CHAPTER 28

WATER-WAVE INTERACTION IN THE SURF ZONE

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Abstract

In the surf zone on beaches, there are strong effects which cannot be described by linear equations. This paper describes a number of wave interactions that can take place, using the simplest set of equations that give an adequate qualitative description. The results are of value for comparison with detailed experimental measurements and for gaining understanding of surf zone processes.

Introduction

The region of breaking and broken waves on a beach is important for the study of waves on beaches, from whatever point of view it is approached. From a theoretical viewpoint the turbulent, unsteady, flow presents many difficulties. The wave motions do not satisfy linear equations, so that the superposition of solutions, which gives such power to linear analysis and makes Fourier techniques so revealing, is not available. Indeed, the high level of turbulence means that accurate equations for describing the flow are not known. In this paper particular solutions of approximate non-linear equations are considered with a view to gaining some understanding of the processes involved.

The equations are the finite-amplitude shallow-water wave equations, which may be written:

$$\frac{\partial u}{\partial t} + \frac{u\partial u}{\partial x} + g\frac{\partial h}{\partial x} = g\frac{\partial h_0}{\partial x}, \tag{1}$$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial t}(hu) = 0, \tag{2}$$

in which, h is the total depth of water, h_0 is the depth below the still water level of the sea bed, which is assumed to be rigid and impermeable, and u is the mean horizontal water velocity. Multiple values arise in solutions of these equations and these are excluded by introducing discontinuities representing bores and satisfying the bore relations:

$$h_1(u_1 - V) = h_2(u_2 - V), \tag{3}$$

$$h_1u_1(u_1 - V) + \tfrac{1}{2}gh_1^2 = h_2u_2(u_2 - V) + \tfrac{1}{2}gh_2^2, \tag{4}$$

These equations of motion are usually derived, e.g. in Peregrine (1972), for the laminar flow of an inviscid fluid, however, similar equations are found for long waves on shear flows (Blythe et al, 1972).

For river flows it is found that turbulent dissipation is adequately represented by the addition of a term $- f u|u|$ to the right-hand side of equation (1). It is likely that adding such a friction term may give quite a good representation of real wave motions. Friction is discussed by Longuet-Higgins (1972) and briefly by Meyer and Taylor (1972). It is not included here since it impedes analysis of the equations and it is expected to have only a quantitative, damping, effect on solutions.

The bore relations are found to be in agreement with experiment for stationary bores (hydraulic jumps), e.g. see Rajaratnam (1967) for a discussion. For travelling bores there are difficulties in measuring the velocity V of the bore because of the strong turbulence at its front. However, Miller's (1968) measurements, despite their wide scatter, are consistent with the theoretical values.

It is important to realise the limitations of the shallow-water theory. These are well illustrated by bores, which are represented mathematically by abrupt discontinuities, but in reality the changes in level and velocity are spread over several times the depth. Similarly the equations of motion are only applicable to variations in velocity and amplitude that are also spread over many times the depth. Close to the shoreline where the mean depth of water is small these are quite acceptable limitations, but where waves have just broken, or are moving shorewards as spilling breakers, the shallow-water theory is inappropriate. Thus this theoretical treatment is most pertinent to what may be called the "inner surf zone".

It is worth noting that the equations do not predict wave breaking. When a solution reaches a point where the slope of the water surface becomes large, and multiple values of the solution are avoided by inserting a discontinuity, the correct interpretation is that the approximation of long gentle waves, which is essential for the equations, is no longer valid. In practice, it is observed that waves often break, and it is reasonable to fit a bore discontinuity. However, if the assumed discontinuity has a depth ratio of less than 1.6 any bore formed will be at least partly undular, and it will be best described by the Boussinesq equations if no breaking occurs.

A number of authors have used this analytical approach to waves on beaches, usually of uniform slope. A bore advancing toward a beach through still water is treated numerically in Keller et al (1960) and by Freeman and Le Mehaute (1964), whilst an asymptotic analysis of its approach to the shore line is in Ho and Meyer (1962). A standing wave solution, without any bore is given by Carrier and Greenspan (1958). Jeffrey (1964,1967) determines the time, if any, at which a sinusoidal wave propagating into still water over a uniform slope would "break". Carrier (1966) gives results for waves approaching a shore when they are sufficiently gentle that they do not break. By limiting attention to very slowly varying depth Varley et al (1971) and Cumberbatch and Wen (1973) deduce a number of results for waves entering still water. Run-up may be discussed in the same theoretical context, results are reviewed by Meyer and Taylor (1972).

