CHAPTER 30

INTERACTION BETWEEN WAVES AND CURRENTS

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ABSTRACT

This paper presents some theoretical results of a general study of the interaction between surface gravity waves and a steady current Assuming irrotational flow and a second order Stokes wave motion, the main objects of the paper have been

a To present a simple graphical method for the computation of the wave length in a current field

b To introduce the concept of the mean energy level for a periodic wave motion with a steady current superimposed

c To utilize this for the calculation of the "current-wave set-down" for a two-dimensional motion with a constant discharge over a gently sloping bottom

d To present a complete set of conservation equations for the case considered under point c

e To present graphs and tables for the variation in length and height of wave for the case considered under point c

No experimental results are presented

1 INTRODUCTION

Aspects of non-linear interaction between gravity waves and a current motion have received increasing attention during the last 10 years The works by Longuet-Higgins and Stewart [5], [6], and Whitham [8] are already classics The mechanism is intimately connected with the so-called radiation stress

However, there are still basic features related to this problem that are not widely known One of these is the important concept of the mean energy level for a periodic, irrotational flow (For a pure wave motion the mean energy level was introduced by Lundgren [7]) One of the objects of the present study is to show how this concept can yield the "current-wave set-down" in a simple way, and to demonstrate how it affects the conservation equations for a two-dimensional current-wave motion propa-

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gating over a gently sloping bed (see Fig 5-A) The conservation equations for wave crests and energy are solved to yield graphs and tables for the variation in wave length and wave height for this situation The wave length graphs and tables can be used, though, for any angle between wave front and current direction provided the current velocity is replaced by its component in the direction of the wave orthogonal A graphical method for the determination of the wave length in a homogeneous current field will also be introduced This method permits a simple discussion of various domains where different solutions are applicable

Energy losses are neglected in this paper The current velocity is assumed to be steady and constant over the water depth, and only surface waves are considered A second order Stokes expansion is used in the calculations

2 NOTATION

c	(m/s)	Wave celerity
c_g	(m/s)	Wave group celerity
D	(m)	"Geometrical water depth" (= $h + \Delta h$, see Fig 4-A)
E	(Nm/m^2)	Mean specific wave energy (= $1/8\ \gamma\ H^2$)
$\overline{E}_f$	(Nm/m/s)	Mean energy flux per unit width
F_w	(N/m)	Radiation stress
g	(m/s^2)	Acceleration due to gravity
H	(m)	Wave height
h	(m)	"Physical water depth" (see Fig 4-A)
Δh	(m)	"Current-wave set-down" (see Fig 4-A)
k	(m^{-1})	Wave number (= $2\ \pi/L$)
L	(m)	Wave length
MEL		Mean energy level (see Fig 4-A)
MWL		Mean water level (see Fig 4-A)
n	(dim less)	c_{gr}/c_r
p	(N/m^2)	Pressure
q	(m^3/m/s)	Discharge per unit width (= $h\ U$)
q^*	(dim less)	Dimensionless discharge per unit width (= $q/(c_o\ L_o)$)
s_v	(dim less)	Slope of straight line in Fig 3-B
T	(s)	Wave period
t	(s)	Time
U	(m/s)	Current velocity (positive in direction of c)
u	(m/s)	Horizontal particle velocity
w	(m/s)	Vertical particle velocity
x	(m)	Horizontal co-ordinate

z	(m)	Vertical co-ordinate
β	(rad)	Angle between wave front and current (interval 0 to $\pi/2$)
γ	(N/m^3)	Specific weight of water
η	(m)	Water surface elevation
ρ	(kg/m^3)	Density of water
ϕ	(m^2/s)	Velocity potential ($u = -\partial\phi/\partial x$, etc)
ω	(s^{-1})	Angular frequency ($= 2\pi/T$)
$\overline{}$		Mean value sign

Suffix a denotes "absolute"

Suffix b denotes "bottom"

Suffix o denotes "deep water with U equal to zero"

Suffix r denotes "relative" (to the current)

Suffix t denotes partial differentiation with respect to time t

3 WAVES IN A HOMOGENEOUS CURRENT FIELD

Consider a region of constant water depth h, where the absolute wave period T_a and the current velocity U are given

WAVE LENGTHS

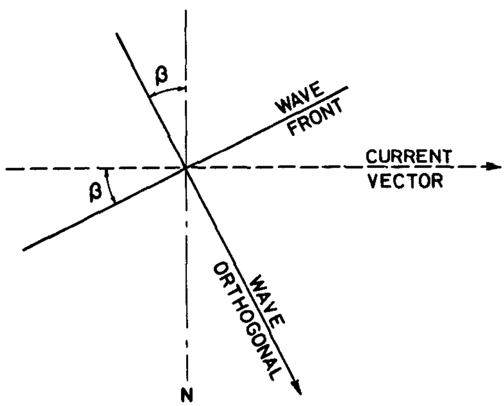

Fig 3-A Definition of the angle β

If **β** is the angle between wave front and direction of current (Fig 3-A), the following four equations are available to determine the unknowns the absolute and relative wave celerities c_a and c_r, the wave length L, and the relative period T_r

$$L = c_a T_a \quad (3\ 1) \qquad\qquad L = c_r T_r \quad (3\ 2)$$

$$c_a = c_r + U \sin\beta \quad (3\ 3) \qquad\qquad c_r = \sqrt{\frac{gL}{2\pi}\tanh kh} \quad (3\ 4)$$

where k is the wave number The current velocity U is considered positive to the same side of the normal N as is indicated by the positive direction of the wave orthogonal β lies, by definition, in the interval $0 \le \beta \le \pi/2$ (3 1) to (3 4) now yield for determination of L

$$\boxed{\sqrt{\frac{h}{L}\tanh kh} = \sqrt{\frac{h}{L_o}}\left[1 - \frac{U \sin\beta\, T_a}{h}\;\frac{h}{L}\right]} \tag{3 5}$$

where L_o is defined as

$$L_o \equiv \frac{g}{2\pi} T_a^2 \tag{3 6}$$

i e the deep water wave length with U equal to zero

(3 5) determines implicitly h/L as a function of h/L_o and $U \sin\beta\, T_a/h$, and the equation can be solved by iteration (see chapter 6) It is not evident, however, whether a solution to (3 5) is unique - or whether it exists at all A graphical representation of (3 5) reveals this Writing (3 5) as

$$F(h/L) = G(h/L) \tag{3 7}$$

it appears that graphically h/L can be found as the value of the abscissa for the intersection of the F- and G-curves, see Fig 3-B The F-curve is unique, and values can be extracted from a "conventional" wave table

