CHAPTER 24

TRANSFORMATION OF NONLINEAR LONG WAVES

by

Nobuo Shuto

Chuo University, Tokyo, Japan.

SYNOPSIS

Kakutani's equation is extended to include the effects of variable width of the channel and the bottom friction. Based on the equation, several solutions are derived and compared with experimental results. For example, Green's law is obtained if the nonlinearity, dispersion and bottom friction are neglected. With the nonlinearity included, it is shown that the wave amplitude follows Green's law and at the same time the wave profile deforms due to the nonlinear effect.

Discussion of the present paper is mainly focused on the effect of the bottom friction. From the experimental results of cnoidal waves in a channel of constant depth and width, on the bottom of which artificial roughnesses are planted, it is shown that the friction coefficient estimated from Kajiura's theories gives good agreements, thus confirming the validity of the method of conversion, proposed in the present paper, between sinusoidal and cnoidal wave motions.

Change in height of cnoidal waves on a slope is also solved. The friction coefficient determined from wave characteristics and bottom conditions, by means of Kajiura's theories and the method of conversion stated above, is used in the comparison with experimental results. Theoretical prediction agrees very well with experimental results.

INTRODUCTION

In shallow water, long waves transform under several effects such as nonlinearity, dispersion, topography and bottom friction. As for the first three effects, the present author has derived an equation and solved in one of his previous papers [1].

In the present paper, an equation which includes all four effects is derived and, therefore, is considered a fundamental equation for water waves in shallow water. In other words, the equation is an extension of Kakutani's equation [2], because if the bottom friction and the effect of the variable width are neglected, it is reduced to Kakutani's one.

The bottom friction here is assumed to be proportional to the square of the horizontal velocity of water particle. This is normally done if the flow is turbulent near the bottom. We encounter two questions as for the expression of the bottom friction. The one is how we select the representative horizontal velocity, and the second problem is how we can estimate the magnitude of the friction coefficient.

In the following derivations, the horizontal velocity of the first order approximation, that is, the horizontal velocity of the linear long waves is used as the representative velocity. Since it has a uniform vertical distribution, it is easy to connect it with the water surface elevation η, for which the equation is derived. At the same time, there is neither ambiguity nor complexity in the definition of the bottom friction.

For the friction coefficient in an oscillatory flow, Jonnson's empirical formula [3] and Kajiura's theories [4], [5] are available. In the present paper, an attempt is made to connect cnoidal waves with Kajiura's theories which assumed sinusoidal waves. It is assumed here that the mean energy dissipation is the same for sinusoidal and cnoidal waves, thus providing the method of conversion of the friction coefficients.

Kajiura derived his first theory, on assuming that the boundary layer thickness is very big and covers the whole water depth. This corresponds to tsunamis or storm surges in natural conditions.

Kajiura's second theory is for wind waves or swells in shallow water. It is assumed that the boundary layer thickness is very thin compared to the water depth.

In the experiments, artificial rectangular roughnesses are planted on the bottom of the channel and on the slope. For a given size and spacing of the roughnesses, Adachi's empirical formula [6] is used to calculate the coefficient of the bottom friction and the roughness length z_0 in steady flow. The friction coefficient thus obtained is always smaller than that required for unsteady flow. Therefore, the roughness length is used to estimate the friction coefficient for unsteady flow, combined with the horizontal velocity of linear long waves, by Kajiura's theories.

Decay of cnoidal waves in a channel of constant depth and width are at first tested in order to examine the validity of the expression of the bottom friction. Dimensions of the roughnesses used in the experiments are big for the water depth, compared with the natural condition. The friction coefficients computed from experimental results are quite big compared with that estimated for steady flow and fall between two Kajiura's theories. Therefore, it is concluded that the present theory combined with Kajiura's theories provides reasonable basis for transformation of nonlinear long waves in shallow water.

Experiments are also carried out on a slope of 1 on 20, on which the same artificial roughnesses are planted. Mean value of the friction coefficient is used to predict the change in wave height and the results agree very well with experimental results.