The philosophy behind the present work is the accumulation of relevant solutions of the non-linear equations that may assist the interpretation and collection of data so that empirical rules for wave properties may be refined. Some relatively simple examples are discussed in this paper, taking a horizontal bottom for simplicity, though the results are directly transferable to situations with variable depth. In particular the analysis of bore interactions suggests a way of plotting field measurements to learn more about actual surf zones.

Introductory Analysis

Analysis is simplified by introducing the "local long-wave velocity" c, defined by

$$c^2 = gh. \tag{5}$$

Then equations (1) and (2), after some simple algebra may be written

$$\left[\frac{\partial}{\partial t} + (u+c)\frac{\partial}{\partial x}\right] (2c+u) = g\frac{dh_0}{dx} , \tag{6}$$

$$\left[\frac{\partial}{\partial t} + (u-c)\frac{\partial}{\partial x}\right] (2c-u) = -g\frac{dh_0}{dx}. \tag{7}$$

Alternatively, these equations take the form

$$\frac{\partial(2c+u)}{\partial S} = g\frac{dh_0}{dx}, \tag{8}$$

$$\frac{\partial(2c-u)}{\partial R} = g\frac{dh_0}{dx}, \tag{9}$$

if characteristic variables R,S defined by

$$2cdR = dx - (u+c)dt, \tag{10}$$

$$2cdS = dx - (u-c)dt, \tag{11}$$

are used. These define two families of characteristic lines:

$$C_+, \quad R = \text{constant}, \quad \frac{dx}{dt} = u+c; \tag{12}$$

$$C_-, \quad S = \text{constant}, \quad \frac{dx}{dt} = u-c. \tag{13}$$

(e.g. see Stoker, 1957, or Abbot, 1969). It is the existence of the relatively simple equations (8) and (9) that permits solutions, or some properties of solutions, to be found in particular cases.

For a horizontal bottom these equations simplify and can be integrated to give:

$$P = 2c+u = \text{constant on } C_+ \text{ lines}, \tag{14}$$

$$Q = 2c-u = \text{constant on } C_- \text{ lines} \tag{15}$$

The variables P and Q are called "Riemann invariants". If one family of characteristics, e.g. C_-, come from a region of uniform flow the appropriate invariant, Q, is equal to the same constant everywhere; such solutions are called "simple waves" Characteristics may be thought of as "elementary" waves of the system of equations - indeed they are the paths in (x,t) space of small amplitude long waves.

If c is used in the bore relations (3), V is given by

$$V = \frac{c_2^2 u_2 - c_1^2 u_1}{c_2^2 - c_1^2}, \tag{16}$$

and then, after substitution for V, equation (4) becomes

$$2c_1^2 c_2^2 (u_1 - u_2)^2 = (c_2^2 - c_1^2)^2 (c_1^2 + c_2^2). \tag{17}$$

"Reflection" from a bore

In a solution of the shallow-water equations with a bore fitted at a discontinuity travelling in the +x direction, the bore and its path are determined by·

(i) the conditions in front of it (i.e. u_1, c_1 or P_1, Q_1), since $V > u_1 + c_1$ it meets two characteristics on that side.

(ii) the value P_2 of P on the C_+ characteristic behind the bore which catches it up since $u_2 + c_2 > V$.

This is shown in the (x,t) plane in figure 1.

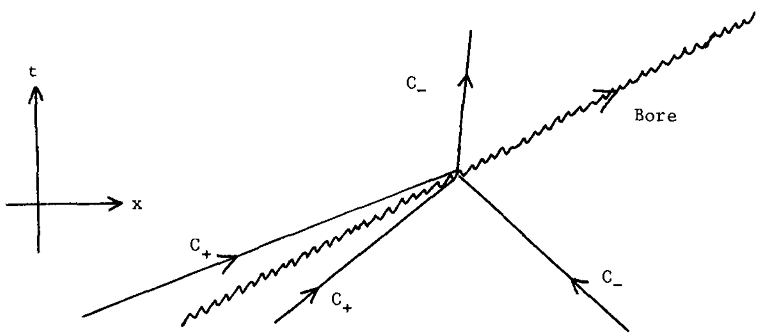

Figure 1. The four characteristics passing through one point on a bore's path.

In particular the velocity V and the value Q_2 of Q on the C_- characteristic behind the bore are determined. Q_2 is always less than Q_1, the difference can easily be evaluated in terms of the strength of the bore,

$$B = c_2/c_1, \quad \text{i.e. } B^2 = h_2/h_1,$$

by using equation (17), and is

$$Q_1 - Q_2 = c_1\left[2(1-B) + \frac{B^2-1}{B}\left(\frac{B^2+1}{2}\right)^{\frac{1}{2}}\right] \tag{18}$$

$$= c_1 q,$$

say. Values of $q=q(B)$ are displayed in figure 2.

