Stanley Park, Vancouver, B.C.

PART III

COASTAL STRUCTURES AND RELATED PROBLEMS

Ross Bay, Victoria, B.C.

CHAPTER 90

STANDING WAVES IN FRONT OF A SLOPING DIKE

by

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Abstract

A solution of two-dimensional long waves on a beach of uniform slope is connected with that in water of constant depth, in order to yield an approximate solution for standing waves in front of a sloping dike. Wave motions are expressed in the Lagrangian description.

The highest possible standing waves as well as the reflection coefficient are calculated according to the Miche's conception. Theoretical results show a good agreement with the experimental results of Murota and Yamada.

It is also predicted that there is a relationship between the wave overtopping quantity and the quantity of water of standing waves above the crest height of the dike.

As for the wave pressure of standing waves, a simple formula in the Eulerian description is derived for relative-dike length $\ell/L < 0.16$ by allowing 6% error.

Introduction

Although a sloping face structure is more practical than a structure with vertical wall, no theoretical work except that of Keller & Keller[1] has been done to solve the standing waves in front of the former. At the point where the slope of the dike intersects the horizontal sea bottom, there is an abrupt change in slope. Difficulty in mathematical approach arises from this point. Since the direction of motion of water particles at the bottom is parallel to the bottom, the water particles should change their direction abruptly at the point. It is hard to satisfy this condition mathematically.

In the following analysis, an approximate theory is developed. The entire region is divided into two regions; [I] region of uniform water depth, and [II] region of a constant slope. Solutions of standing long waves in each regions are obtained at first independently and are connected so that the horizontal and vertical motions of water particles are continuous across the boundary between the two regions.

There are several theories for waves in water of uniform depth. Solution for long waves to the first order approximation is applied, assuming that the relative water depth in front of the structure is shallow enough and the motion is small, although the result is found applicable to such a higher waves as waves breaking on the slope.

Only a few wave theories are available for waves on a sloping beach. Miche[2] obtained a solution for waves over beaches sloping at special angles, by using the Lagrangian description. In his paper, surface waves were discussed. Since nodal line in case of standing surface waves are not straight lines in vertical direction, Miche's theory introduces another complexity if it is connected with a solution of waves in water of uniform depth. Lewy[3], or Isaacson[4] extended the Miche's theory for surface waves, but for slopes of more general angles or for all angles. Carrier & Greenspan[5] solved long wave motion on a sloping beach, using the shallow water theory in terms of the Eulerian description. As the horizontal coordinate in their result is a function of local wave height which varies with time, it is not convenient to use their solution in the case, because no such a theoretical result is available for case of long waves in water of uniform depth. Keller & Keller[1] applied the Isaacson's solution to the practical problems similar to the present problem. They obtained a result which corresponds to wave run-up height.

The author[6),7)] obtained a theory of long waves on a sloping beach in the Lagrangian description. Although his theory is of the first order of approximation, the result was comparable to that of Carrier & Greenspan. In addition, it is easy to obtain long wave motion in water of uniform depth. Therefore, this theoretical result is used in the present paper.

Theory

Let us consider the two-dimensional motion of an inviscid fluid. The position of the water particle, which is at the point (x_o,y_o) at t=0 is (x,y) at the time t=t. The pressure acting on this water particle is denoted by p. The still water surface is chosen as the horizontal axis and the y-axis is taken positive upwards.

Equations of continuity and motion are

$$\frac{\partial(x,y)}{\partial(x_o,y_o)} = 1 \tag{1}$$

$$\frac{\partial^2 x}{\partial t^2} = -\frac{1}{\rho}\frac{\partial(p,y)}{\partial(x_o,y_o)} \tag{2}$$

$$\frac{\partial^2 y}{\partial t^2} = -g-\frac{1}{\rho}\frac{\partial(x,p)}{\partial(x_o,y_o)} \tag{3}$$

where the symbol $\frac{\partial(u,v)}{\partial(x_o,y_o)}$ denotes the Jacobian

$$\frac{\partial(u,v)}{\partial(x_o,y_o)} = \begin{vmatrix} \frac{\partial u}{\partial x_o}, & \frac{\partial u}{\partial y_o} \\ \frac{\partial v}{\partial x_o}, & \frac{\partial v}{\partial y_o} \end{vmatrix} \tag{4}$$

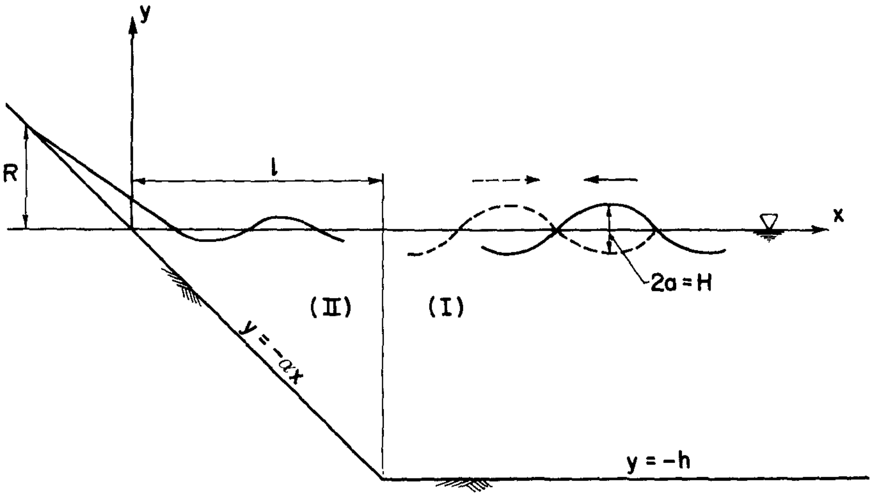

Fig.1. Definition Sketch

Let x_1, y_1 and p_1 be the small fluctuations around the zeroth order terms x_o, y_o and p_o, then