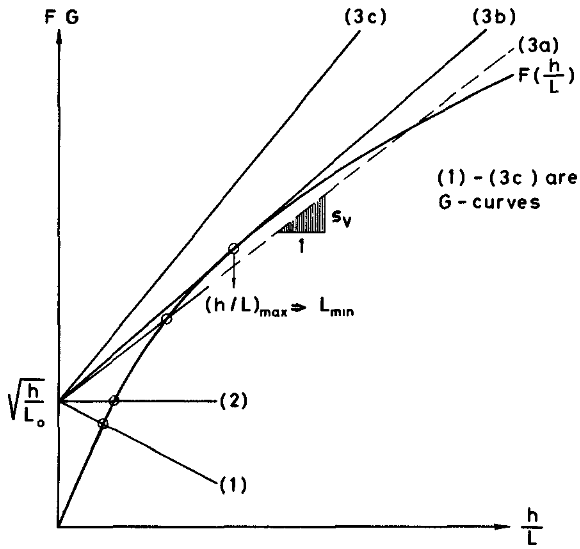

Fig 3-B Graphical determination of L (schematical)

The G-curves are straight lines through the point $(0, \sqrt{h/L_o})$, with a slope of 1 horizontal to s_v vertical, where

$$s_v = - \sqrt{\frac{h}{L_o}} \quad \frac{U \sin \beta \, T_a}{h} = - \sqrt{2\pi} \, \frac{U \sin \beta}{\sqrt{g\,h}} = - \frac{U \sin \beta}{c_o \, \sqrt{h/L_o}} \tag{3 8}$$

Here c_o is defined as

$$c_o \equiv \frac{g}{2\pi} T_a \tag{3 9}$$

i e the deep water wave celerity with U equal to zero

Fig 3-B now enables us to discuss the solution(s) to (3 5) in detail (h, T_a and β are assumed constant, so s_v becomes directly proportional to $-U$) For U positive we have case (1), and a unique solution exists for L This also applies to case (2), where U is zero For a negative current (case (3)), there are three possibilities In case (3a) there are theoretically two solutions for L If it is argued that the variation of L must, for physical reasons, be continuous for $U \to 0$, only one solution is possible This means that the larger value for L should be chosen, as shown in the figure In case (3b) the G-curve is a tangent to the F-curve, and U has attained its minimum value (always negative), as has the wave length, $L = L_{min}$ This case of "maximum counter-current" corresponds to

$$\sqrt{\frac{h}{L} \tanh kh} = \frac{1}{1-n} \sqrt{\frac{h}{L_o}} \tag{3 10}$$

with

$$n = \frac{c_{gr}}{c_r} = \frac{1}{2} \left[1 + \frac{2kh}{\sinh 2kh} \right] \tag{3 11}$$

c_{gr} being the relative wave group celerity From (3 5) and (3 10) we then find that the absolute wave group celerity is zero for $L = L_{min}$

$$c_{ga} = c_{gr} + U \sin \beta = 0 \tag{3 12}$$

(It will be shown in chapter 5 that in the case of a two-dimensional wave motion ($\sin \beta = 1$) progressing against a current over a gently sloping bed, the wave height reaches infinity at the water depth at which (3 12) is fulfilled) In case (3c) there is no solution for L It is readily seen from Fig 3-B that - other things being equal - a positive current "lengthens" the waves whereas a negative current "shortens" them

A full account of the graphical method, including its practical application, is found in [4] Solutions to (3 5) (with $\sin \beta = 1$) are presented in chapter 6

For deep-water waves (for instance $h/L > 0\ 5$) (3 5) can be solved explicitly

$$L = \frac{1}{4} L_o \left[1 + \sqrt{1 + 4 \, \frac{U \sin \beta}{c_o}} \right]^2 \tag{3 13}$$

MOMENTUM AND PRESSURE FORCES ("STRESSES")

In this paper wave heights will be calculated only for two-dimensional flow Stresses and energy fluxes will therefore be calculated only for the special case $\beta = \pi/2$ Taking mean values over period T_a, the following total force (normal "stress") over depth h and per unit width is found in a section at right angles to the current vector (and so in this case also to the direction of wave propagation)

$$\sigma = \frac{1}{2} \gamma h^2 + \rho h U^2 + F_w \tag{3 14}$$

In this expression, F_w is the radiation stress

$$F_w = \frac{1}{16} \gamma H^2 \left[1 + 2 \frac{2kh}{\sinh 2kh}\right] = \left(2 \frac{c_{gr}}{c_r} - \frac{1}{2}\right) E \tag{3 15}$$

where E is the mean specific wave energy The first two terms in (3 14) are readily discernible

ENERGY FLUX

The mean energy flux over depth h and per unit width depends upon the zero level for potential energy, since a net current is present Taking the mean water level as level we find for $\beta = \pi/2$ in a section at right angles to the current vector

$$\overline{E}_{f,MWL} = \frac{1}{2} \rho h U^3 + (U + c_{gr}) E + U F_w \tag{3 16}$$

This expression has already been given by Longuet-Higgins and Stewart [5]

The paradox arises that although (3 16) may be interpreted physically, it is of no direct use for the calculation of the wave height variation on a non-uniform current (because of the variation in the mean water level) This will be elucidated in chapters 4 and 5

In these and the subsequent calculations the following restriction on a Stokes wave must be borne in mind For long waves the "Ursell parameter" (HL^2/h^3) must not exceed a certain number

4 THE MEAN ENERGY LEVEL

As a computation of the wave height variation requires a knowledge of the mean energy flux at any station corresponding to the same horizontal datum, it is imperative to determine the variation in the MWL over an "arbitrary" (here gently sloping) bottom In other words, the "set-down" of the MWL is wanted The discussion is confined to periodic, irrotational flows These two conditions are written in the frames in Fig 4-A (For simplicity we consider two-dimensional flow only)

In the figure we have a sloping, known bottom and a sloping, but unknown, MWL And we have a horizontal datum, z = 0 The mean water level is given by

$$\overline{\eta} = z_b + h \tag{4 1}$$

so the determination of the MWL calls in fact for a definition of the water depth h In the present case, which is a second order theory in wave height, but is also a "zero-order bottom-slope theory", it is natural to define the water depth from the mean bottom pressure