FORMULATION

The equations to be solved are;

$$\left.\begin{aligned} & u_x + v_y + w_z = 0 \\ & u_t + uu_x + vu_y + wu_z = -\frac{1}{\rho} p_x + \frac{1}{\rho} \tau_z \\ & v_t + uv_x + vv_y + wv_z = -\frac{1}{\rho} p_y \\ & w_t + uw_x + vw_y + ww_z = -\frac{1}{\rho} p_z - g \end{aligned}\right\} \text{------ (1)}$$

in which the x-axis is taken horizontally and parallel to the direction of wave propagation, the y-axis horizontally and normal to the x-axis, and the z-axis vertically and positive upwards. It is assumed that the bottom friction mainly contributes in the x-direction only.

Boundary conditions are as follows.
On the free surface $z = h_0 + \eta$,

$$\left.\begin{aligned} p &= 0 \\ \eta_t + u\eta_x + v\eta_y &= w \\ \tau &= 0 \end{aligned}\right\} \text{----------------- (2)}$$

On the sea bottom $z = h(x)$

$$\left.\begin{aligned} uh_x &= w \\ \tau &= \tau_b \end{aligned}\right\} \text{----------------- (3)}$$

At the side wall of the channel $y = \pm b(x)$,

$$ub_x = \pm v \qquad \text{----------------- (4)}$$

In these expressions, it is assumed that the centerline of the channel coincides with the x-axis and the water depth does not vary in the y-direction.

Equations and conditions are expressed in dimensionless form by using Johnson's method [7] and are expanded into series by Kakutani's method of perturbation [2]. The details are not stated here because they are almost the same as are given in one of the author's previous papers, except that the bottom friction is developed as $\tau = \varepsilon^{1/2} \tau_0 + \varepsilon^{3/2} \tau_1 + $ ----. With this expression, the effect of the bottom friction does not appear in the first order approximation but in the second approximation.

Solutions of the first order approximation are given as follows.

$$u_0 = \frac{1}{V_0}\eta_0, \; v_0 = 0, \; w_0 = \frac{1}{d}\eta_{0\xi}(1 - d - z), \; p_0 = \eta_0 \qquad (5)$$

where every quantities are expressed in terms of the elevation of free surface η_0. Up to this point, no restriction is given to the wave profile.

Second order approximation is solved and every terms are expressed again in terms of η_0 and are integrated once with respect to the y-direction in order to include the side wall condition in the equation, and vertically once thus taking into consideration the bottom and free surface conditions. Finally we have the following equation for the first order surface elevation in dimensional expression.

$$\eta_x + \frac{3}{2} g^{-1/2} d^{-3/2} \eta\eta_\xi + \frac{1}{6} g^{-3/2} d^{1/2} \eta_{\xi\xi\xi} + \frac{1}{4} d^{-1} d_x \eta + \frac{1}{2} b^{-1} b_x\eta + \frac{1}{2} C_1 d^{-2} \eta|\eta| = 0 \qquad (6)$$

in which the subscript $_0$ is omitted for simplicity. Letter subsripts in the equation denote differentiation with respect to them.

The first term in the equation denotes the spatial rate of change in wave profile and others are the causes. The second term is the effect of finite amplitude, referred hereafter, as the effect of nonlinearity. The third is the effect of dispersion. The fourth is the effect of the variable water depth. With these four terms only, the equation is reduced to Kakutani's equation.

The fifth term gives the effect of the variable width of the channel. With this term included, the equation is extended to two dimensional cases which are important in practical problems such as refraction problem. The bottom friction is given by the last term in which the bottom friction is expressed as $\tau = \rho C_1 u|u|$.

Notations used are as follows. The term x denotes the horizontal distance, ξ a modified time defined by $\xi = \int(1/\sqrt{gd})dx - t$, t the actual time, g the acceleration of gravity, d the undisturbed water depth, b the width of the channel and C_1 the friction coefficient.