$$x = x_o + x_1 + \dots \quad , \quad y = y_o + y_1 + \dots \quad ,\text{and} \quad p = p_o + p_1 + \dots \tag{5}$$

We assume the long wave motion in which $\frac{\partial^2 y}{\partial t^2}$ is negligible compared with other terms up to the first order approximation.

The terms of the zeroth order approximation yield the equations,

$$1 = 1 \tag{6}$$

$$\frac{\partial p_o}{\partial x_o} = 0 \tag{7}$$

$$\frac{\partial p_o}{\partial y_o} = -\rho g \tag{8}$$

and the first order terms yield the equations

$$\frac{\partial x_1}{\partial x_o} + \frac{\partial y_1}{\partial y_o} = 0 \tag{9}$$

$$\frac{\partial^2 x_1}{\partial t^2} + \frac{1}{\rho}\frac{\partial p_1}{\partial x_o} + g\frac{\partial y_1}{\partial x_o} = 0 \tag{10}$$

$$\frac{1}{\rho}\frac{\partial p_1}{\partial y_o} - g\frac{\partial x_1}{\partial x_o} = 0 \tag{11}$$

The water particles forming the free surface at the initial instant remain on the free surface throughout the succeeding motion. The water pressure acting on these particles is taken equal to the atmospheric pressure.

$$p = p_o + p_1 + \ldots = 0 \qquad \text{for particles satisfying } y_o = 0 \tag{12}$$

The water particles initially on the bottom cannot depart from the bottom. In the region [I], the vertical coordinates of these particles should satisfy the following equation

$$y = -h = y_o + y_1 + \ldots$$

which yields

$$y_o = -h, \; y_1 = 0 \qquad \text{for particles satisfying } y_o = -h \tag{13}$$

In the region [II], the slope of the dike is given by

$$y = -\alpha x$$

and bottom condition is written as

$$y_o = -\alpha x_o, \; y_1 = -\alpha x_1 \qquad \text{for particles satisfying } y_o = -\alpha x_o \tag{14}$$

Equations (1) through (3) yield the zeroth order solution

$$p_o = -\rho g y \tag{15}$$

which indicates the hydrostatic pressure distribution.

Equation (9) is integrated from the bottom to y_o

$$g y_1 = -g\int_{\text{bottom}}^{y_o} \frac{\partial x_1}{\partial x_o}\, dy_o - g\alpha x_1(x_o, -\alpha x_o; t)\,\delta_{i\,II} \tag{16}$$

where $\delta_{i\,II}$ is $\delta_{I\,II} = 0$ and $\delta_{II\,II} = 1$ for i=I, II. Equation (16) satisfies the bottom boundary condition.

Equation (11) is integrated from the free surface to y_o so as to satisfy the proposed boundary condition on the free surface, and is given by

$$\frac{p_1}{\rho} = g\int_o^{y_o} \frac{\partial x_1}{\partial x_o} dy_o \tag{17}$$

Substituting Eqs. (16) and (17) into Eq.(10), we have

$$\frac{\partial^2 x_1}{\partial t^2} + \frac{\partial}{\partial x_o}\left[g\int_o^{bottom} \frac{\partial x_1}{\partial x_o} dy_o - g\alpha x_1(x_o, -\alpha x_o; t)\, \delta_{i\ II} \right] = 0 \tag{18}$$

which determines the required solution.

In the region [I], Eq.(18) is written as

$$\frac{\partial^2 x_1}{\partial t^2} + g\frac{\partial}{\partial x_o}\int_o^{-h} \frac{\partial x_1}{\partial x_o} dy_o = 0 \tag{19}$$

Let $x_1 = X(x_o)\ Y(y_o)\ T(t)$. From Eq.(19), $Y(y_o)$ is concluded to be constant. Terms X and T should satisfy the equation

$$XT'' - gh\ X''T = 0$$

where each prime denotes the differentiation once with respect to the particular independent variable. A solution of standing waves is given by

$$\left.\begin{aligned} T &= \cos \sigma t \\ X &= \cos\left(\frac{\sigma}{\sqrt{gh}} x_o + \delta\right) \end{aligned}\right\} \tag{20}$$

where $\frac{\sigma}{\sqrt{gh}} = k$ is the wave number and δ the phase lag which is included because the point $x_o = 0$ does not always coincide with the antinode of the standing waves in the region [I].