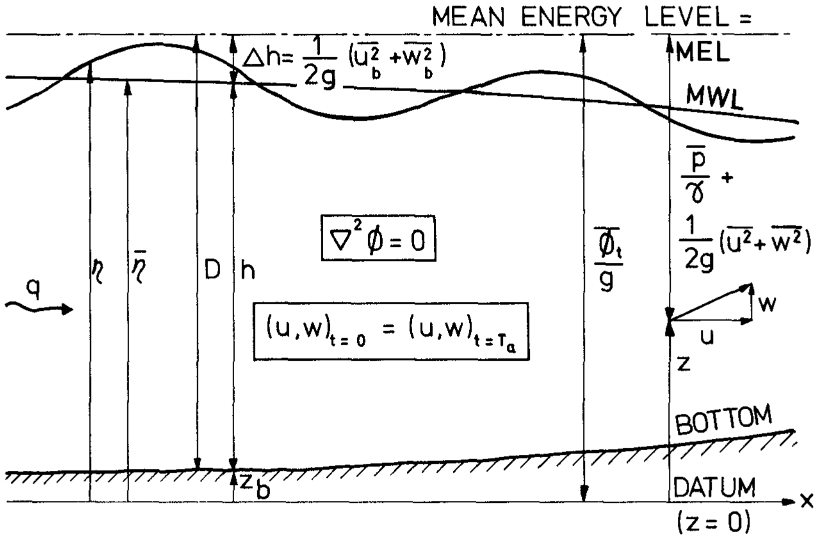

Fig 4-A Illustration of the "set-down" (schematical)

$$h = \frac{1}{\gamma}\overline{p_b} \tag{4 2}$$

We still have not utilized the fact, that the flow is irrotational and periodic The Bernoulli equation - presented with mean values over period T_a - is therefore introduced

$$z + \frac{\overline{p}}{\gamma} + \frac{1}{2g}[\overline{u^2} + \overline{w^2}] = \frac{1}{g}\overline{\phi_t} = \text{const} \tag{4 3}$$

(ϕ is defined, so that $u_1 = -\partial\phi/\partial x_1$) The fact that the right hand side is independent of x and z is easily seen from

$$\frac{\partial}{\partial x_1}\left(\overline{\frac{\partial\phi}{\partial t}}\right) = \overline{\frac{\partial}{\partial t}\left(\frac{\partial\phi}{\partial x_1}\right)} = 0 \tag{4 4}$$

since the flow is periodic

This means that a constant horizontal level is now found which is inherently connected with the flow itself - and independent of the arbitrary datum From the datum we can mark $\overline{\phi_t}/g$ vertically, and we will arrive at the same level for any x This level will be called the <u>mean energy level</u> (MEL), since it contains the three terms that are analogous to conventional steady hydraulics Combining (4 1), (4 2) and (4 3) (at the bottom) we find the MWL

$$\overline{\eta} + \frac{1}{2g}(\overline{u_b^2} + \overline{w_b^2}) = \frac{1}{g}\overline{\phi_t} = \text{const} \tag{4 5}$$

Since the constant in (4 5) is the distance from the datum up to the MEL, the set-down of the MWL - as defined by (4 1) and (4 2) - is

$$\boxed{\Delta h \equiv D - h = \frac{1}{2g} (\overline{u_b^2} + \overline{w_b^2})} \tag{4 6}$$

(D could be termed the "geometrical depth", in contrast to h, which is a "physical depth") Assuming the discharge q to be finite anywhere, it appears from (4 6) that the set-down Δh is in fact the depression of the MWL at an arbitrary depth below the MWL at infinite depth And this depression equals the mean velocity head at the bottom)

The actual analytical expression for the set-down in our two-dimensional combination of a steady current and a wave motion is found as follows It is assumed that locally, we can use the velocity potential corresponding to a horizontal bottom This is - correct to second order - with MWL as datum

$$\phi = - U x + \frac{Hc_r}{2} \frac{\cosh k(z + h)}{\sinh kh} \sin (\omega_a t - kx) + \frac{3}{32} \frac{c_r}{k} (k H)^2 \frac{\cosh 2k(z + h)}{\sinh^4 kh} \sin 2(\omega_a t - kx) + \frac{g H^2}{8 h c_r} x + \left(\frac{1}{2} U^2 + \frac{g H^2}{16 h} \frac{2kh}{\sinh 2kh} - \frac{g H^2}{8 h} \frac{U}{c_r}\right) t \tag{4 7}$$

(4 7) assumes $\overline{\eta} = 0$, so (4 5) - (4 7) yield directly

$$\boxed{\Delta h = \frac{U^2}{2g} + \frac{H^2}{16 h} \frac{2kh}{\sinh 2kh} - \frac{H^2}{8 h} \frac{U}{c_r}} \tag{4 8}$$

for two-dimensional flow It will be observed in (4 8) that, in addition to the "current set-down" (first term) and the "wave set-down" (second term, see [7] and [1]), an interaction term appears (This last term was unfortunately missing from two previous publications from our laboratory Progress Report No 6 and a loose enclosure in Progress Report No 7 This error was corrected in [3] p 20, where some other Corrigenda were also presented) Note that although (4 8) contains a negative term if U is positive, Δh can never become negative, according to (4 6)

It should be added that the set-down for a pure wave motion has been measured in a wave channel by Bowen et al [1] They found that the theory predicts the set-down outside the surf zone very well (They also found the set-up inside the surf zone)

5 TWO-DIMENSIONAL WAVE TRANSFORMATION

The two-dimensional current-wave system, which now will be considered in some detail, is shown schematically in Fig 5-A The five main unknown quantities that we want to calculate are The wave length L, the current velocity U, the set-down Δh, the "physical water depth" h and the wave height H (c_a, c_r and T_r can hereafter be found from (3 1) - (3 4), with $\sin \beta = 1$) We assume that we know the absolute wave period T_a, the discharge q, the bottom topography D = D(x), and the deep-water wave height H_o (where the current is zero)

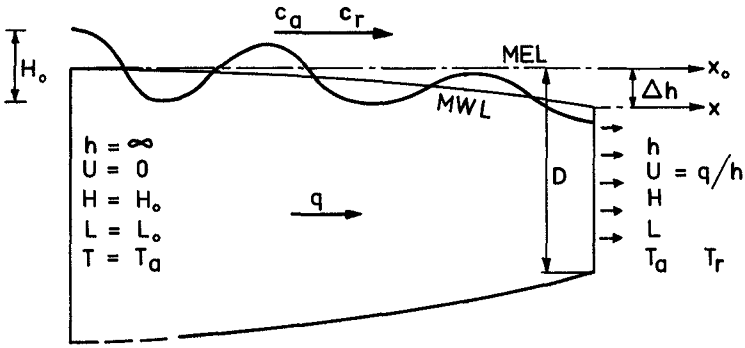

Fig 5-A Definition sketch for wave transformation

CONSERVATION EQUATIONS

To solve the problem, we must set up all the relevant conservation equations Only the energy concept presents us with some difficulty in this connection As pointed out in chapter 3, the expression for the mean energy flux with MWL as zero level cannot be used in a non-uniform flow since this level is constantly changing However, the existence of the MEL, which was found in the preceding chapter to be independent of the horizontal coordinate(s), overcomes this difficulty

In accordance with Fig 5-A we find the following expression for the mean energy flux with MEL as reference

$$\bar{E}_{f,MEL} = \bar{E}_{f,MWL} - (\gamma\ h\ \Delta h)\ U \tag{5 1}$$

$$= (1 + \frac{U}{c_r})(U + c_{gr})\ E \tag{5 2}$$

using (3 16) and (4 8) (An erroneous expression for $\bar{E}_{f,MEL}$ was presented in some earlier publications from this laboratory, see the comments to (4 8)) As, according to our assumptions, (5 2) must be constant, we can now write down the complete set of conservation equations