EXAMPLES OF SOLUTION

Decay of linear long waves

Equation (6), with the second and third terms neglected, is reduced to

$$\frac{\eta_x}{\eta} + \frac{1}{2}\frac{b_x}{b} + \frac{1}{4}\frac{d_x}{d} + \frac{C_1}{2}\frac{|\eta|}{d^2} = 0 \qquad (7)$$

and the solution is

$$\eta\, b^{1/2}\, d^{1/4} = \eta_0\, b_0^{1/2}\, d_0^{1/4}\, exp[- \int_{x_0}^{x} \frac{C_1}{2} \frac{|\eta|}{d^2}\, dx] \qquad \text{-------- (8)}$$

If the friction term is neglected, the equation is reduced to Green's law of shoaling.

For a sinusoidal wave in a channel of constant depth and width, when the effect of bottom friction is assumed small enough to be replaced by an equivalent linear friction, then for a small travel distance x, the wave profile is given by

$$\eta = \eta_0\, exp[- \frac{4a_0}{3} \frac{x}{d^2}] \qquad \text{--------------(9)}$$

where a_0 is the initial wave amplitude.

Decay of shallow water waves due to bottom friction

It is assumed here that the depth and width of the channel are constant. Equation (6) is reduced to the following expression.

$$\eta_x + \frac{3}{2} g^{-1/2} d^{-3/2} \eta\eta_\xi + \frac{C_1}{2} d^{-2} \eta|\eta| = 0 \qquad \text{------------- (10)}$$

For the positive η. the equation is simply expressed by

$$\eta_x + A\eta\eta_\xi + B\eta^2 = 0 \qquad \text{------------- (11)}$$

and its solution which satisfies the boundary condition $\eta = f(-t)$ at $x = x_0$ is given by

$$\eta = \frac{f[\xi + \frac{A}{B}\, ln\{1 - B\eta(x - x_0)\}]}{1 + B(x - x_0)\; f[\, \xi + \frac{A}{B}\, ln\{1 - B\eta(x - x_0)\}]} \qquad \text{------- (12)}$$

For the negative η, B in the equation is replaced by -B. In these expression $A = 3/2 \cdot g^{-1/2}\, d^{-3/2}$, and $B = C_1 d^{-2}/2$.

CHANGE IN WAVE HEIGHT OF CNOIDAL WAVES

Equation

For simplicity, equation (6) is written as

$$\eta_x + \alpha_1 \eta \eta_\xi + \alpha_2 \eta_{\xi\xi\xi} + \alpha_3 \eta |\eta| + \alpha_4 \eta = 0 \qquad (13)$$

The wave profile is divided into two parts; the one is the principal part η_0 which is cnoidal waves and the other minor part a modification.

$$\eta_0 = H\, cn^2\beta + \delta, \quad \beta = A[\xi - Bx], \quad A = \frac{1}{k}\left[\frac{3}{4}\frac{gH}{d^2}\right]^{1/2} \qquad (14)$$

It is assumed that the principal part of the wave profile always keeps the cnoidal shape. Main change can occur in its wave height, phase and the position of wave trough. And even if the higher order term is included, the latter should not increase secularly with respect to x. Otherwise, at some time later, the magnitude of the higher order term exceeds that of the basic cnoidal waves and, then, the perturbation applied here is no longer valid. As for the details of the manipulation, one can refer References [1], [8] and [9].

Under this assumption, the following equation is derived and gives the change in wave height.

$$\frac{1}{H}\frac{dH}{dx} - \frac{1}{2U}\frac{dU}{dx} + \alpha_4 + \frac{\delta}{H}\frac{I_2}{I_1}\left[\frac{1}{H}\frac{dH}{dx} - \frac{1}{U}\frac{dU}{dx} + \alpha_4\right] + \alpha_3 H \frac{I_0}{I_1} = 0 \qquad (15)$$

where U^2 is a kind of Ursell's parameter defined by gHT^2/d^2, and others ar are as follows.