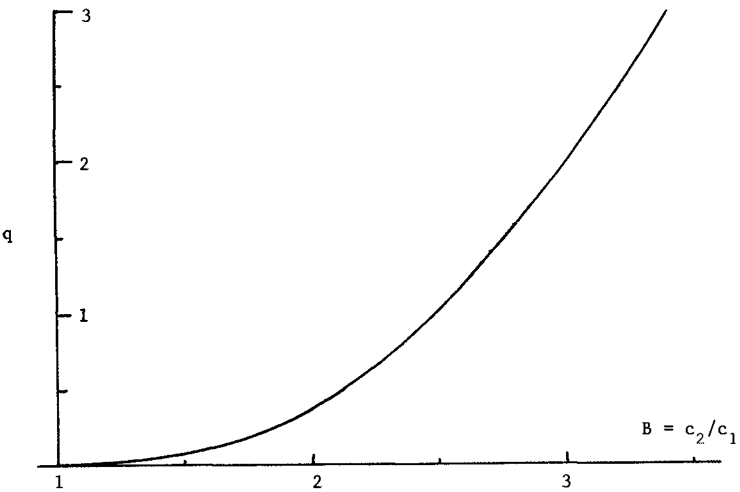

Figure 2. The change in Q at a bore, divided by c_1.

One way to interpret this result is to consider a bore travelling over a flat bottom into uniform conditions P_1, Q_1, with the wave behind the bore being of only limited extent before the same uniform conditions are recovered. Then as the C_- characteristic from the bore propagates into these uniform conditions they are disturbed because Q is less than Q_1. In fact, for such a wave

$$u = u_1 + \tfrac{1}{2}c_1 q \quad \text{and} \quad c = c_1(1 - \tfrac{1}{4}q).$$

and thus since $q > 0$ and $c^2 = gh$, the wave coming back from a bore is a wave of depression. However, the C_- characteristic only travels backwards if $u - c < 0$, but $u - c = u_1 - c_1 + \frac{3}{4} c_1 q$ and is less than zero when

$$q < \frac{4}{3}\left(1 - \frac{u_1}{c_1}\right).$$

(e.g. for the case $u_1 = 0$, $q < 4/3$ implies $B < 2.695$ or $h_2/h_1 < 7.263$)

This wave of depression can be thought of as a form of "reflection" from a bore.

On a beach, such a reflected wave would start when a wave breaks, at least, if it breaks in sufficiently shallow water. However, it would be propagating in the direction of increasing depth and from equation (19) one may deduce that this causes Q to increase Thus for waves incident on a beach the amplitude of the wave would be due to both these opposing influences.

Small amplitude waves and bores

A small amplitude wave is most naturally described in the context of the finite-amplitude shallow-water equations as a small change in P, (or Q) travelling along the C_+ (or C_-) characteristics. There are three ways in which a small amplitude wave can meet a bore travelling in the +x direction, corresponding to the three characteristics; C_+ and C_- in front and C_+ from behind the bore. In all three cases the resulting wave propagates along the C_- characteristic behind the bore. Thus for a wave which originally travelled along a C_+ characteristic, interaction with a bore reverses its direction of travel.

The magnitude of changes in the waves may be found by evaluating $dQ_2/dP_1, dQ_2/dQ_1, dQ_2/dP_2$ at the bore, where suffices 1 and 2 refer to values in front of and behind the bore respectively, and in each derivative the other pair of Riemann invariants are kept constant. A straightforward algebraic manipulation of the differential of the bore relation (17) together with the definitions of P and Q gives

$$\frac{dQ_2}{dQ_1} = \frac{B[U(B^4+B^2+2) + 2(B^4-1)]}{U(2B^4+B^2+1) + 2B(B^4-1)}, \tag{19a}$$

$$\frac{dQ_2}{dP_1} = \frac{B[U(B^4+B^2+2) - 2(B^4-1)]}{U(2B^4+B^2+1) + 2B(B^4-1)}, \tag{19b}$$

$$\frac{dQ_2}{dP_2} = \frac{-U(2B^4+B^2+1) + 2B(B^4-1)}{U(2B^4+B^2+1) + 2B(B^4-1)}, \tag{19c}$$

where $U = (u_2-u_1)/c$. Note, that in terms of U and B the bore relation (17) is

$$2B^2U^2 = (B^2-1)^2(B^2+1). \tag{20}$$

From these results all changes in the wave's properties may be found.