Wave crests $\omega_a = \omega_r + k\ U$	(5 3)
Mass $\frac{d}{dx}(h\ U) = 0$	(5 4)
Momentum $\frac{dF_w}{dx} + \frac{d}{dx}(\rho\ h\ U^2) - \gamma\ h\ \frac{d(\Delta h)}{dx} = 0$	(5 5)
Energy $\frac{d}{dx}\left[\left(1 + \frac{U}{c_r}\right)\left(U + c_{gr}\right)E\right] = 0$	(5 6)
Bottom topography $h + \Delta h = D$	(5 7)

(5 3) is the conventional way of expressing wave crest conservation In order to get to a practical formula, the expression for the relative wave celerity (3 4) must be used The mass and momentum equations have a direct physical interpretation and require no further comment ($-d(\Delta h)/dx$ is the slope of the MWL) The energy equation is less obvious It should be noted, though, that (5 6) is a special case of Garrett's adiabatic invariant expression,[2] It is directly seen from (5 6), that H reaches infinity at the water depth at which the absolute wave group celerity is zero It is interesting to recognize this "dynamical" limit as the "kinematical" limit (found in chapter 3), yielding the minimum wave length The last conservation equation may seem a trifle sophisticated, if not superfluous This is in a way true for a pure wave motion Our degree of approximation does not permit a discrimination between D and h for the determination of L, H and Δh for this case However, when a current is present, the first term in (4 8) is by definition a zero-order term, and adjustment should be made for this, as described later

PRACTICAL EQUATIONS

The conclusion of the foregoing discussion is that although the conservation equations are fundamental, they are not applicable for direct calculations A new set of practical equations are therefore

$$\sqrt{\frac{h}{L}\tanh kh} = \sqrt{\frac{h}{L_o}}\left[1 - q^* \frac{h/L}{(h/L_o)^2}\right] \tag{5 8}$$

$$h\,U = q \tag{5 9}$$

$$\Delta h = \frac{U^2}{2g} + \frac{H^2}{16\,h}\,\frac{2kh}{\sinh 2kh} - \frac{H^2}{8\,h}\frac{U}{c_r} \tag{5 10}$$

$$\frac{H}{H_o} = \left[2\,\frac{c_r}{c_o}\left(\frac{c_{gr}}{c_r} + \frac{q^*}{(c_r/c_o)(h/L_o)}\right)\left(1 + \frac{q^*}{(c_r/c_o)(h/L_o)}\right)\right]^{-\frac{1}{2}} \tag{5 11}$$

$$h + \Delta h = D \tag{5 12}$$

with

$$q^* \equiv \frac{q}{c_o\,L_o} \quad (5\ 13) \qquad \text{and} \qquad \frac{c_r}{c_o} = \sqrt{\frac{h/L_o}{h/L}\tanh kh} \quad (5\ 14)$$

(5 8) is simply a rewriting of (3 5) with $\sin\beta = 1$ (Now assumed to be valid for a gradually varying water depth) (5 9) and (5 11) are (5 4) and (5 6) integrated The momentum equation is replaced by the Δh-expression (4 8) (which in fact can be deduced directly from the conservation equations)

The equations are solved as follows for given values of D, q, T_a and H_o (not all combinations are admissable) First h and U are found from (5 9) and (5 12), with $\Delta h = U^2/2g$ and subcritical flow assumed L_o, c_o and q^* are calculated from (3 6), (3 9) and (5 13) Next, L is found from (5 8), either by iteration, or graphically, as explained in chapter 3 (with $s_v = -\,q^*/(h/L_o)^{3/2}$), or read from Fig 6-A, or interpolated in Table 6-a or 6-b H is calculated from (5 11), (3 11) and (5 14), or read from Fig 6-B A new value of Δh is then calculated from (5 10), (5 9) and (5 12) hereafter yield the final value of U, which should be used to correct the first term in (5 10), to give the final value of Δh (and h) In this way we should have taken account of all terms $O(H^2)$ (The calculations can of course be repeated in an iterative manner to give "full numerical reciprocity" between D and h It should be borne in mind, though, that the increase in accuracy is formal) If h were given (measured), instead of D, the calculations and considerations are straightforward

6 TABLES AND GRAPHS FOR L AND H

WAVE LENGTHS

(5 8) was solved numerically by a Newton-Raphson iteration method on a digital computer The results are plotted in Fig 6-A ("constant discharge curves")

For q^* positive, the h/L_o - limits have been so chosen that the Froude number is smaller than one, and h/L goes up to about 0 5 Outside the latter limit, deep-water expressions will normally suffice Here (3 13) gives (with $\sin\beta = 1$)

$$\frac{L}{L_o} = \frac{1}{4}\left[1 + \sqrt{1 + 4\frac{q^*}{h/L_o}}\right]^2 \qquad (6\ 1)$$

(The former limit is of course unrealistic Near such a high Froude number the curvature of the mean water surface becomes so large that the assumptions for our theory become invalid) The straight line corresponding to h/L equal to 0 05 indicate that shallow-water conditions are not "typical" for the higher q^*-values The shallow-water wave length is found from (5 8)

$$\frac{L}{L_o} = \sqrt{\frac{2\pi\ h}{L_o}} + \frac{q^*}{h/L_o} \qquad (6\ 2)$$

Of particular interest is the existence of maxima and minima for a number of positive q^*-values This is due to the fact that the steady increase in current velocity - as the depth decreases - tends to lengthen the waves, while the decrease in depth tends, in itself, to have the opposite effect The possibility of horizontal tangents can also be seen directly from (3 11) and the following expression, which is valid for constant values of T_a and q

$$\frac{dL}{dh} = \frac{L}{2\ h}\ \frac{(2c_{gr} - c_r) - 2\ U}{c_{gr} + U} \qquad (6\ 3)$$

For q^* negative, the upper limit for h/L_o was chosen (arbitrarily) as 0 61 The lower limit was chosen so that the wave length L comes as close to L_{min} as the selected values of h/L_o allow (see Table 6-b) In the limit $L = L_{min}$ (see (3 10) and the pertaining discussion) the numerical value of the Froude number will be smaller than one, see [4] It is immediately obvious that for q^* smaller than - 0 05 (appr) deep-water conditions prevail, and (6 1) can be used It will also be seen that in the case of a negative discharge we have the same trend as for no current L decreases monotonously with decreasing depth

In Table 6-a, numerical values of L/L_o are presented for $|q^*| \leq 0\ 02$ (It should finally be mentioned that Fig 6-A and Tables 6-a and 6-b can also be applied in the three-dimensional case, Fig 3-A Here we take as "current velocity" in (5 13) the component in the direction of the wave orthogonal)