$$\left.\begin{aligned}
\alpha_3 &= \frac{C_1}{2} d^{-2} \\
\alpha_4 &= \frac{1}{2} b^{-1} b_x + \frac{1}{4} d^{-1} d_x \\
\delta &= -\frac{H}{k^2}\left[\frac{E}{K} - k'^2\right] \\
I_0 &= \int_{-K}^{K} \left(cn^2\beta + \frac{\delta}{H}\right)^2 \left|cn^2\beta + \frac{\delta}{H}\right| d\beta \\
I_1 &= \int_{-K}^{K} cn^4\beta\, d\beta \\
I_2 &= \int_{-K}^{K} cn^2\beta\, d\beta
\end{aligned}\right\} \qquad (16)$$

There are difficulties in the above equation if one want to solve it analytically. They are due to two terms, $\delta I_2/HI_1$ and I_0/I_1.

Approximations of $\delta I_2/HI_1$ and I_0/I_1 for large Ursell's parameter

For Ursell's parameter larger than 50, the following approximation was obtained in Ref. [1],

$$\frac{\delta I_2}{HI_1} = -\frac{2\sqrt{3}}{U} \qquad (17)$$

and if we allow errors of 3.5 % at most, this approximation is extended down to $U^2 = 40$.

As for another coefficient I_0/I_1, we have to approximate it in a form convenient for integration. At first the integral I_0 is rewritten as follows for convenience of numerical computation.

$$\begin{aligned} I_0 &= \int_{-k}^{k} (cn^2\beta + \frac{\delta}{H})^2 \, |cn^2\beta + \frac{\delta}{H}| \, d\beta \\ &= 2\int_0^{k} (cn^2\beta + \frac{\delta}{H})^2 \, |cn^2\beta + \frac{\delta}{H}| \, d\beta \\ &= 2\int_0^{\beta_1} (cn^2\beta + \frac{\delta}{H})^3 \, d\beta - 2\int_{\beta_1}^{k} (cn^2\beta + \frac{\delta}{H})^3 \, d\beta \\ &= 4\int_0^{\beta_1} (cn^2\beta + \frac{\delta}{H})^3 \, d\beta - 2\int_0^{k} (cn^2\beta + \frac{\delta}{H})^3 \, d\beta \end{aligned} \qquad (18)$$

where β_1 is the value which satisfies

$$cn^2\beta_1 + \frac{\delta}{H} = 0 \qquad (19)$$

The above equation is approximated for large Ursell's parameter, that is, for large K, by

$$I_0 = \frac{16}{15} - \frac{16}{\sqrt{3}} U^{-1} + 32\, U^{-2} \qquad (20)$$

Figure 1 shows the comparison between numerical value of I_0 and its approximation, Eq.(20). Taking into consideration the fact that Eq.(20) is no longer applicable down to $U^2 = 100$ and is still inconvenient for integration, a set of I_0/I_1 shown in Table 1 is proposed as the approximation for integration.

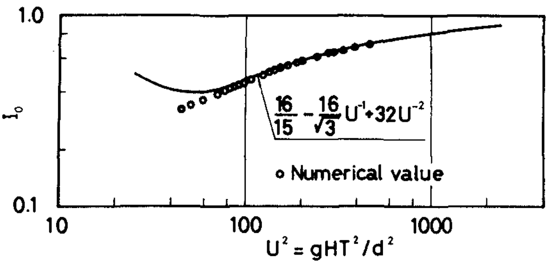

Fig.1. I_0 as a function of Ursell's parameter

U^2	I_0 / I_1
40 - 200	$0.05254\ U^{4/5}$
200 - 1 000	$0.1516\ U^{2/5}$
1 000 - 2 500	$0.3025\ U^{1/5}$
2 500 -	$\frac{4}{5}(1 - \frac{5\sqrt{3}}{U})$

Table 1. Approximate values of I_0/I_1

Case 1. Constant depth and constant width

To examine the validity of the expression of the bottom friction and to correlate the friction coefficient for cnoidal waves to Kajiura's theories, solutions without the topographical effect are derived and are to be compared with experimental results. Since d is constant in this case, Ursell's parameter varies with wave height H. Then, we have

$$U^2 = \frac{gT^2}{d^2} H = \frac{H}{D}, \qquad \frac{H_x}{H} = 2\frac{U_x}{U} \qquad \text{---------------} \quad (21)$$

where D is d^2/gT^2 and takes a constant value, because the wave period T is also considered constant.