Solution for y_1 and p_1 are obtained by Eqs.(16) and (17). With appropriate coefficients, a set of solutions is given for standing waves in the region [I] as follows:

$$\left.\begin{aligned} x_1 &= \frac{2a}{kh} \cos (kx_o + \delta) \cos \sigma t \\ y_1 &= \frac{2a}{h} (y_o + h) \sin (kx_o + \delta) \cos \sigma t \\ p_1 &= -\rho g \frac{2a}{h} y_o \sin (kx_o + \delta) \cos \sigma t \end{aligned}\right\} \tag{21}$$

where a denotes the amplitude of the incident waves, $\sigma = 2\pi/T$ the frequency and $k = 2\pi/L$ the wave number.

Equation (18) is written, in the region [II], as

$$\frac{\partial^2 x_1}{\partial t^2} + g \frac{\partial}{\partial x_o} \int_o^{-\alpha x_o} \frac{\partial x_1}{\partial x_o} dy_o - g\alpha \frac{\partial}{\partial x_o} x_1(x_o, -\alpha x_o; t) = 0 \tag{22}$$

Since the second and third terms in the above equation are functions independent of y_o, x_1 is a function of x_o and t only. Equation (22) is reduced to

$$\frac{\partial^2 x_1}{\partial t^2} - g\alpha \frac{\partial^2}{\partial x_o^2} (x_o x_1) = 0 \tag{23}$$

Let $x_1 = X(x_o)\ T(t)$ and we have

$$\frac{T''}{T} = \frac{g\alpha\,[x_o X'' + 2X']}{X} = -\sigma^2 \tag{24}$$

Functions X and T for standing waves are given by

$$\left.\begin{aligned} T &= \cos \sigma t \\ X &= \frac{1}{\sqrt{x_o}} J_1\left(\frac{2\sigma}{\sqrt{g\alpha}} \sqrt{x_o}\right) \end{aligned}\right\} \tag{25}$$

First order terms are given as follows for standing waves on a sloping beach, with a coefficient A to be determined later.

$$\left.\begin{aligned} X_1 &= \frac{A}{\sqrt{x_o}} J_1\left(\frac{2\sigma}{\sqrt{g\alpha}} \sqrt{x_o}\right) \cos \sigma t \\ Y_1 &= A\left[(y_o + \alpha x_o) \frac{\sigma}{\sqrt{g\alpha}} \frac{1}{x_o} J_2\left(\frac{2\sigma}{\sqrt{g\alpha}} \sqrt{x_o}\right) - \frac{\alpha}{\sqrt{x_o}} J_2\left(\frac{2\sigma}{\sqrt{g\alpha}} \sqrt{x_o}\right)\right] \cos \sigma t \\ P_1 &= -\frac{\rho g \sigma}{\sqrt{g\alpha}} \frac{y_o}{x_o} A\ J_2\left(\frac{2\sigma}{\sqrt{g\alpha}} \sqrt{x_o}\right) \cos \sigma t \end{aligned}\right\} \tag{26}$$

where y_1 and p_1 are determined by Eqs.(18) and (17), and J_n is the n-th Bessel Function of the first kind.

Two coefficients, A and δ, which are undetermined connect the two regions. Consider the particles which are, at the initial instant, at the point $x_o = \ell$ which is the boundary between two regions. Two expressions of the horizontal movements of these particles should be the same. As for the vertical movement, we cannot connect it for all water particles locating on $x = \ell$. However, the most important factor to be considered is the vertical displacement of the water particle on the free surface. As the first order pressure is the hydrostatic pressure, any discontinuity in water levels between the two regions might have a big influence. Therefore, this condition is adopted as one of the conditions. We have

$$\ell + \frac{2a}{kh} \cos (k\ell+\delta) \cos \sigma t = \ell + \frac{A}{\sqrt{\ell}} J_1\left(\frac{2\sigma}{\sqrt{g\alpha}} \sqrt{\ell} \right) \cos \sigma t \tag{27}$$

and

$$\frac{2a}{kh} \sin (k\ell+\delta) \cos \sigma t = \frac{A}{kh} \left[\frac{\sqrt{\alpha}}{\sqrt{g}} \sigma J_2\left(\frac{2\sigma}{\sqrt{g\alpha}} \sqrt{\ell}\right) - \frac{\alpha}{\sqrt{\ell}} J_1\left(\frac{2\sigma}{\sqrt{g\alpha}} \sqrt{\ell}\right)\right] \cos \sigma t \tag{28}$$

Eliminating δ, A is determined as

$$A = \frac{2a}{kh} \sqrt{\ell} \left[J_o^2 \left(4\pi \frac{\ell}{L} \right) + J_1^2 \left(4\pi \frac{\ell}{L} \right)\right]^{-1/2} \tag{29}$$

in which the following relation is used.

$$\frac{2\sigma}{\sqrt{g\alpha}} \sqrt{\ell} = \frac{4\pi}{\sqrt{gh}\, T} \sqrt{\frac{h\ell}{\alpha}} = 4\pi \frac{\ell}{L} \tag{30}$$

Run-up Height

Wave run-up on the dike is given by the vertical location of a particle corresponding to $x_o = 0$ and $y_o = 0$.

$$y\big|_{x_o = 0,\ y_o = 0} = A\left[\frac{\sigma}{\sqrt{g\alpha}} \frac{y_o}{x_o} J_2\left(\frac{2\sigma}{\sqrt{g\alpha}} \sqrt{x_o}\right) - \frac{\alpha\sigma}{\sqrt{g\alpha}} J_o\left(\frac{2\sigma}{\sqrt{g\alpha}} \sqrt{x_o}\right)\right] \cos\sigma t\Big|_{x_o=0,y_o=0}$$

$$= -A\sigma \sqrt{\frac{\alpha}{g}} \cos \sigma t \tag{31}$$

Relative wave run-up height is given by

$$\frac{R}{2a} = \left[J_o^2 \left(4\pi \frac{\ell}{L}\right) + J_1^2 \left(4\pi \frac{\ell}{L}\right) \right]^{-\frac{1}{2}} \tag{32}$$

The result coincides with the amplification factor of Keller & Keller.