For example, consider a wave dQ_1 meeting the bore on a C_- characteristic. The relative change in amplitude is

$$\frac{dh_2}{dh_1} = \frac{c_2 dc_2}{c_1 dc_1},$$

but $dc_1 = \frac{1}{4} dQ_1$ and $dc_2 = \frac{1}{4} dQ_2$ since $dP_1 = dP_2 = 0$ and thus

$$\frac{dh_2}{dh_1} = B\frac{dQ_2}{dQ_1}$$

Since both $B > 1$ and $dQ_2/dQ_1 > 1$, this wave is always increased in amplitude.

From the geometry of the characteristics in the (x,t) plane, shown in figure 3, the change in length of a small amplitude wave may be calculated;

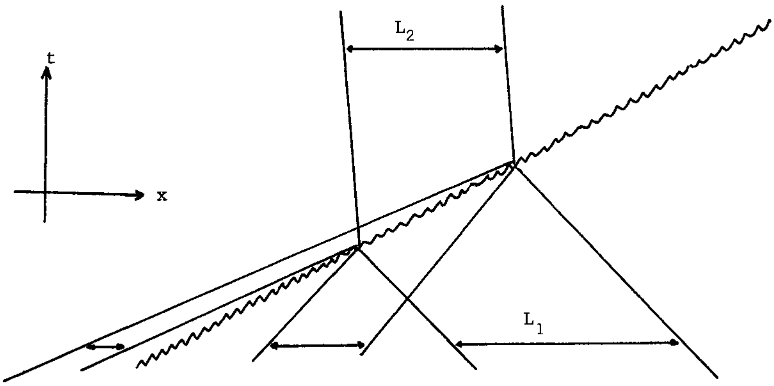

Figure 4. Arrangement of characteristics when a long small amplitude wave meets a bore.

$$\frac{L_2}{L_1} = \frac{B(B^2-1)+U}{B^2-1+B^2U} \quad \text{in this case,} \tag{21a}$$

$$= \frac{B(B^2-1)+U}{B^2U-(B^2-1)} \quad \text{for an incident wave } dP_1, \tag{21b}$$

$$\text{and} \quad = \frac{B(B^2-1)+U}{B(B^2-1)-U} \quad \text{for an incident wave } dP \tag{21c}$$

A better measure of the change in a wave is the change in its volume (area for a two-dimensional wave), i.e.

$$dA = Ldh.$$

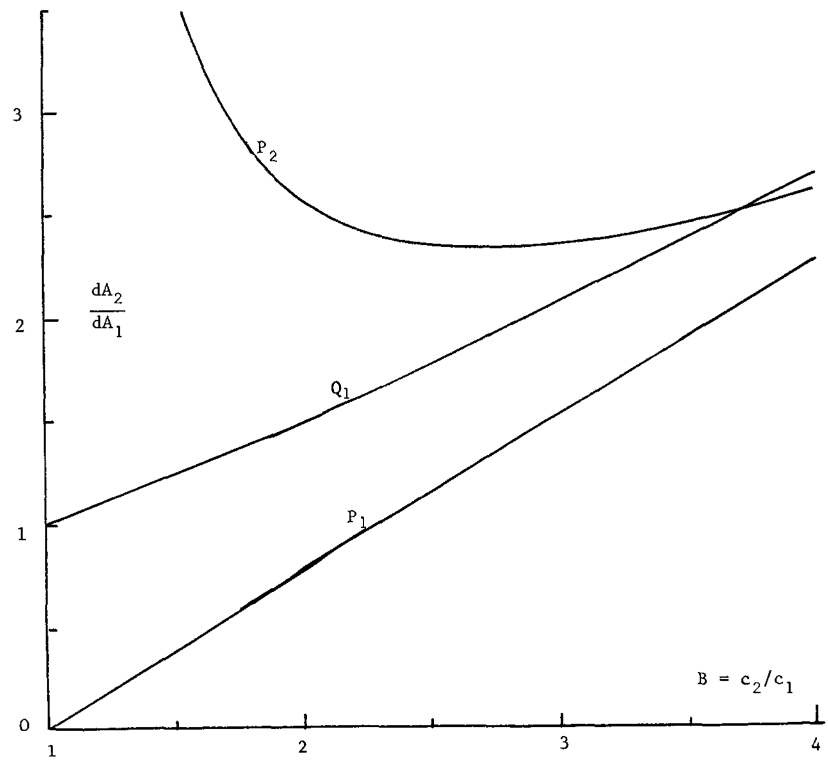

Figure 4. The relative change in volume when a small amplitude wave meets a bore. The curves are labelled with the corresponding Riemann invariants of the incident wave.