For Ursell's parameter between 40 and 2500, the coefficient I_0/I_1 is expressed by

$$\frac{I_0}{I_1} = FU^{m/5} \qquad \text{------------------} \quad (22)$$

where F and m are constant for given ranges of Ursell's parameter and are given in Table 1.

Equation (15) is now reduced to

$$\frac{3}{2}\left[U^{-1} - \frac{4}{\sqrt{3}} U^{-2} \right] \frac{dU}{dx} + \alpha_3 DFU^{(10 + m)/5} = 0 \quad \text{------- (23)}$$

or further to

$$\left[U^{-1-(10 + m)/5} - \frac{4}{\sqrt{3}} U^{-1-(15 + m)/5} \right] dU + \frac{2}{3}\alpha_3 DF dx = 0$$

$$\text{-------------------- (24)}$$

and the solution is given by

$$\left(\frac{U_0}{U} \right)^{(10 + m)/5} \left[1 - \frac{4}{\sqrt{3}} \frac{10 + m}{15 + m} \frac{1}{U}\right] - \left[1 - \frac{4}{\sqrt{3}} \frac{10 + m}{15 + m} \frac{1}{U_0} \right]$$

$$= \frac{10 + m}{15} \frac{C_1}{gT^2} Fx\ U_0^{(10 + m)/5} \quad \text{-------------------- (25)}$$

where the subscript $_0$ denotes the value at a reference point x = o.

Although the wave height is given by Eq.(25), the equation is still a little complicated. More simple relationship, if obtained, is convenient for practical purpose. Therefore, the Ursell's parameter in the first square bracket is replaced by U_0. A numerical examination shows that if the wave height H remains larger than $0.8H_0$, this replacement yields 3 % errors at most. The wave height H_0 is the wave height at the beginning of an interval where the bottom friction is not negligible and H is the wave height at its end. With the distance x of the interval well chosen so as to satisfy this restriction, wave height can be predicted by simple formulae as shown in Table 2.

For Ursell's parameter larger than 2500, Eq.(15) becomes of the following form.

$$\left(\frac{3}{2} \frac{1}{U} - \frac{2\sqrt{3}}{U^2} \right) \frac{dU}{dx} + \frac{4}{5} \left(1 - \frac{5\sqrt{3}}{U}\right)\alpha_3 DU^2 = 0 \quad \text{------------ (26)}$$

For this range of Ursell's parameter, $5\sqrt{3}\ U^{-1}$ is not larger than 0.17 and the following approximation is used in the integration.

$$\left(1 - \frac{4}{\sqrt{3}} \frac{1}{U}\right)/\left(1 - 5\sqrt{3}\ \frac{1}{U}\right) \simeq 1 + \frac{11}{\sqrt{3}} \frac{1}{U}$$

The solution is given as follows.

$$\left(\frac{U_0}{U}\right)^2\left[1+\frac{22}{3\sqrt{3}}\frac{1}{U}\right]-\left[1+\frac{22}{3\sqrt{3}}\frac{1}{U_0}\right]=\frac{16}{15}\alpha_3 D U_0^2 x \qquad \text{---------} \quad (27)$$

Again the fact that $22/3\sqrt{3}\cdot U^{-1}$ is smaller than 0.15 is taken into consideration and U in the first square bracket is replaced by U_0. The result is also listed in Table 2.

For a solitary wave as a limit when Ursell's parameter tends to infinity, we have

$$H = H_0/\left(1+\frac{8}{15}\frac{C_1 H_0 x}{d^2}\right) \qquad \text{---------------------} \quad (28)$$

If one prefer Manning's n in place of C_1 used in the present paper, he can replace C_1 by $gn^2d^{-1/3}$ although there is an unsolved problem, that is, how to determine the value of n for unsteady flow.