Figure 2 shows the relationship between $\frac{R}{2a}$ and $\frac{\ell}{L}$, and its asymptotic expressions:

$$\frac{R}{2a} = \sqrt{2}\,\pi \left(\frac{\ell}{L}\right)^{\frac{1}{2}} \qquad \text{for large value of } \frac{\ell}{L} \tag{33}$$

$$\frac{R}{2a} = 1 + 2\pi^2 \left(\frac{\ell}{L}\right)^2 \qquad \text{for small value of } \frac{\ell}{L} \tag{34}$$

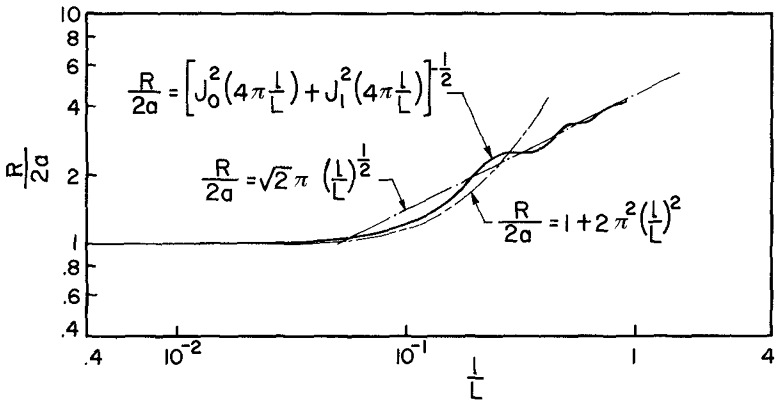

Fig.2. Relative Wave Run-up

Breaking Condition

Local wave steepness at the wave front cannot be steeper than the inclination of the slope. Therefore,

$$\frac{dy}{dx}\Big|_{x_o=0,\ y_o=0} = \frac{\partial y/\partial x_o}{\partial x/\partial x_o}\Big|_{x_o=0,\ y_o=0} = -\alpha \tag{35}$$

gives the critical condition. This is the Miche's criterion for breaking. From Eqs.(5) and (26),

$$\frac{\partial y}{\partial x_o}\Big|_{x_o=0,\ y_o=0} = A\frac{\sigma^3}{g\sqrt{g\alpha}}\cos\sigma t$$

and

$$\frac{\partial x}{\partial x_o}\Big|_{x_o=0,\ y_o=0} = 1 - \frac{\sigma^3 A}{2g\alpha\sqrt{g\alpha}}\cos\sigma t$$

Equation (35) is written as

$$-\alpha + \frac{\sigma^3 A}{2g\sqrt{g\alpha}}\cos\sigma t = \frac{\sigma^3 A}{g\sqrt{g\alpha}}\cos\sigma t$$

Introducing the critical amplitude a_m of the incident wave which just breaks on the slope, the breaking condition is given by

$$\frac{2a_m}{L}\frac{1}{\alpha} = \frac{2}{\pi}\left(4\pi\frac{\ell}{L}\right)^{-1}\left[J_o^2\left(4\pi\frac{\ell}{L}\right) + J_1^2\left(4\pi\frac{\ell}{L}\right)\right]^{\frac{1}{2}} \tag{36}$$

Figure 3 shows the relation above with the two approximate expressions;

$$\frac{2a_m}{L}\frac{1}{\alpha} = \left(\frac{1}{2\pi^2}\right)^{3/2}\left(\frac{\ell}{L}\right)^{-3/2} \quad \text{for large value of } \frac{\ell}{L} \tag{37}$$

$$\frac{2a_m}{L}\frac{1}{\alpha} = \frac{1}{2\pi^2}\left(\frac{\ell}{L}\right)^{-1} - \frac{\ell}{L} \quad \text{for small value of } \frac{\ell}{L} \tag{38}$$

Figure 4 provides a comparison of theoretical results with experimental results by Murota and Yamada.[8] For a relative depth $h/L = \alpha\ell/L$ and an incident wave steepness $2a/L$, an angle, α, of the slope on which waves just break is uniquely determined by Eq.(36), and is expressed in degree in the figure.

