise one can find the inverse effect of groynes as in practice.

2° the influence of diffraction on the lee-side of groynes.
The author has the feeling that diffraction does not really change the effect of a groyne system, but only has minor effects in the immediate vicinity.

3° variable wave direction
This causes changing boundary conditions near the groynes.
Most influence will be found near the first groyne of a groyne system, where this will generate short-period moving sandwaves on the beach. These sandwaves have a short wavelength and will decay at a short distance of the groyne. At a long distance of the groynes one only finds the effect of mean wave conditions and no influence of variations. Therefore, also with varying wave conditions, most of what has been said, especially about long-period processes, remains its validity with changing wave conditions.

4° non-linearity in the transport equation.
According to the author, this is mostly of minor importance, except if anywhere the angle of wave incidence along beach or inshore becomes

about 45° or more. In thie case the matter of instability, mentioned in the introduction becomes important. A point where this can occur is in fig. 8 on the inshore, right in front of the groyne.

Of course everyone will be interested in the values of the coastal constants.

The following is not more than a reasonable quees, because serious investigations have not yet been done.

For some parts of the Dutch coast, $q/D \approx 0,4 \times 10^6$ m^3/m depth/year/radian and q_y might be 1 to 10 m/year at a depth D_1 of 3 m.

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APPENDIX

1. THE DERIVATION OF PELNARD-CONSIDÈRE

If x is the main coastal direction and y is in seaward direction, the angle of wave incidence is nearly (taking $\operatorname{arctg} \frac{\partial y}{\partial x} \approx \frac{\partial y}{\partial x}$): $\alpha = \alpha_o - \frac{\partial y}{\partial x}$

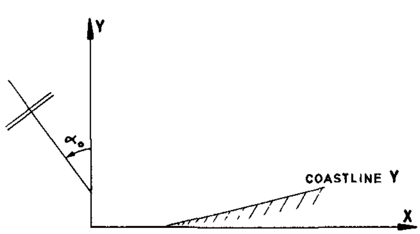


FIG 17

The littoral drift Q is a function of the angle of wave incidence and can be put into a Taylor seriee:

$$Q = Q_o + \frac{dQ}{d\alpha}(\alpha - \alpha_o) + \ldots.$$

in which Q_o denotes the traneport Q if the angle of wave incidence is α_o This gives in linear approximation:

$$Q = Q_o - q\frac{\partial y}{\partial x}$$

in which $q = \frac{dQ}{d\alpha}$ for $\alpha = \alpha_o$.

The equation of continuity says that the sedimentation is equal to the decrease of littoral drift:

$$\frac{dQ}{dx} + D\frac{\partial y}{\partial t} = 0.$$

Substituting Q gives the equation of Pelnard-Considère:

$$\frac{\partial y}{\partial t} = \frac{q}{D}\frac{\partial^2 y}{\partial x^2}.$$

It will be seen, that this derivation remains its validity when Q denotes the mean yearly transport along a coast.

In practice, the transport is zero if the angle of wave incidence is zero. In order to get this correct in the mathematical model, it has sense to choose Q_o less than the transport when the angle of wave incidence is α_o (fig. 18)

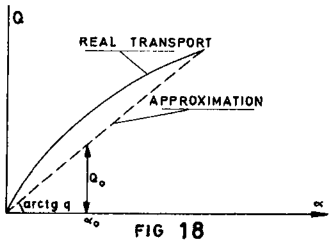


FIG 18

2. DERIVATION OF THE FORMULAE FOR THE TWO LINE SYSTEM

The equations of continuity are:

$$\left.\begin{aligned} -\frac{dQ_1}{dx} - Q_y &= D_1\frac{\partial y_1}{\partial t} \\ -\frac{dQ_2}{dx} + Q_y &= D_2\frac{\partial y_2}{\partial t} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (12)$$

Substituting the dynamical equations (3) and (5) for Q_1, Q_2 and Q_y gives:

$$\left.\begin{aligned} q_1\frac{\partial^2 y_1}{\partial x^2} - q_y(y_1 - y_2) &= D_1\frac{\partial y_1}{\partial t} \\ q_2\frac{\partial^2 y_2}{\partial x^2} - q_y(y_2 - y_1) &= D_2\frac{\partial y_2}{\partial t} \end{aligned}\right\} \quad \ldots\ldots (13)$$