$40<gHT^2/d^2<200$

$$\left(\frac{H_0}{H}\right)^{7/5}-1 = 0.0490\,\frac{C_1\,x}{gT^2}\left(\frac{gH_0T^2}{d^2}\right)^{7/5}\Big/\left[1-1.70\left(\frac{gH_0T^2}{d^2}\right)^{-1/2}\right]$$

$200 < gHT^2/d^2 < 1000$

$$\left(\frac{H_0}{H}\right)^{6/5}-1 = 0.121\,\frac{C_1\,x}{gT^2}\left(\frac{gH_0T^2}{d^2}\right)^{6/5}\Big/\left[1-1.63\left(\frac{gH_0T^2}{d^2}\right)^{-1/2}\right]$$

$1000 < gHT^2/d^2 < 2500$

$$\left(\frac{H_0}{H}\right)^{11/10}-1 = 0.222\,\frac{C_1\,x}{gT^2}\left(\frac{gH_0T^2}{d^2}\right)^{11/10}\Big/\left[1-1.59\left(\frac{gH_0T^2}{d^2}\right)^{-1/2}\right]$$

$2500 < gHT^2/d^2$

$$\frac{H_0}{H}-1 = \frac{8}{15}\frac{C_1\,H_0x}{d^2}\Big/\left[1+4.23\left(\frac{gH_0T^2}{d^2}\right)^{-1/2}\right]$$

Table 2. Decay of wave height in case of $\alpha_4 = 0$

Case 2. Cnoidal waves in general case

Almost similar calculation is possible for this case except that the water depth in Ursell's parameter is also variable. The following relationship should be substituted in place of Eq.(21).

$$\frac{2}{U}\frac{dU}{dx} = \frac{1}{H}\frac{dH}{dx} - \frac{2}{d}\frac{dd}{dx} \qquad (29)$$

The equation to be solved is

$$\frac{1}{U}\frac{dU}{dx} + \frac{1}{3}\frac{1}{b}\frac{db}{dx} + \frac{3}{2}\frac{1}{d}\frac{dd}{dx} + \frac{2}{3}\frac{\delta}{H}\frac{I_2}{I_1}\left[\frac{1}{U}\frac{dU}{dx} + \frac{1}{2}\frac{1}{b}\frac{db}{dx} + \frac{9}{4}\frac{1}{d}\frac{dd}{dx}\right]$$

$$+ \frac{2}{3}\alpha_3 H\frac{I_0}{I_1} = 0 \qquad (30)$$

For Ursell's parameter between 40 and 2500, the equation is further reduced to the following with the aid of Eqs.(17) and (22).

$$\frac{1}{U}\frac{dU}{dx}\left[1 - \frac{4}{\sqrt{3}}\frac{1}{U}\right] + \frac{1}{2}\left[\frac{1}{b}\frac{db}{dx} + \frac{9}{2}\frac{1}{d}\frac{dd}{dx}\right]\left[\frac{2}{3} - \frac{4}{\sqrt{3}}\frac{1}{U}\right]$$

$$+ \frac{2}{3}\frac{C_1}{2gT^2}F\,U^{(10+m)/5} = 0 \qquad (31)$$

The solution is given by

$$U^2(U - 2\sqrt{3})\, b\, d^{9/2}\, exp\left[\int\frac{C_1F}{gT^2}\frac{U^{(15+m)/5}}{U - 2\sqrt{3}}dx\right] = \text{const.} \qquad (32)$$

The value of U in the exponential function of the equation can be replaced by U_0 under the same consideration as in case of constant depth and width of the channel. In terms of wave height H, the relationships are shown in Table 3.

For Ursell's parameter larger than 2500, the equation is given by

$$\frac{2}{U}\frac{dU}{dx} + \frac{1}{U - 2\sqrt{3}}\frac{dU}{dx} + \frac{1}{b}\frac{db}{dx} + \frac{9}{2}\frac{1}{d}\frac{dd}{dx} + \frac{4}{5}\frac{C_1}{gT^2}U^2\frac{U - 5\sqrt{3}}{U - 2\sqrt{3}} = 0 \qquad (33)$$

and the solution is

$$\frac{U^2(U - 2\sqrt{3})\ b\ d^{9/2}}{U_0{}^2(U_0 - 2\sqrt{3})\ b_0 d_0{}^{9/2}} = exp[-\frac{4}{5}\frac{C_1}{gT^2} U^2 \frac{U - 5\sqrt{3}}{U - 2\sqrt{3}} dx] \qquad (34)$$

In terms of H, the result is also shown in Table 3.