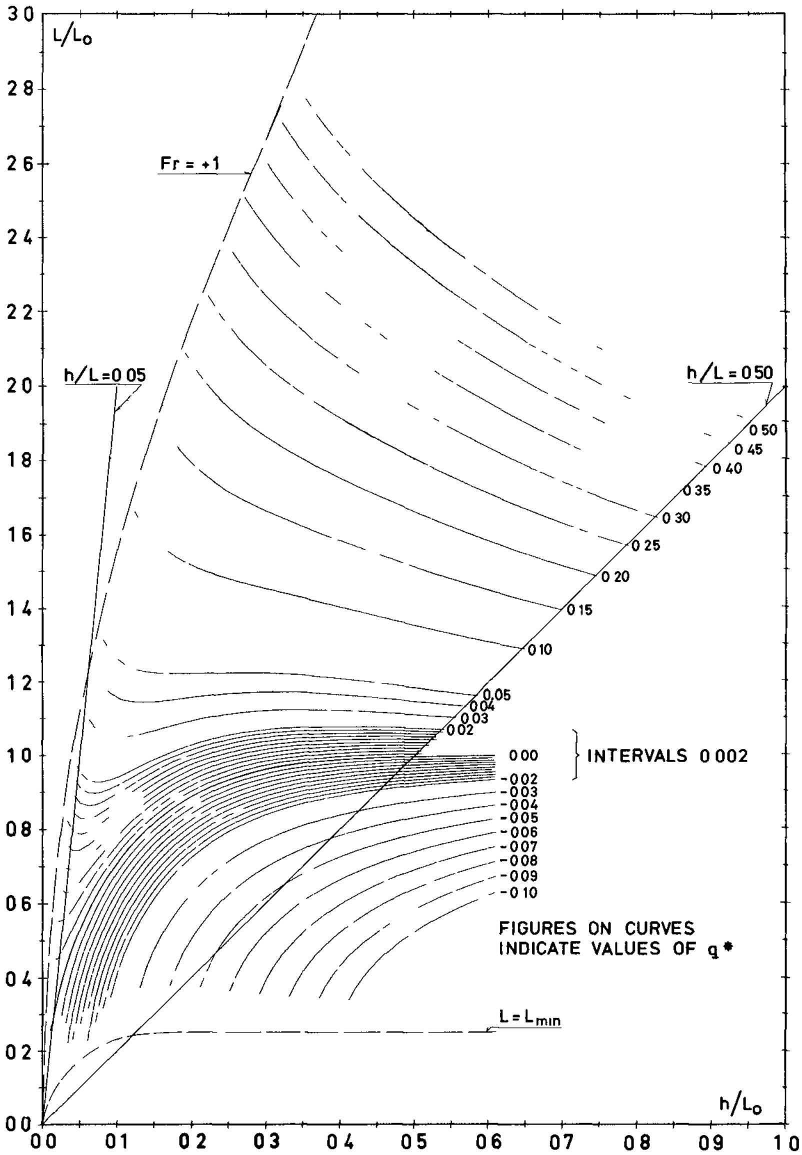

Fig 6-A "Constant discharge curves" for L/L_o

h/L_o \ q^*	0	+0 002	+0 004	+0 006	+0 008	+0 010	+0 012	+0 014	+0 016	+0 018	+0 020
0 00	0 000										
0 01	0 248	0 450									
0 02	0 347	0 450	0 551	0 652							
0 03	0 420	0 491	0 560	0 628	0 696	0 763	0 830				
0 04	0 480	0 534	0 587	0 639	0 691	0 742	0 793	0 844	0 895	0 946	0 996
0 05	0 531	0 575	0 618	0 661	0 703	0 745	0 786	0 828	0 869	0 910	0 951
0 06	0 575	0 613	0 649	0 686	0 722	0 757	0 792	0 827	0 862	0 897	0 931
0 07	0 614	0 647	0 679	0 711	0 743	0 774	0 804	0 835	0 865	0 896	0 9[illegible]6
0 08	0 649	0 679	0 707	0 736	0 764	0 792	0 819	0 846	0 874	0 900	0 927
0 09	0 681	0 707	0 734	0 759	0 785	0 810	0 835	0 860	0 884	0 909	0 9[illegible]3
0 10	0 709	0 734	0 758	0 782	0 805	0 828	0 851	0 874	0 896	0 919	0 941
0 11	0 735	0 758	0 780	0 802	0 824	0 846	0 867	0 888	0 909	0 9[illegible]0	0 950
0 12	0 759	0 780	0 801	0 822	0 842	0 862	0 882	0 902	0 921	0 941	0 960
0 13	0 780	0 800	0 820	0 840	0 859	0 878	0 897	0 915	0 934	0 952	0 970
0 14	0 800	0 819	0 838	0 856	0 875	0 893	0 910	0 928	0 946	0 963	0 980
0 15	0 818	0 836	0 854	0 872	0 889	0 906	0 923	0 940	0 957	0 973	0 990
0 16	0 835	0 852	0 869	0 886	0 903	0 919	0 935	0 951	0 967	0 983	0 999
0 17	0 850	0 867	0 883	0 899	0 915	0 931	0 946	0 962	0 977	0 992	1 007
0 18	0 864	0 880	0 896	0 911	0 927	0 942	0 957	0 972	0 986	1 001	1 015
0 19	0 877	0 892	0 907	0 922	0 937	0 952	0 966	0 981	0 995	1 009	1 0[illegible]3
0 20	0 888	0 903	0 918	0 933	0 947	0 961	0 975	0 989	1 003	1 016	1 030
0 21	0 899	0 914	0 928	0 942	0 956	0 970	0 983	0 997	1 010	1 023	1 036
0 22	0 909	0 923	0 937	0 950	0 964	0 977	0 990	1 004	1 017	1 029	1 04[illegible]
0 23	0 918	0 931	0 945	0 958	0 971	0 984	0 997	1 010	1 023	1 035	1 048
0 24	0 926	0 939	0 952	0 965	0 978	0 991	1 003	1 016	1 028	1 040	1 052
0 25	0 933	0 946	0 959	0 972	0 984	0 996	1 009	1 021	1 033	1 045	1 057
0 26	0 940	0 953	0 965	0 977	0 990	1 002	1 014	1 025	1 037	1 049	1 060
0 27	0 946	0 958	0 971	0 983	0 995	1 006	1 018	1 030	1 041	1 053	1 064
0 28	0 952	0 964	0 976	0 987	0 999	1 011	1 022	1 033	1 045	1 056	1 067
0 29	0 957	0 968	0 980	0 992	1 003	1 014	1 025	1 037	1 048	1 059	1 069
0 30	0 961	0 973	0 984	0 995	1 006	1 018	1 029	1 039	1 050	1 061	1 072
0 31	0 965	0 977	0 988	0 999	1 010	1 020	1 031	1 042	1 052	1 063	1 073
0 32	0 969	0 980	0 991	1 002	1 012	1 023	1 034	1 044	1 054	1 065	1 075
0 33	0 972	0 983	0 994	1 004	1 015	1 025	1 036	1 046	1 056	1 066	1 076
0 34	0 975	0 986	0 996	1 007	1 017	1 027	1 037	1 047	1 057	1 067	1 077
0 35	0 978	0 988	0 999	1 009	1 019	1 029	1 039	1 049	1 059	1 068	1 078
0 36	0 980	0 991	1 001	1 011	1 020	1 030	1 040	1 050	1 059	1 069	1 079
0 37	0 983	0 992	1 002	1 012	1 022	1 032	1 041	1 051	1 060	1 070	1 079
0 38	0 984	0 994	1 004	1 014	1 023	1 033	1 042	1 051	1 061	1 070	1 079
0 39	0 986	0 996	1 005	1 015	1 024	1 033	1 043	1 052	1 061	1 070	1 079
0 40	0 988	0 997	1 006	1 016	1 025	1 034	1 043	1 052	1 061	1 070	1 079
0 41	0 989	0 998	1 007	1 017	1 026	1 035	1 043	1 052	1 061	1 070	1 079
0 42	0 990	0 999	1 008	1 017	1 026	1 035	1 044	1 05[illegible]	1 061	1 070	1 078
0 43	0 991	1 000	1 009	1 018	1 027	1 035	1 044	1 052	1 061	1 069	1 078
0 44	0 992	1 001	1 010	1 018	1 027	1 035	1 044	1 052	1 061	1 069	1 077
0 45	0 993	1 002	1 010	1 019	1 027	1 036	1 044	1 052	1 060	1 068	1 077
0 46	0 994	1 002	1 011	1 019	1 027	1 036	1 044	1 05[illegible]	1 060	1 068	1 076
0 47	0 995	1 003	1 011	1 019	1 027	1 035	1 043	1 051	1 059	1 067	1 075
0 48	0 995	1 003	1 011	1 019	1 027	1 035	1 043	1 051	1 059	1 067	1 074
0 49	0 996	1 004	1 012	1 020	1 027	1 035	1 043	1 051	1 058	1 066	1 074
0 50	0 996	1 004	1 012	1 020	1 027	1 035	1 043	1 050	1 058	1 065	1 073
0 51	0 997	1 004	1 012	1 020	1 027	1 035	1 042	1 050	1 057	1 065	1 072
0 52	0 997				1 027	1 034	1 042	1 049	1 057	1 064	1 071
0 53	0 997						1 041	1 049	1 056	1 063	1 070
0 54	0 998									1 062	1 069
0 55	0 998										
0 56	0 998										
0 57	0 998										
0 58	0 999										
0 59	0 999										
0 60	0 999										