Adding both equations gives:

$$\frac{\partial^2(q_1y_1 + q_2y_2)}{\partial x^2} = \frac{\partial(D_1y_1 + D_2y_2)}{\partial t}$$

which can be written as:

$$\frac{q}{D}\frac{\partial^2 y}{\partial x^2} + \frac{D_1 D_2}{D^2}\left(\frac{q_1}{D_1} - \frac{q_2}{D_2}\right)\frac{\partial^2 (y_1 - y_2)}{\partial x^2} = \frac{\partial y}{\partial t} \quad \ldots\ldots (14),$$

in which $\frac{q}{D} = \frac{q_1 + q_2}{D_1 + D_2}$ and $y = \frac{1}{D}(D_1 y_1 + D_2 y_2)$.

y is the "coastline" of Pelnard-Considère. We will confine ourselvss to to cases where $\frac{q_1}{D_1} = \frac{q_2}{D_2}$, which means, that if bsach and inshore have the same curvature, they fill up equable and the profile does not change.
In this case the second left-hand term of (14) is zero and we have back eq. (1) of Pslnard-Considère.

By dividing the squations (13) by D_1 and D_2 respectively and subtracting ons finds:

$$\frac{q}{D}\frac{\partial^2 y_-}{\partial x^2} - \frac{q_y D}{D_1 D_2} y_- = \frac{\partial y_-}{\partial t} \quad \ldots\ldots\ldots (15)$$

in which $y_- = y_1 - y_2$ .... (16)

Eq. (15) is the equation for the offshore transport, which equals $q_y\, y_-$.
By using the auxiliary variable y_e, equal to:

$$y_e = y_-\, e^{\frac{q_y D}{D_1 D_2} t} \quad \ldots\ldots\ldots\ldots (17)$$

this can bs written as:

$$\frac{q}{D}\frac{\partial^2 y_e}{\partial x^2} = \frac{\partial y_e}{\partial t} \quad \ldots\ldots\ldots\ldots (18)$$

(1) and (18) both represent the diffusion-, warmth-, or conductivity equation, for which many numerical or graphical integration processes are availabls, for instance the method of Schmidt [5].
By substituting ths appropriats boundary conditions for y and y_s, one can find y and y_s at every time and place, from which y_1 and y_2.

Some problems can be solved analytically, of which soms examples will be given.

From (17) it will be seen, that the tims scals of this kind of processes is highly dependent of a reference time T_o:

$$T_o = \frac{D_1 D_2}{q_y D} \quad \ldots\ldots\ldots\ldots (19)$$

3. THE PROBLEM OF FIG. 8. $D_1 = D_2$, $q_1 = q_2$.

Boundary conditions: a $y_1 = y_2 = 0$ for $x = \infty$ and $0 < t < \infty$
b $y_2 = 0$ for $x = 0$ and $0 < t < \infty$

$$\underline{c}\quad \frac{\partial y_1}{\partial x} = \tan\alpha = \frac{Q_{o1}}{q_1} \quad \text{for } x = 0 \text{ and } 0 < t < \infty$$

$$\underline{d}\quad y_1 = y_2 = 0 \qquad \text{for } 0 < x < \infty \text{ and } t = 0$$

First the equations (1) and (15) are mads dimensionless by substituting $x' = x/L_o$, $t' = t/T_o$ and $y_{1,2}' = y_{1,2} \cot\alpha$, in which L_o and T_o are defınsd in (6) and (19) respectively. In ths following the accents will be omitted. Denoting the Laplace transform of y' with $\bar{y}$, the Laplacs transforms of the new equations are for the given boundary conditions:

$$\left.\begin{aligned} \frac{\partial^2 \bar{y}}{\partial x^2} &= s\,\bar{y} \\ \frac{\partial^2 \bar{y}_-}{\partial x^2} - \bar{y}_- &= s\,\bar{y}_- \end{aligned}\right. \qquad \ldots\ldots\ldots\ldots (20)$$

Solving eq. (20) and substituting the boundary conditions give:

$$\left.\begin{aligned} \bar{y} &= \frac{-e^{-x\sqrt{s}}}{s(\sqrt{s} + \sqrt{s+1})} \\ \bar{y}_- &= \frac{-2e^{-x\sqrt{s+1}}}{s(\sqrt{s} + \sqrt{s+1})} \end{aligned}\right\} \qquad \ldots\ldots (21)$$