Reflection Coefficient

According to the Miche's definition, the reflection coefficient, r, of a slope is given by

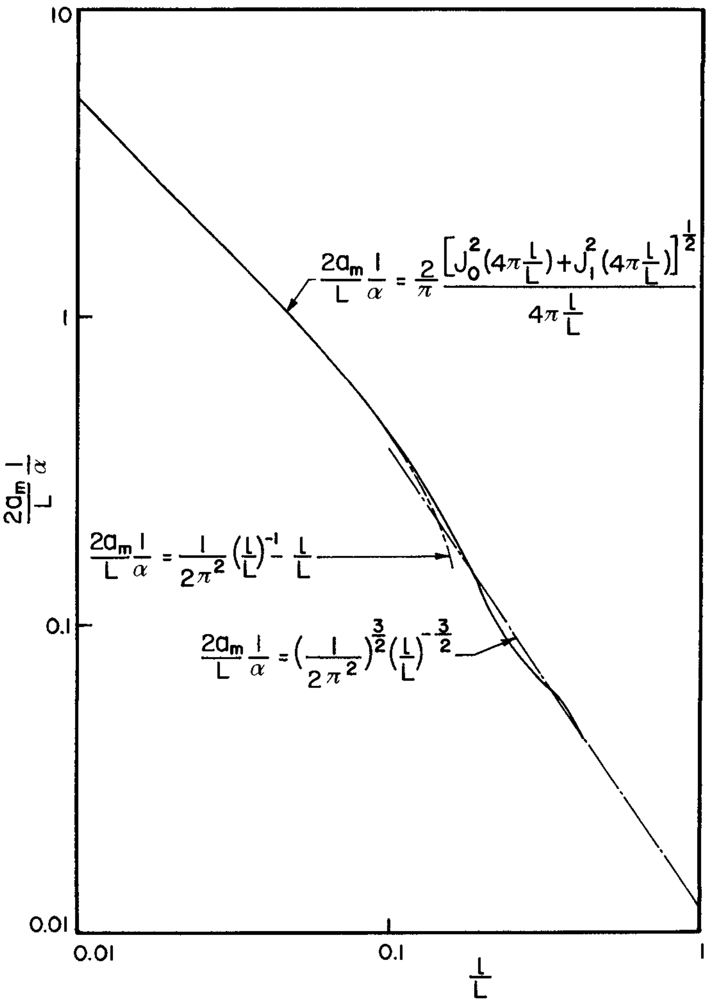

Fig.3. Breaking Condition

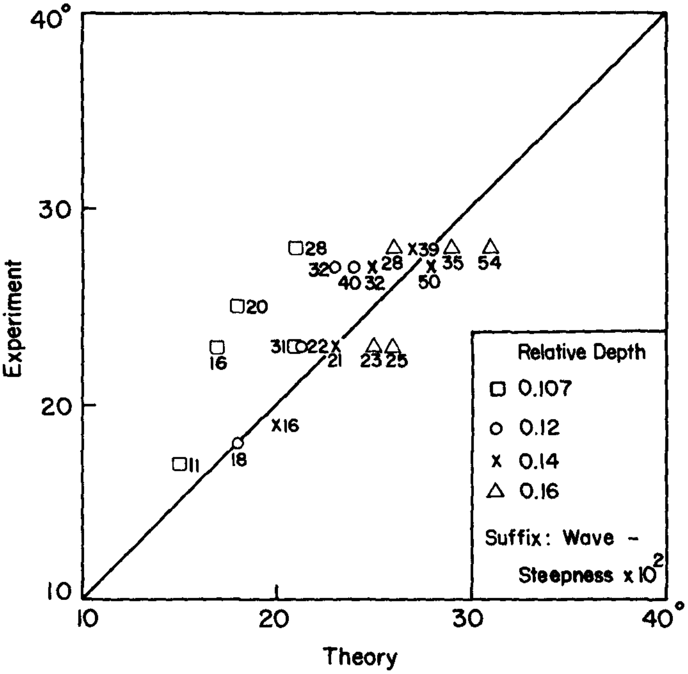

Fig.4. Theoretical Breaking Condition Compared With Murota & Yamada's Experiment.

$$\left.\begin{aligned} r &= 1 && \text{for } H_i \leq 2\,a_m \\ &= \frac{2\,a_m}{H_i} && \text{for } H_i \geq 2\,a_m \end{aligned}\right\} \qquad (39)$$

in which H_i is the incident wave height and $2a_m$ is the critical height of the incident wave which just breaks on the slope.

Wave smaller than $2a_m$ can be totally reflected by the slope, while a part of waves corresponding to $2a_m$ is considered to be reflected in case of bigger waves and the rest is considered to be lost due to the breaking.