Values of dA_2/dA_1 for the three different types of incident wave are shown in figure 4. The change in volume can be explained ny noting that there is a small change dV in the velocity of the bore while it is interacting with the incident wave. A number of cases lead to an amplification of the wave. These results only apply to waves long enough to be described by the shallow-water equations.

The energy density at a point is

$$E = \tfrac{1}{2}\rho u^2 h + \rho g h(\tfrac{1}{2}h - h_0) = \rho(u^2 + c^2 - 2c_0{}^2)c^2/2g, \qquad (22)$$

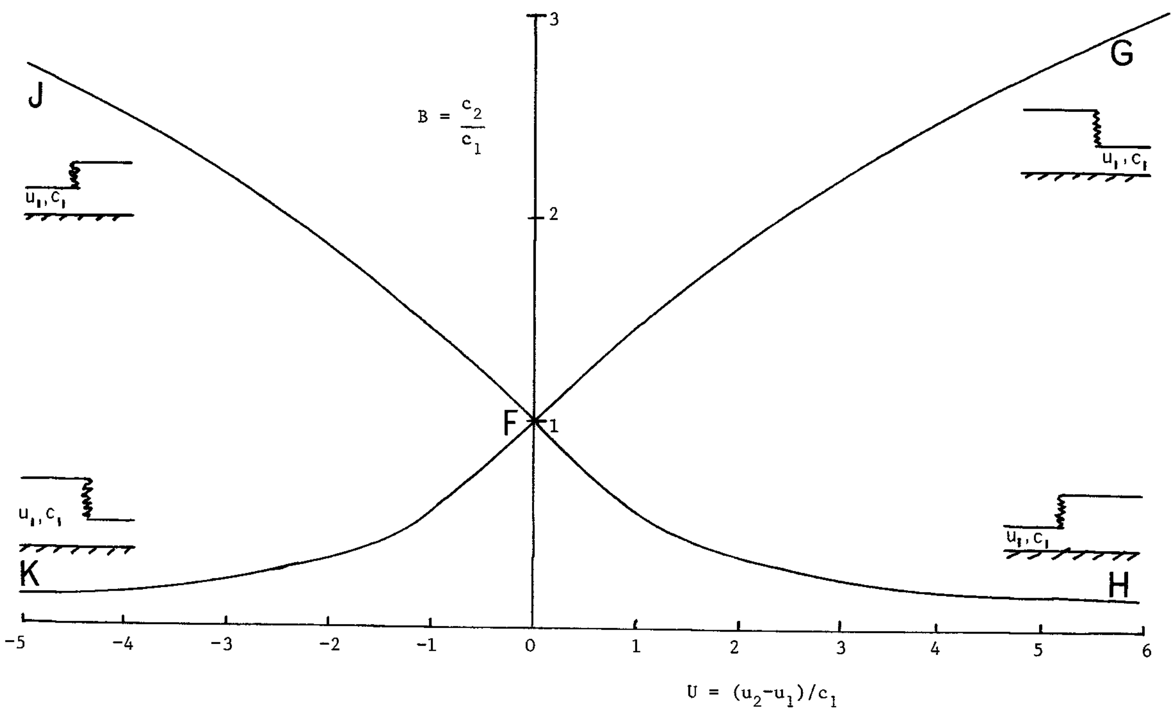

Figure 5. The bore relation, equation (17), in the (u,c) plane. The coordinates are given in terms of non-dimensional variables. The branch KFG represents a bore facing the +x direction, and the branch JFH represents a bore facing the −x direction, as indicated in the small sketches.

so that the energy of an infinitesimal wave is given by

$$gdE/\rho = c^2udu + 2(c^2-c_0^2)cdc$$
$$= \tfrac{1}{2}(c^2-c_0^2 + cu)cdP + \tfrac{1}{2}(c^2-c_0^2-cu)cdQ. \qquad (23)$$

When an infinitesimal wave traverses any finite amplitude flow dE varies, on the other hand for any flow over a flat bottom at least dP or dQ will be constant, e.g. see Peregrine (1967) for a treatment of reflection of an infinitesimal wave by a sloping beach. Thus the energy of an infinitesimal wave is not very useful in this context.

Interaction of bores

When one bore meets, or catches up with, another the result of their interaction may include one or two bores. The interaction is most easily solved by considering the (u,c) plane. The bore relation (17) defines a pair of lines in the (u,c) plane for each point (u_1, c_1). The points on these lines are possible values of (u_2,c_2). This pair of lines is plotted in figure 5 for the point $(0,1)$ (effectively the relation (20) between B and U), for other values of u_1 the lines are simply translated and for other values of c_1 the scale is changed appropriately

When one bore catches another the initial configuration is as shown in figure 6 where bore B_2 is necessarily travelling faster than B_1. This can be displayed in the (u,c) plane by representing the states (u_i,c_i) for $i = 1,2,3$ by points labelled F_i, $i = 1,2,3$ with lines representing the relevant bore relations joining each state, as in figure 7. This makes it clear that after the bores meet a single bore from F_1 to F_3 is not possible.