NOTE BELOW DOTTED STEP-CURVE WE HAVE h/L>0 5

Table 6-a Values of L/L_o (positive discharge)

h/L_o \ q^*	0	-0 002	-0 004	-0 006	-0 008	-0 010	-0 012	-0 014	-0 016	-0 018	-0 020
0 00	0 000										
0 01	0 248										
0 02	0 347	0 240									
0 03	0 420	0 348	0 270	0 169							
0 04	0 480	0 425	0 367	0 304	0 227						
0 05	0 531	0 486	0 439	0 390	0 336	0 272					
0 06	0 575	0 537	0 497	0 456	0 413	0 365	0 310	0 223			
0 07	0 614	0 581	0 546	0 511	0 474	0 434	0 391	0 341	0 275		
0 08	0 649	0 619	0 589	0 557	0 524	0 490	0 454	0 414	0 369	0 311	
0 09	0 681	0 654	0 626	0 597	0 568	0 537	0 505	0 471	0 434	0 392	0 341
0 10	0 709	0 684	0 659	0 633	0 606	0 578	0 549	0 519	0 487	0 452	0 412
0 11	0 735	0 712	0 689	0 664	0 640	0 614	0 588	0 560	0 531	0 500	0 467
0 12	0 759	0 737	0 715	0 693	0 670	0 646	0 622	0 596	0 570	0 542	0 512
0 13	0 780	0 760	0 739	0 718	0 697	0 674	0 652	0 628	0 604	0 578	0 551
0 14	0 800	0 781	0 761	0 741	0 721	0 700	0 679	0 656	0 634	0 610	0 585
0 15	0 818	0 800	0 781	0 762	0 743	0 723	0 703	0 682	0 660	0 638	0 615
0 16	0 835	0 817	0 800	0 781	0 763	0 744	0 725	0 705	0 685	0 664	0 642
0 17	0 850	0 833	0 816	0 799	0 781	0 763	0 745	0 726	0 706	0 687	0 666
0 18	0 864	0 848	0 831	0 815	0 798	0 780	0 763	0 745	0 726	0 707	0 688
0 19	0 877	0 861	0 845	0 829	0 813	0 796	0 779	0 762	0 744	0 726	0 708
0 20	0 888	0 873	0 858	0 843	0 827	0 811	0 794	0 778	0 761	0 743	0 726
0 21	0 899	0 884	0 870	0 855	0 839	0 824	0 808	0 792	0 776	0 759	0 742
0 22	0 909	0 895	0 880	0 866	0 851	0 836	0 821	0 805	0 789	0 773	0 757
0 23	0 918	0 904	0 890	0 876	0 862	0 847	0 832	0 817	0 802	0 786	0 770
0 24	0 926	0 912	0 899	0 885	0 871	0 857	0 843	0 828	0 813	0 798	0 783
0 25	0 933	0 920	0 907	0 894	0 880	0 866	0 852	0 838	0 824	0 809	0 794
0 26	0 940	0 927	0 914	0 901	0 888	0 875	0 861	0 847	0 833	0 819	0 805
0 27	0 946	0 934	0 921	0 908	0 896	0 882	0 869	0 856	0 842	0 829	0 815
0 28	0 952	0 939	0 927	0 915	0 902	0 890	0 877	0 864	0 850	0 837	0 823
0 29	0 957	0 945	0 933	0 921	0 908	0 896	0 883	0 871	0 858	0 845	0 832
0 30	0 961	0 950	0 938	0 926	0 914	0 902	0 890	0 877	0 865	0 852	0 839
0 31	0 965	0 954	0 942	0 931	0 919	0 907	0 895	0 883	0 871	0 859	0 846
0 32	0 969	0 958	0 947	0 935	0 924	0 912	0 901	0 889	0 877	0 865	0 853
0 33	0 972	0 961	0 950	0 939	0 928	0 917	0 906	0 894	0 882	0 871	0 859
0 34	0 975	0 965	0 954	0 943	0 932	0 921	0 910	0 899	0 887	0 876	0 864
0 35	0 978	0 968	0 957	0 946	0 936	0 925	0 914	0 903	0 892	0 881	0 869
0 36	0 980	0 970	0 960	0 949	0 939	0 928	0 918	0 907	0 896	0 885	0 874
0 37	0 983	0 973	0 962	0 952	0 942	0 932	0 921	0 911	0 900	0 889	0 879
0 38	0 984	0 975	0 965	0 955	0 945	0 935	0 924	0 914	0 904	0 893	0 883
0 39	0 986	0 977	0 967	0 957	0 947	0 937	0 927	0 917	0 907	0 897	0 887
0 40	0 988	0 978	0 969	0 959	0 950	0 940	0 930	0 920	0 910	0 900	0 890
0 41	0 989	0 980	0 971	0 961	0 952	0 942	0 933	0 923	0 913	0 903	0 894
0 42	0 990	0 981	0 972	0 963	0 954	0 944	0 935	0 925	0 916	0 906	0 897
0 43	0 991	0 983	0 974	0 965	0 955	0 946	0 937	0 928	0 919	0 909	0 900
0 44	0 992	0 984	0 975	0 966	0 957	0 948	0 939	0 930	0 921	0 912	0 902
0 45	0 993	0 985	0 976	0 967	0 959	0 950	0 941	0 932	0 923	0 914	0 905
0 46	0 994	0 986	0 977	0 969	0 960	0 951	0 943	0 934	0 925	0 916	0 908
0 47	0 995	0 986	0 978	0 970	0 961	0 953	0 944	0 936	0 927	0 919	0 910
0 48	0 995	0 987	0 979	0 971	0 963	0 954	0 946	0 938	0 929	0 921	0 912
0 49	0 996	0 988	0 980	0 972	0 964	0 956	0 947	0 939	0 931	0 923	0 914
0 50	0 996	0 989	0 981	0 973	0 965	0 957	0 949	0 941	0 933	0 924	0 916
0 51	0 997	0 989	0 981	0 974	0 966	0 958	0 950	0 942	0 934	0 926	0 918
0 52	0 997	0 990	0 982	0 974	0 967	0 959	0 951	0 943	0 936	0 928	0 920
0 53	0 997	0 990	0 983	0 975	0 968	0 960	0 952	0 945	0 937	0 929	0 922
0 54	0 998	0 990	0 983	0 976	0 968	0 961	0 953	0 946	0 938	0 931	0 923
0 55	0 998	0 991	0 984	0 976	0 969	0 962	0 954	0 947	0 940	0 932	0 925
0 56	0 998	0 991	0 984	0 977	0 970	0 963	0 955	0 948	0 941	0 934	0 926
0 57	0 998	0 992	0 985	0 978	0 970	0 963	0 956	0 949	0 942	0 935	0 928
0 58	0 999	0 992	0 985	0 978	0 971	0 964	0 957	0 950	0 943	0 936	0 929
0 59	0 999	0 992	0 985	0 979	0 972	0 965	0 958	0 951	0 944	0 937	0 930
0 60	0 999	0 992	0 986	0 979	0 972	0 966	0 959	0 952	0 945	0 938	0 932