The functions of (21) can be splitted up into fractions. Then terms arise as $\frac{e^{-x\sqrt{s+1}}}{\sqrt{s}}$, which can be developed into series of the kind

$$s^{-n+\frac{1}{2}}\; s^{-x\sqrt{s}} ,$$

of which the inverse are integrals of the error function. From this, one finds the final solution. In the following, the coordinates ars given for $x > 0$ (the eroded part). For $x < 0$, there is antisymmetry. The solution is:

$$y_{1,2} = \frac{1}{2\sqrt{\pi t}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n-\frac{1}{2}} \cdot \frac{t^n}{n!}\, Y_{2n-1}\left(\frac{x}{2\sqrt{t}}\right) \mp \frac{e^{-t}}{\sqrt{\pi t}} \sum_{n=1}^{\infty} \frac{t^n}{n!}\, Y_{2n-1}$$

$$\left(\frac{x}{2\sqrt{t}}\right) \pm \tfrac{1}{2}\, s^{-x} \operatorname{erfc}\left(-\sqrt{t} + \frac{x}{2\sqrt{t}}\right) \mp \tfrac{1}{2}\, e^{x} \operatorname{erfc}\left(\sqrt{t} + \frac{x}{2\sqrt{t}}\right) \quad \ldots\ldots (22)$$

(22) <u>is in abridgsd notation</u>: $x = x/L_o$ (cf (6)), $t = t/T_o$ (cf (19)) and $y_{1,2} = y_{1,2} \cot\alpha$.

The upper sign gives y_1, the lower sign y_2.

In (22), the meaning of Y_n is: $Y_n(x) = 2^n\, \Gamma(\frac{n}{2} + 1)\, i^n \operatorname{srfc} x$.

i^n erfc x has nothing to do with complex numbers, but denotes the n^{th} integral of the complementary error function.

Thus $\Upsilon_{-1}(x) = e^{-x^2}$. The functions Υ_n are shown in fig. 19. We refer to [6], also for tabels and recurrence formulae (page 300 till 318).

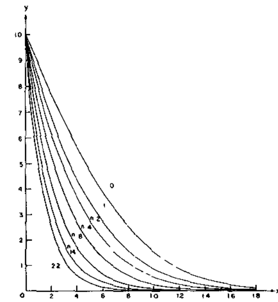


Fig. 19 Repeated Integrals of the Error Function

The erosion of the coast at x = 0 is:

$$[y_1]_{x=0} = 2 \tan\alpha \left(\frac{1 - e^{-t/T_o}}{\sqrt{\pi t/T_o}} - \operatorname{erf}\sqrt{\frac{t}{T_o}} \right) \quad . . (23)$$

in which $\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_o^{\infty} e^{-x^2} dx$ and $\operatorname{erfc} x = 1 - \operatorname{erf} x$.

4. STATIONARY CASE WITH GROYNES: FIG. 9.

In the stationary case $\frac{\partial y_1}{\partial t}$ and $\frac{\partial y_2}{\partial t}$ in (13) are zero.

This gives the general solution:

$$\left.\begin{aligned} y_1 &= Ae^{x/L_o} + Be^{-x/L_o} + Cx + E \\ y_2 &= -\frac{q_1}{q_2}\left(Ae^{x/L_o} + Be^{-x/L_o}\right) + Cx + E \end{aligned}\right\} \quad . . . . . . (24)$$

Boundary conditions for fig. 9:

a antisymmetry $A = -B$

b $y_1 = y_2 = 0$ for $x = 0$: $E = 0$

c $Q_1 = 0$ for $x = L$: $\left(\frac{\partial y_1}{\partial x}\right)_{x=L} = \frac{Q_{o1}}{q_1} = \tan\alpha$

d $y_2 = 0$ for $x = L$

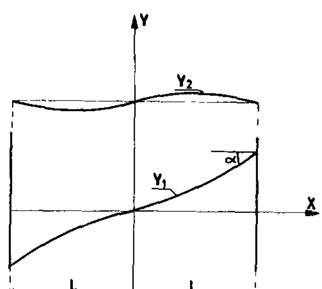


FIG 20

Result:

$$\left.\begin{aligned} y_1 &= \tan\alpha \frac{x + \frac{q_2}{q_1} L \frac{\sinh x/L_o}{\sinh L/L_o}}{1 + \frac{q_2}{q_1} \frac{L/L_o}{\operatorname{tgh} L/L_o}} \\ y_2 &= \tan\alpha \frac{x - L \frac{\sinh x/L_o}{\sinh L/L_o}}{1 + \frac{q_2}{q_1} \frac{L/L_o}{\operatorname{tgh} L/L_o}} \end{aligned}\right\} \quad . . (25)$$