The alternative form of transition between two levels of water is a simple wave of depression. That is a long wave propagating in the +x(or -x) direction

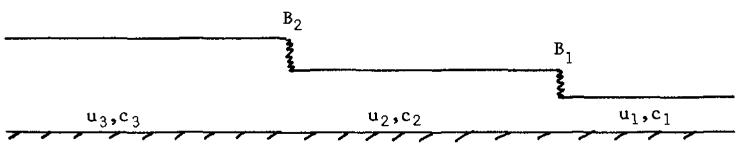

Figure 6. One bore catching another.

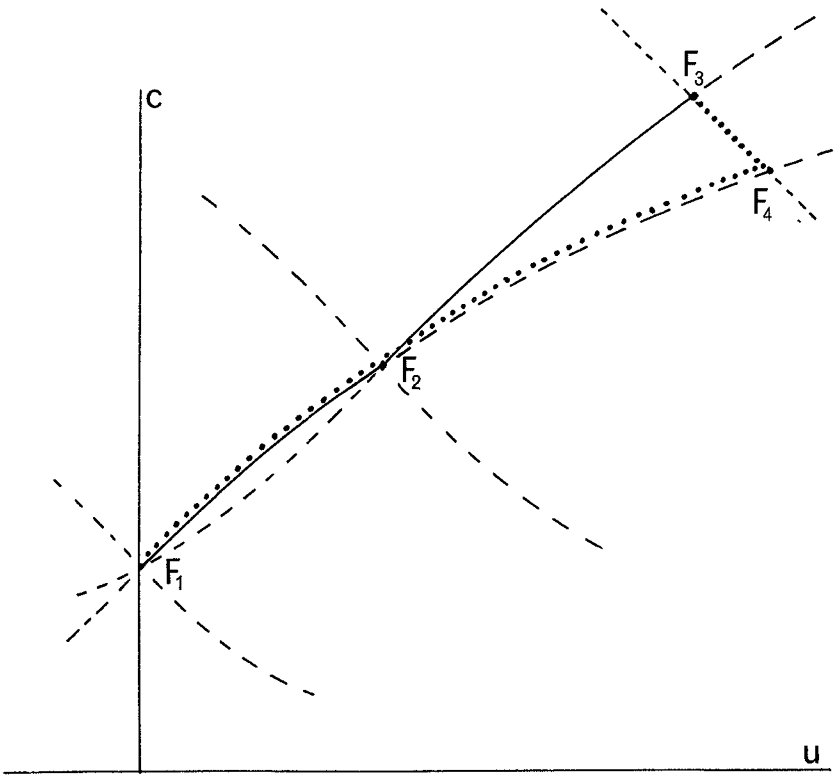

Figure 7. One bore catching another: the (u,c) plane. Bore B_1 connects states F_1 and F_2 , bore B_2 connects states F_2 and F_3 , the corresponding bore relations are shown by a continuous line. After the interaction the single bore B_3 connects states F_1 and F_4 and a simple wave connects states F_4 and F_3 as indicated by the dotted line.