With C_1 set equal to zero, that is, with no friction, the right-hand sides of the results in Table 3 are reduced to unity and they coincide with the result of the author's previous paper which discussed the shoaling of cnoidal waves.

$$\frac{Hb\ d^{5/2}\ (\sqrt{gHT^2/d^2} - 2\sqrt{3})}{H_0 b_0 d_0{}^{5/2}\ (\sqrt{gH_0T^2/d_0^2} - 2\sqrt{3})}$$

$40 < gHT^2/d^2 < 200$

$$= exp[-\frac{0.05254\ C_1 x}{gT^2} (\frac{gH_0T^2}{d_0^2})^{9/5} /\{\sqrt{gH_0T^2/d_0^2} - 2\sqrt{3}\}]$$

$200 < gHT^2/d^2 < 1000$

$$= exp[-\frac{0.1516\ C_1 x}{gT^2} (\frac{gH_0T^2}{d_0^2})^{17/10} /\{\sqrt{gH_0T^2/d_0^2} - 2\sqrt{3}\}]$$

$1000 < gHT^2/d^2 < 2500$

$$= exp[-\frac{0.3025\ C_1 x}{gT^2} (\frac{gH_0T^2}{d_0^2})^{8/5} /\{\sqrt{gH_0T^2/d_0^2} - 2\sqrt{3}\}]$$

$2500 < gHT^2/d^2$

$$= exp[-\frac{4C_1 H_0 x}{5\ d_0^2} \{\sqrt{gH_0T^2/d_0^2} - 5\sqrt{3}\}/\{\sqrt{gH_0T^2/d_0^2} - 2\sqrt{3}\}]$$

Table 3. Change in wave height of cnoidal waves under the effects of variable water depth, variable channel width and bottom friction

EXPERIMENTAL PROCEDURE

Experiments were carried out in a wave flume 50 m long, 1 m wide and 1 m high. First series of the experiments was carried out in a channel of constant depth. Water depths during the experiments were kept 10 cm, 20 cm and 30 cm. Wave period was kept constant as 2 sec. Wave heights were varied between 2 cm at minimum and 12 cm at maximum. At the end of the channel, a permeable slope of 1 on 25 was installed,packed with waste films. Even with this wave absorber, we had normally 7-8 % reflection from this end of the channel. The maximum reflection observed during experiments was about 10 %.

Artificial rectangular roughnesses were planted on the bottom in the middle part of the channel. Height and width of a roughness are the same and 8 mm. Its length is 1 m and can cover the whole width of the channel. Spacing of the roughnesses is 8 cm. The roughnesses were arranged at right angles to the direction of wave propagation. Total length of roughened bed measured along the direction of wave propagation is 10 m. From the beginning of this area, wave heights are measured at every 1 m intervals. Since the reflection is not completely negligible, we draw, by using the experimental results, average curves of the change in wave height, from which the friction coefficient C_1 is estimated.

In the second series of the experiments, a slope of 1 on 20 was installed. Total horizontal length of the slope was 6 m. Water depths at the toe of the slope were 40 cm, 45 cm and 50 cm. From the upper end of the slope, continues another horizontal bed, on which water depths during the experiments were 10 cm, 15 cm and 20 cm. The same roughnesses as in the first series of experiments were planted on the surface of the slope. Wave height was measured at every 50 cm intervals on the slope. Wave periods were varied between 2 sec and 10 sec. Wave heights at the toe of the slope were varied from 1.5 cm to 10 cm.