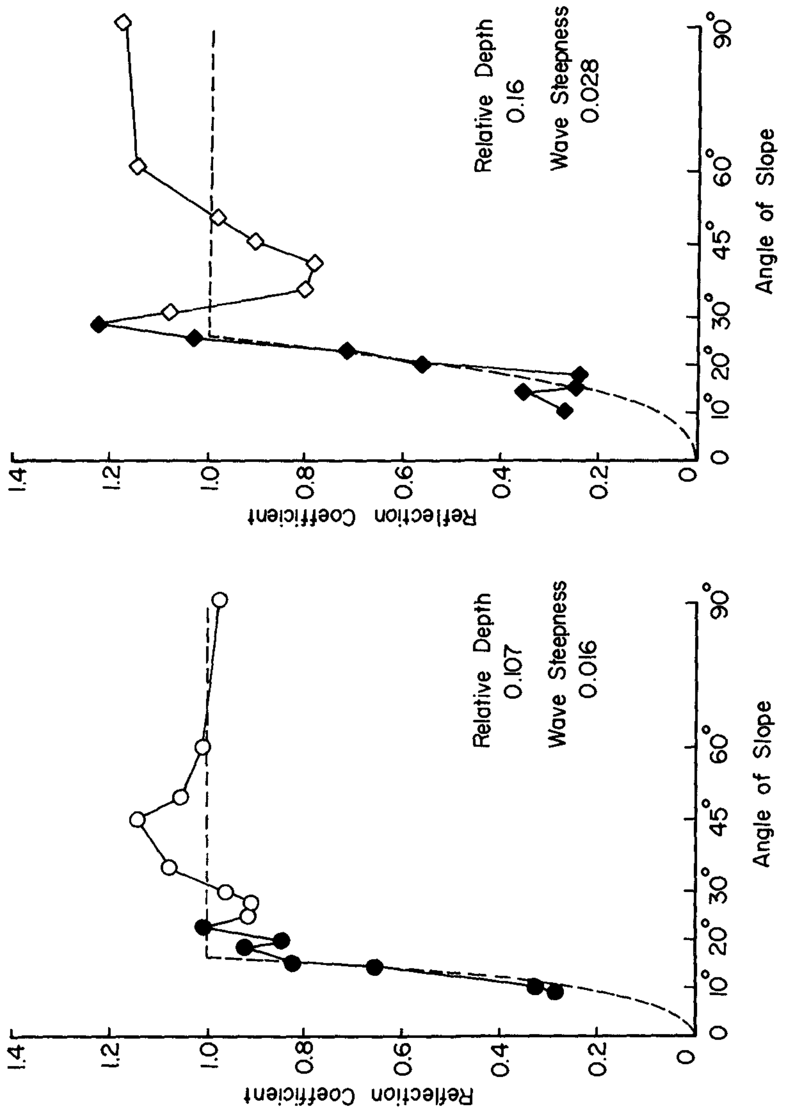

Fig.5 (a) Reflection Coefficient

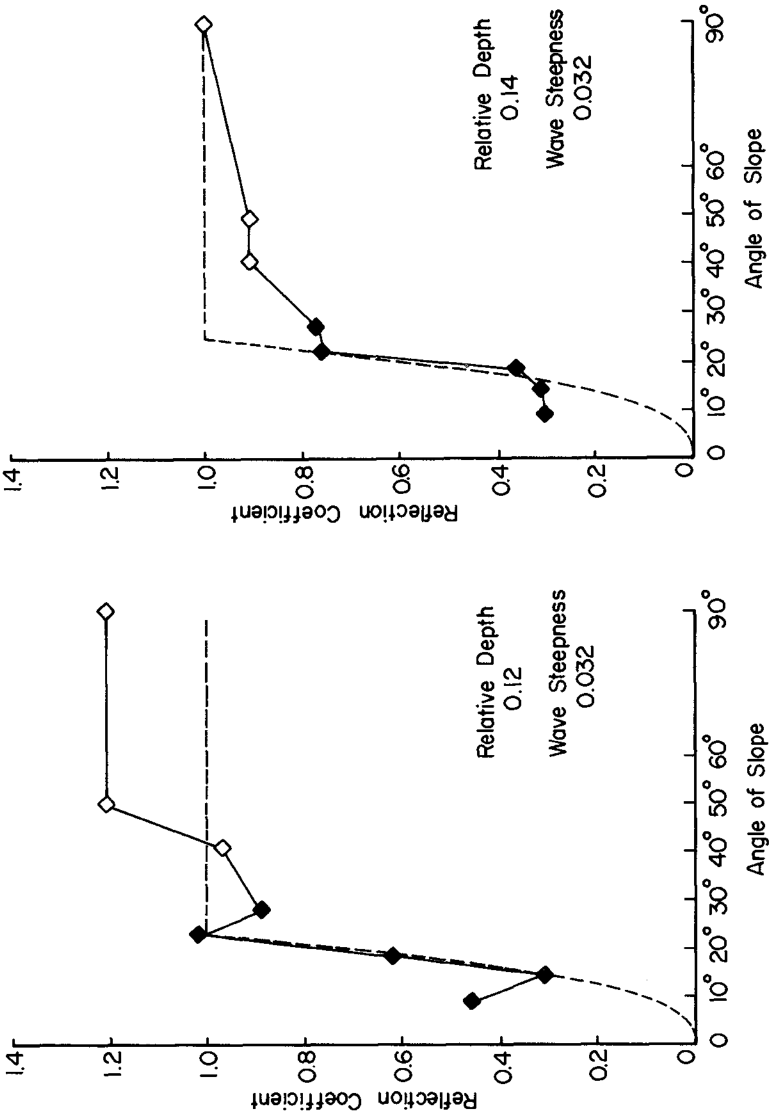

Fig.5 (b) Reflection Coefficient

Figures 5 - (a), (b) show examples of the comparison of theoretical result with the experimental results of Murota and Yamada. They carried out experiments in a wave flume which was divided into two parallel parts of the same width by a separator set along the centerline. At the end of the one part they placed a slope, while at the same end of another part a wave absorber was set to generate only the progressive waves in this part. Wave gauges were set in both parts at the same distance from the wave generator. Signals from wave gauges were electrically subtracted to separate the reflected waves from standing waves. For waves of bigger height, effect of nonlinear interaction may affect the magnitude of the reflected waves as was pointed by Goda and Abe[9], but in their experimental results no correction was made. This is considered a reason why they obtained the reflection coefficients bigger than unity under some conditions. In the figures, black marks correspond to waves which break on the slope, white marks correspond to non-breaking waves and the broken lines are theoretical results.