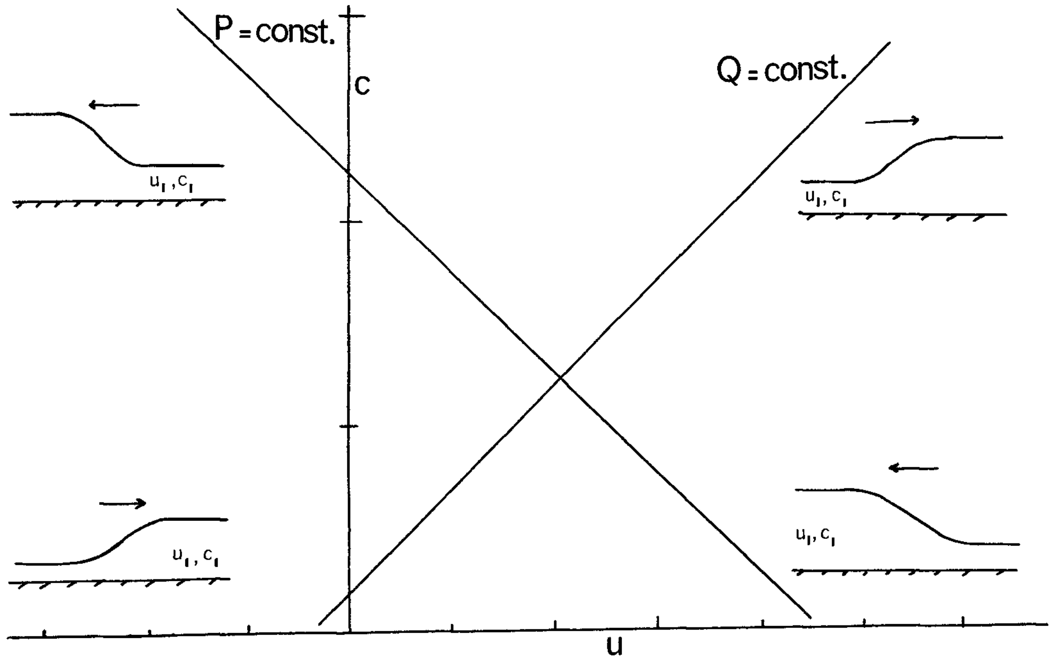

Figure 8. Simple waves of depression in the (u,c) plane, Units arbitrary, but in ratio indicated. Waves Q = constant propagate in the +x direction, and P = constant in the −x direction, as indicated in the small sketches.

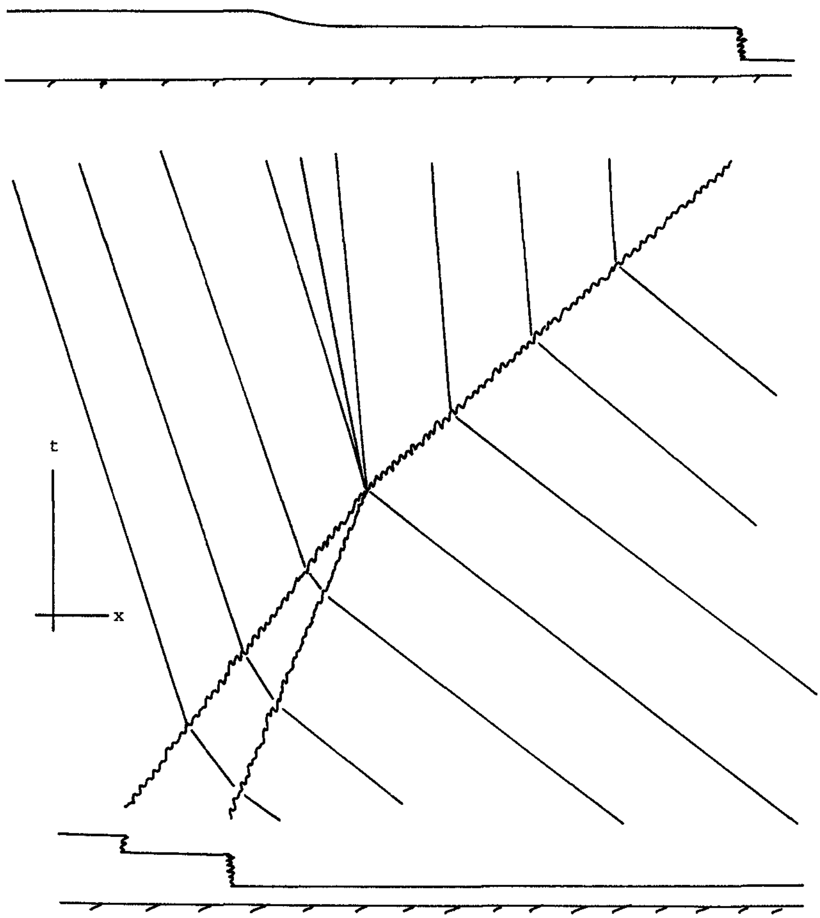

Figure 9 Diagram of (x,t) plane showing one bore catching another, with the C- characteristics and the bore paths shown. Below and above the diagram are representations of the initial and final states.