5. STATIONARY CASE WITH ONE GROYNE: FIG. 8d AND FIG. 9.

This case can be found from the former by taking $L = \infty$ and $x' = L + x$. The coordinates of the eroded part are:

$$\left.\begin{aligned} y_1 &= - L_o \tan\alpha \left(\frac{q_1}{q_2} + e^{-x/L_o}\right) \\ y_2 &= - L_o \tan\alpha \frac{q_1}{q_2} \left(1 - e^{-x/L_o}\right) \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (26)$$

6. LITTORAL DRIFT ON A PROTECTED AREA.

In order to return from the two-line system to the one-line eystem, we look for a transport formula for a protected area of the kind:

$$Q = Q_o' - q' \frac{\partial y'}{\partial x} \quad \ldots\ldots\ldots\ldots \quad (27)$$

in which y' gives the overall coastal direction of a protected area (fig. 21) and Q_o' denotes the transport if the overall coastal direction is parallel to the x-axis, as in fig. 9. By applying the transport formula (5) to (25) we find ($\tan\alpha = \frac{Q_{o1}}{q_1}$):

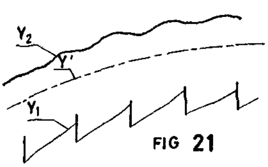


FIG 21

$$Q_o' = Q_{o2} + Q_{o1} \frac{1 - \dfrac{\text{tgh } L/L_o}{L/L_o}}{1 + \dfrac{q_1}{q_2} \dfrac{\text{tgh } L/L_o}{L/L_o}} \quad \ldots \quad (28)$$

If the overall coastal direction $\frac{\partial y'}{\partial x}$ hae a certain value, the transport changes because Q_{o1} and Q_{o2} in (28) have to be replaced by

$$Q_{o1} - q_1 \frac{\partial y'}{\partial x} \quad \text{and} \quad Q_{o2} - q_2 \frac{\partial y'}{\partial x}.$$

From this one can find q' and p^2:

$$p^2 = \frac{q_1 + q_2}{q'} = 1 + \frac{q_1}{q_2} \frac{\text{tgh } L/L_o}{L/L_o} \quad \ldots\ldots \quad (29)$$

7. THE SANDWAVES OF FIG. 13.

Differential equations: for the unprotected part (1) and (15), for the protected part (13) with $q_1 = 0$. Assumed is no littoral drift along the protected beach.

In this harmonical case, the derivates to t like $\frac{\partial y}{\partial t}$ can be replaced by $i\omega y$, by example

Boundary conditions:

a $y_1 = y_2 = 0$ at $x = \infty$

b $y_1 = y_2 = e^{-K_+'x} \cos(\omega t - K_+'x)$ at $x = -\infty$, in which

$K_+' = \sqrt{\frac{\omega D}{2q}}$

c $\frac{\partial y_1}{\partial x} = 0$ for $x = -0$

d y_2 continuous and differentiable at $x = 0$

Solution:

See adjacent page. The "offshore transport wave" is the solution of the equation (15) for y_-, the incoming and reflected wave are solutions of (1). The solution is highly dependent of the value of ωT_o. This value defines the short-period waves ($\omega T_o << 1$) and large-period waves ($\omega T_o >> 1$).

8. INFLUENCE OF OBLIQUE WAVES INCIDENCE (FIG. 15). FORMULAE.

Accretion at unprotected coast (branch y_I in fig. 15b):

$$y_I = \frac{1}{\sqrt{\pi}} \frac{p-1}{p} \frac{Q_{o1}}{q_1} \left\{ \sqrt{\frac{4qt}{D}} e^{-x^2 D/4qt} + x\sqrt{\pi} + x\sqrt{\pi} \operatorname{erf}\left(x\sqrt{\frac{D}{4qt}}\right) \right\}$$

Littoral drift Q at unprotected coast (branch y_I)

$$Q = Q_o - \frac{p-1}{p} . q \frac{Q_{o1}}{q_1} \left\{ 1 + \operatorname{erf}\left(x\sqrt{\frac{D}{4qt}}\right) \right\}$$

Accretion at protected coast (branch y'_{II} in fig. 15b):

$$y'_{II} = y_I(-px)$$

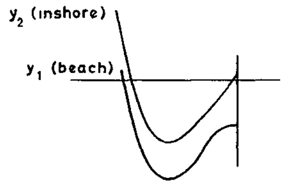
Littoral drift Q at protected coast (branch y'_{II} in fig. 15b)