NOTE BELOW DOTTED STEP-CURVE WE HAVE h/L>0 5

Table 6-b Values of L/L_o (negative discharge)

WAVE HEIGHTS

The wave heights - calculated from (5 8) and (5 11) - are plotted in Fig 6-B Note how the "shape" of the curves corresponding to a constant positive discharge differs from that corresponding to $q^* = 0$ The outmost left part of these curves correspond to Froude numbers close to one (see also Fig 6-A, with comments) For negative discharges we end with infinite wave heights in the limit $L = L_{min}$

7 NUMERICAL EXAMPLES

In the examples results are only given with normal slide-rule accuracy

EX 7-a CALCULATION OF L, H, Δh, h AND U FROM T_a, q, D AND H_o

Two-dimensional flow is assumed, and the following quantities are given

$T_a = 8\ 0$ s, $q = 10\ 1\ m^2/s$, $D = 9\ 5$ m, $H_o = 1\ 6$ m (corresponding to a deep-water steepness $S_o = 1\ 6\ \%$)

In the first approximation, H is put equal to zero in (5 10), so we find from (5 9) and (5 12), by iteration

$\Delta h_1 = 0\ 0583$ m, $h_1 = 9\ 4417 \approx 9\ 44$ m, $U_1 = 1\ 07$ m/s

From (3 6), (3 9) and (5 13) $L_o = 1\ 56\ 8\ 0^2 = 100$ m, $c_o = 1\ 56\ 8\ 0 = 12\ 5$ m/s, $h/L_o = 0\ 0944$, $q^* = 0\ 00808$

By interpolation in Table 6-a we then find $L/L_o = 0\ 795 \Rightarrow$ $\underline{L = 79\ 5\ m}$ (without the current, the wave length would be 69 5 m, for the same values of D and T_a)

$h/L = 9\ 44/79\ 5 = 0\ 119$, so we find from (5 14) and (3 11) $c_r/c_o = 0\ 709 \Rightarrow$ $\underline{c_r = 8\ 86\ m/s}$ (without the current the wave celerity would be 8 69 m/s), and $c_{gr}/c_r = 0\ 853$

(5 11) then gives $H/H_o = 0\ 804 \Rightarrow$ $\underline{H = 1\ 29\ m}$ (without the current the wave height would be 1 50 m, for the same value of H_o)

The new value of Δh from (5 10) $\Delta h_2 = 0\ 0583$ m $+ 0\ 0078$ m $- 0\ 0027$ m $\approx 0\ 0634$ m $\Rightarrow h_2 \approx 9\ 44$ m $= h_1$ So within slide-rule accuracy we find $\underline{U = 1\ 07\ m/s}$, $\underline{\Delta h = 0\ 0634\ m}$ (without the current the set-down would be 0 0094 m), $\underline{h = 9\ 4366\ m\ (\approx 9\ 44\ m)}$

Finally from (3 1) and (3 2) $\underline{c_a} = 79\ 5/8\ 0 = \underline{9\ 93\ m/s}$ and $\underline{T_r} = 79\ 5/8\ 86 = \underline{8\ 98\ s}$

EX 7-b CALCULATION OF WAVE HEIGHT FROM BOTTOM PRESSURE CELL

This example demonstrates how the wave height H can be determined for a two-dimensional flow, when the below mentioned parameters have been measured It also shows specifically the effect of a current