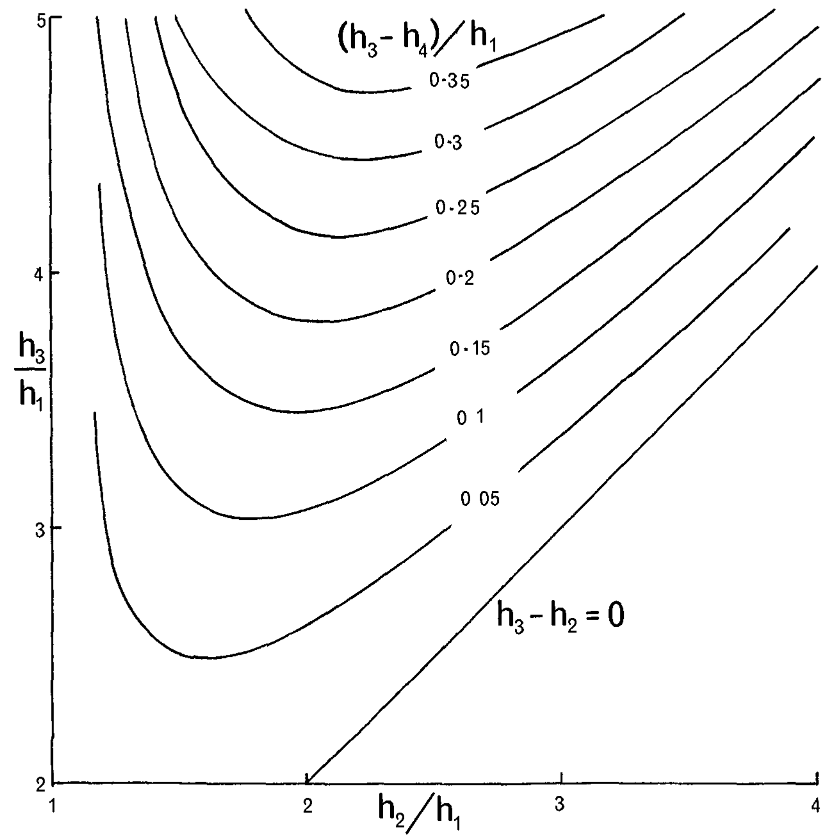

Figure 10. The amplitude of the simple wave resulting from a bore of depth h_3 catching one of depth h_2 propagating into water of depth h_1. The resulting bore has depth h_4 behind it.

with Q (or P) constant. The possible (u,c) states are thus connected by the straight lines $Q = 2c-u$ = constant, (or $P = 2c+u$ = constant) as shown in figure 8. (In the figures of the (u,c) plane the unit for c is chosen to be twice that for u, thus the lines P,Q constant are ate 45° to the u and c axes.) Referring again to figure 7, the only simple wave which can propagate back into the conditions (u_3,c_3) is one with P = constant. Thus after the bores have met the (u,c) diagram is as indicated by the dotted line in figure 7, and the result is a bore plus a simple wave. Figure 9 is a diagram showing characteristics in the (x,t) plane for such an interaction.

The difference of levels may be calculated, using equation (17) and equation (14) applied to the states F_3 and F_4. Numerical results for $(h_3-h_4)/h_1$, shown in figure 10, indicate that the simple wave has a relatively small amplitude e.g. when $h_3 = 5h_1$ the maximum value of (h_3-h_4) is $0.4h_1$ thus $(h_3-h_4) = 0.08h_3$.

Other interactions between bores and simple waves may be analysed in a similar manner. The result of each interaction is given in the table. In the cases marked * the end result depends on the relative sizes of the bore and the wave and the interaction takes a very long time to reach its final state.

r.h.s. / l.h.s	B_+	B_-	W_+	W_-
B_+	W_- B_+	B_- B_+	*	W_- B_+
B_-	do not meet	B_- W_+	do not meet	*
W_+	*	*	do not meet	W_- W_+
W_-	do not meet	*	do not meet	do not meet

B = bore, W = simple wave of depression, suffix $\pm$ represents travel in $\pm$ x direction.

A similar approach is used to study shock wave interaction in gas dynamics (e.g. Courant and Friedrichs, 1948, section IIID). Other solutions of gas-dynamic problems may be transferred to the water-wave context. For example, this paper is confined to two-dimensional examples but the intersection of bores at an angle to each other may be solved by methods used for intersecting shock waves.

Analysis of observations

In the previous section the (u,c) plane is used to analyse bore interactions. It can also be of value in interpreting actual wave motion. If measurements of velocity, u, and of total depth, giving c, are available as functions of time for a point in the surf zone, then a corresponding trajectory can be plotted in the (u,c) plane. If the shore line is in the +x direction the variation of Q will indicate the amount of reflection whereas the variation of P will correspond to the incident waves.

Conclusions

The examples give an indication of some of the non-linear processes acting in the surf zone. An appropriate next step is to use numerical modelling in conjunction with further analytical work to synthesize a qualitative picture of all wave action in the surf zone. There is no intrinsic difficulty in finding numerical solutions and there is plenty of scope for further analysis.

Comparison with experiment and with prototype observations is desirable. Firstly to obtain a good representation of turbulent dissipation, and then to assess the importance of other factors such as a mobile, porous bed and effects at the instantaneous shoreline.

The most intractable problem is probably that of providing an adequate description of wave breaking in order to give a good representation of the input to the surf zone.

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