Wave Overtopping

In Figure 6, the rate of overtopping is compared with the volume of water above the crown height of the dike at the time when a crest of standing waves appears on the dike. Wave profile in front of the dike is drawn for water particles $y_o = 0$ at the time $\cos \sigma t = -1$ by using Eqs.(5) and (26), at first assuming the crown height of the dike is high enough to allow no overtopping. Then, the volume of water above the crest height is calculated. The experimental results are taken from one of the author's experiments. The figure suggests there is a close relationship between wave overtopping and the volume of water determined by the method above.

Wave Pressure

Pressure acting on the slope when maximum wave run-up occurs is given as follows by setting $\cos \sigma t = -1$, $y_o = -\alpha x_o$ and $y_1 = -\alpha x_1$ in Eqs.(5), (15) and (26).

$$\left.\begin{aligned} p &= \rho g \alpha x_o - \frac{\rho g \sigma}{\sqrt{g\alpha}} \alpha A \, J_2\left(\frac{2\sigma}{\sqrt{g\alpha}} \sqrt{x_o}\right) \\ x &= x_o - \frac{A}{\sqrt{g\alpha}} J_1\left(\frac{2\sigma}{\sqrt{g\alpha}} \sqrt{x_o}\right) \end{aligned}\right\} \qquad (40)$$

In order to calculate the pressure acting at x, the initial position, x_o, of the particle is determined by the second equation of Eq.(40), and is substituted into the first equation of Eq.(40).

As the procedure above is troublesome, an approximate expression is derived as follows. Introduction of non-dimensional variables defined as

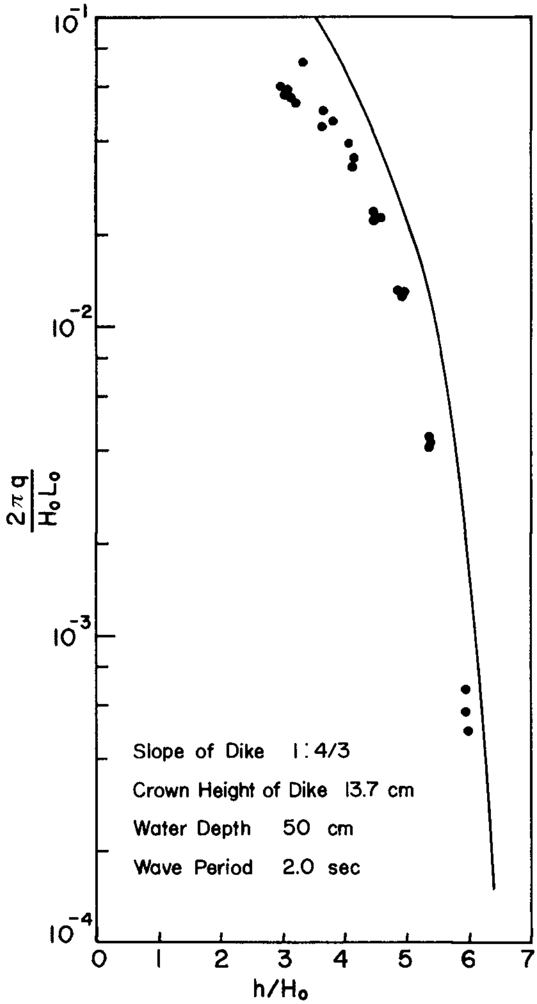

Fig.6. Wave Overtopping

$$\sqrt{X_o} = \frac{2\sigma}{\sqrt{g\alpha}} \sqrt{x_o} , \quad \sqrt{X} = \frac{2\sigma}{\sqrt{g\alpha}} \sqrt{x} , \quad P = \frac{4\sigma^2}{g\alpha} \frac{p}{\rho g\alpha}$$

and

$$D = \frac{4\sigma^2}{g\alpha} \frac{\sigma}{\sqrt{g\alpha}} A$$

reduces Eq.(40) to the followings

$$\left.\begin{aligned} P &= X_o - D \; J_2(\sqrt{X_o}) \\ X &= X_o - 2D\frac{1}{\sqrt{X_o}} J_1 (\sqrt{X_o}) \end{aligned}\right\} \qquad (41)$$

For small X_o Eq.(41) is approximated by

$$\left.\begin{aligned} P &= X_o - D\frac{X_o}{4}\left[\frac{1}{2} - \frac{X_o}{24}\right] \\ X &= X_o - D\left[1 - \frac{X_o}{8}\right] \end{aligned}\right\} \qquad (42)$$

Error in the value of pressure given by Eq.(42) is less than 6% for $X_o \leq 4$, which is equivalent to $\ell/L \leq 0.16$

An Eulerian expression of the pressure is given by

$$P = \frac{8 - D}{8 + D} (X+D) + \frac{2}{3}D \frac{(X+D)^2}{(8+D)^2} \qquad (43)$$

Figure 7 shows the wave induced pressure for case of D = 0.5. First term of the above equation is a simple expression accurate enough in this case. Including the second term, Eq.(43) coincides with the theoretical result, Eq.(41).