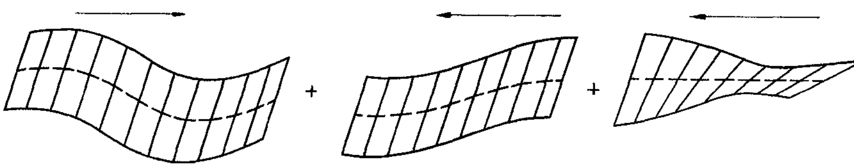
$$Q = Q_o - \frac{p-1}{p} q . \frac{Q_{o1}}{q_1} \left\{ 1 + \frac{1}{p} \operatorname{erf}\left(px\sqrt{\frac{D}{4qt}}\right) \right\}$$

in which: $\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_o^x e^{-t^2} dt.$

For the scour hole (branch y'_{III} and y_{IV}), the formulae are similar.

UNPROTECTED COAST



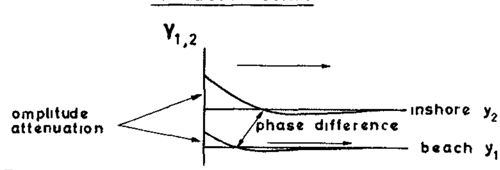


Incoming wave (equilibrium profile) + Reflected wave (equilibrium profile) + Offshore transport wave (no equilibrium profile)

$$y_1 = Re\left[\left(e^{-k_+x} + \frac{C_2}{C_1} e^{+k_+x} + \frac{D_2}{D}\frac{C_3}{C_1} e^{k_-x}\right)e^{i\omega t}\right]$$

$$y_2 = Re\left[\left(e^{-k_+x} + \frac{C_2}{C_1} e^{+k_+x} - \frac{D_1}{D}\frac{C_3}{C_1} e^{k_-x}\right)e^{i\omega t}\right]$$

PROTECTED COAST



$$y_1 = Re\left[\frac{e^{-kx+i\omega t}}{C_1\left(1+\frac{D_1}{qy}\, i\omega\right)}\right]$$

$$y_2 = Re\left[\frac{1}{C_1}\, e^{-kx+i\omega t}\right]$$

IN WHICH

$$k_+ = \sqrt{\frac{D}{q} i\omega} = \sqrt{\frac{\omega D}{2q}} + i\sqrt{\frac{\omega D}{2q}} = \frac{1}{L_o}\sqrt{i\omega T_o}$$

$$k_- = \sqrt{\frac{D}{q}\left(\frac{1}{T_o} + i\omega\right)} = \frac{1}{L_o}\sqrt{1+i\omega T_o}$$

$$k = \sqrt{i\omega\frac{D}{q_2}\left(\frac{1+i\omega T_o}{1+\frac{D_1}{qy} i\omega}\right)} = \frac{1}{L_o}\sqrt{\frac{1+i\omega T_o}{1+\frac{D_2}{i\omega D}}}$$

$$C_1 = \frac{1}{2} + \frac{D_1}{2D}\frac{k}{k_-} + \frac{D_2}{2D}\frac{k}{k_+}$$

$$C_2 = \frac{1}{2} + \frac{D_1}{2D}\frac{k}{k_-} - \frac{D_2}{2D}\frac{k}{k_+}$$

$$C_3 = \frac{k}{k_-}$$

9. EROSION OF BEACH AND INSHORE ACCORDING TO TWO-LINE SYSTEM (fig. 16). FORMULAE.

Differential equations for $t > 0$: (13) with $q_1 = 0$.

Results:

$t < 0$ (groynes not yet constructed):

$$y_{1,2} = ax^2 + \frac{2aq_2}{q_y} \cdot \frac{D_1 D_2}{D^2} \cdot \frac{D}{D_2} \cdot \frac{t}{T_o}$$

$t > 0$ (groynes constructed):

$$y_1 = ax^2 + \frac{2aq_2}{q_y} \cdot \frac{D_1 D_2}{D^2} \cdot \left(e^{-t/T_o} - 1 + \frac{t}{T_o} \right)$$

$$y_2 = ax^2 + \frac{2aq_2}{q_y} \cdot \frac{D_1 D_2}{D^2} \left\{ e^{-t/T_o} - 1 + \frac{t}{T_o} + \frac{D}{D_2} \left(1 - e^{-t/T_o}\right) \right\}$$