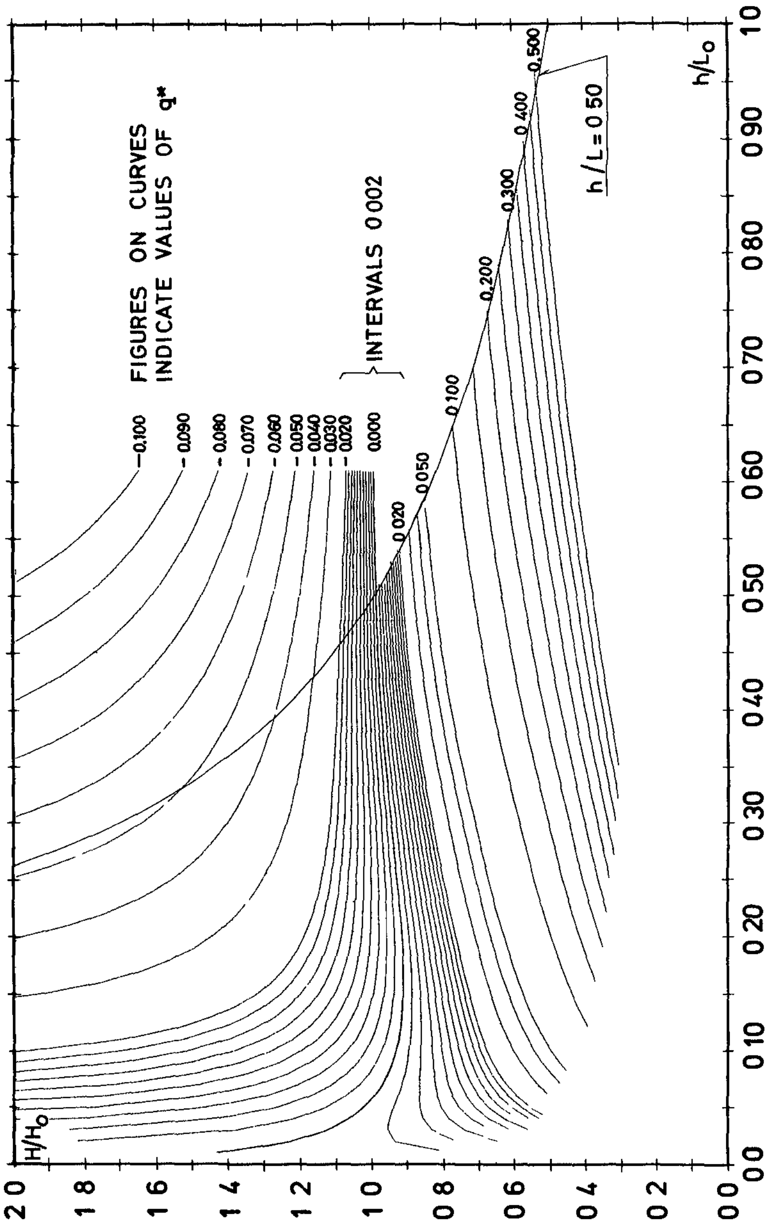

Fig 6-B "Constant discharge curves" for H/H_o

Measured parameters

$h = 8\ 50$ m

$U = -0\ 70$ m/s (i e a counter-current)

$T_a = 6\ 0$ s

and max $\Delta p_b = 3400\ N/m^2$ ($1\ N = 1$ kg m/s^2), where Δp_b is the difference between actual pressure and hydrostatic pressure corresponding to MWL ($\equiv \gamma\ h$) at the bottom, (here considered measured with a pressure cell at the bottom)

The calculations proceed thus

$$L_o = \frac{9\ 81}{2\ \pi}\ 6\ 0^2 = 56\ 2\ m \qquad (3\ 6)$$

$$h/L_o = 8\ 50/56\ 2 = 0\ 151$$

$$c_o = \frac{9\ 81}{2\ \pi}\ 6\ 0 = 9\ 36\ m/s \qquad (3\ 9)$$

$$q^* = 8\ 50\ (-0\ 70)/(9\ 36\ 56\ 2) = -0\ 0113 \qquad (5\ 13)$$

Using linear interpolation in Table 6-b we obtain

$$L/L_o = 0\ 712$$

hence $h/L \equiv (h/L_o)/(L/L_o) = 0\ 151/0\ 712 = 0\ 212$

Using the first order expression only, we have for the wave amplitude

$$H/2 = \max \Delta p_b \quad \cosh kh/\gamma$$

$$\cosh kh = 2\ 03$$

and so the wave height becomes

$$\underline{H} = 2\ 3400\ 2\ 03/(1000\ 9\ 81) = \underline{1\ 41\ m}$$

If the current were not noticed, then $L/L_o = 0\ 820$ from Table 6-a or 6-b and $h/L = 0\ 151/0\ 820 = 0\ 184$

$$\cosh kh = 1\ 75$$

$$\underline{H} = 2\ 3400\ 1\ 75/(1000\ 9\ 81) = \underline{1\ 21\ m}$$

$$\frac{\Delta H}{H} = \frac{1\ 41 - 1\ 21}{1\ 41}\ 100\ \% = 14\ \%$$

i e in this case, neglection of the current will give wave height 14 % too small (A positive current will have the opposite effect)

It is not difficult to find realistic sets of values for h, U and T_a that lead to higher values of $\Delta H/H$ (as defined above) This aspect shows the importance of considering the effect of a possible current The phenomenon may be one of the factors which can obscure the direct comparison in a coastal zone between wave heights as measured at the water surface and as calculated from pressure cell recordings at the bottom

8 REFERENCES

[1] Bowen, A J , Inman, D L , and Simmons, V P (1968) Wave "Set-Down" and Set-Up J Geophys Res 73, 2569-2577

[2] Garrett, C J R (1967) Discussion The Adiabatic Invariant for Wave Propagation in a Nonuniform Moving Medium Proc Roy Soc , A, 299, 26-27

[3] Jonsson, I G (ed), (1969) Index to Reports 1-19 (1961-1969) Basic Res -Prog Rep 20 Coastal Engrg Lab and Hydraulic Lab , Techn Univ Denmark

[4] Jonsson, I G (1970) Graphical Determination of the Wave Length in a Homogeneous Current Field Basic Res -Prog Rep 21, 19-26 Coastal Engrg Lab and Hydraulic Lab , Techn Univ Denmark

[5] Longuet-Higgins, M S , and Stewart, R W (1960) Changes in the Form of Short Gravity Waves on Long Waves and Tidal Currents J Fluid Mech 8, 565-583

[6] Longuet-Higgins, M S , and Stewart, R W (1961) The Changes in Amplitude of Short Gravity Waves on Steady Non-Uniform Currents J Fluid Mech 10, 529-549

[7] Lundgren, H (1963) Wave Thrust [≡ Radiation Stress] and Wave Energy Level Int Assoc Hydr Res , Proc 10th Congress 1, 147-151, London

[8] Whitham, G B (1962) Mass, Momentum and Energy Flux in Water Waves J Fluid Mech 12, 135-147