The term D is also a function of ℓ/L, and it is approximated by, for small relative dike length, ℓ/L,

$$D = \frac{8\pi}{\alpha} \frac{H}{L} 4\pi \frac{\ell}{L}\left[1 + 2\pi^2\left(\frac{\ell}{L}\right)^2\right] \qquad (44)$$

where H = 2a.

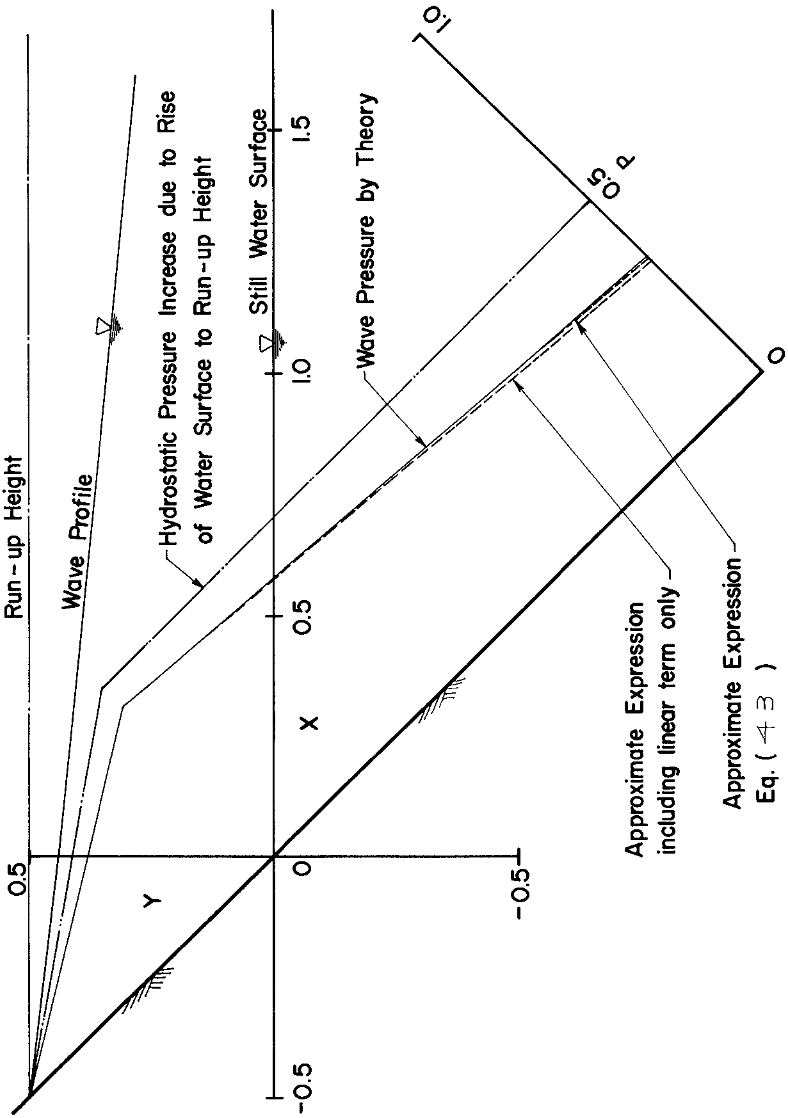

Fig.7. Wave Pressure Distribution At The Time When Wave Crest Appears At The Dike.

With dimensional variables, total pressure is written in the Eulerian description as follows.

$$\frac{p}{\rho g} = \frac{\alpha - 4\pi^2 \frac{H}{L}\frac{\ell}{L}\left\{1 + 2\pi^2\left(\frac{\ell}{L}\right)^2\right\}}{\alpha + 4\pi^2 \frac{H}{L}\frac{\ell}{L}\left\{1 + 2\pi^2\left(\frac{\ell}{L}\right)^2\right\}}\left[\alpha x + 2H\left\{1 + 2\pi^2\left(\frac{\ell}{L}\right)^2\right\}\right]$$

$$+ \frac{4}{3} H \left\{1 + 2\pi^2\left(\frac{\ell}{L}\right)^2\right\}\left[\frac{2\pi^2 \frac{\ell}{L}\frac{\alpha x}{L} + 4\pi^2 \frac{H}{L}\frac{\ell}{L}\left\{1 + 2\pi^2\left(\frac{\ell}{L}\right)^2\right\}}{\alpha + 4\pi^2 \frac{H}{L}\frac{\ell}{L}\left\{1 + 2\pi^2\left(\frac{\ell}{L}\right)^2\right\}}\right]^2 \qquad (45)$$

Conclusion

A theory is developed to describe the standing long wave motion in front of a sloping dike. Relative run-up height, breaking condition, reflection coefficient and pressure distribution are given. The breaking condition and the reflection coefficient are compared with the experimental results, and good agreements between the theory and experiments are shown. A method is proposed to predict the wave overtopping. Comparison with experimental results shows utility of this method.

Although the theory has several approximations from mathematical and physical point of view, it provides a useful basis for practical design.

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