COMPARISON BETWEEN THEORY AND EXPERIMENT

Characteristics of roughness in steady flow

According to Adachi's empirical formula [6], the effect of the rectangular ribs arranged at right angles to the stream is expressed in terms of Nikuradse's equivalent sand roughness k_s' by the following relationship.

$$k_s' = 30\ k\ m\ (\ R_1/k\ )^{-\theta} \qquad \text{------------} \quad (35)$$

where R_1 is the hydraulic radius of the channel, k is the height of a roughness, s is the spacing of the roughnesses, m and θ are functions of s and k which were experimentally determined by Adachi.

$$m = 0.79\ (s/k\ )^{-0.26}$$

$$\theta = 0.02\ (s/k\ )^{0.8} \qquad \text{---------------- (36)}$$

The roughness length z_0 is defined by $k_s'/30$. For example, $z_0 = 0.23$ cm for d = 30 cm, k = 8 mm and s = 8 cm.

The friction coefficient C_1 for steady flow is computed by

$$C_1 = [\ 6.0 + 5.75\ log_{10}\ (\ d/k_s'\ )\ ]^{-2} \qquad \text{---------------- (37)}$$

and is 0.0227, 0.0137 and 0.0108 for water depth 10 cm, 20 cm and 30 cm, respectively, in the first series of the experiments.

Estimation of the friction coefficient by Kajiura's theories

Field data and experimental results obtained by many researchers show that the friction coefficient under wave action is different from that in steady flow. Kajiura established two theories, on considering the average state of turbulence over one wave period and adopting the assumption of the eddy viscosity analogous to that for the steady turbulent flow. According to his theories, the frictional coefficient is given as a function of certain dimensionless parameters constructed from known quantities of wave and bottom conditions.

Kajiura's theories are based on the assumption that the oscillatory motion is sinusoidal and the present theory used the cnoidal waves. Direct substitution of the results of Kajiura's theory is not recommended. In order to connect two different definitions of the friction coefficients, mean energy dissipation of sinusoidal motion is equated with that of cnoidal waves.

For the case of long period waves [4], in which the flow is fully turbulent, the mean energy dissipation is

$$< E > = \frac{\rho}{2} \frac{8}{3\pi} \hat{C}_{k1} cos\theta\ \hat{U}\ \hat{\bar{u}}^2 \qquad \text{---------------- (38)}$$

where $\hat{\bar{u}}$ is the amplitude of the mean velocity vertically averaged over the water depth. The angle θ denotes the phase lead of the bottom stress relative to the vertically averaged velocity and $\hat{U}$ is the amplitude of a formally defined velocity corresponding to the pressure gradient.

For the case of short period waves [5], in which the thickness of the bottom frictional layer is very thin compared with the total depth of water, the mean energy dissipation is given by

$$\langle E \rangle = \frac{\rho}{2} \hat{C}_{k2} \cos\theta \, \hat{U}^3 \quad \text{------------} \quad (39)$$

where $\hat{U}$ is the amplitude of the horizontal velocity at the top of the bottom frictional layer. The angle θ denotes the phase lead of the bottom stress relative to the velocity at the top of the frictional layer.

For cnoidal waves in the present paper, the mean energy dissipation is given by

$$\langle E \rangle = \frac{\rho}{2} \, 16 \, I \, C_1 \hat{u}^3 \quad , \quad I = I_0 / 2K \quad \text{------------} \quad (40)$$

where $\hat{u}$ is the amplitude of the horizontal velocity which is vertically uniform and I is given in Table 4 in simple expressions convenient for practical application.

Equating these three formulas, the coefficient used in the present paper can be estimated from Kajiura's theories.

$$\hat{C}_{k2} \cos\theta = \frac{8}{3\pi} \hat{C}_{k1} \cos\theta \left(\frac{\hat{u}}{\hat{U}} \right)^2 = 16 \, I \, C_1 \left(\frac{\hat{u}}{\hat{U}} \right)^3 \quad \text{----} \quad (41)$$

$gHT^2/d^2 = U^2$	I
40 - 200	$0.08089 \, U^{-1/5}$
200 - 1000	$0.2334 \, U^{-3/5}$
1000 - 2500	$0.4657 \, U^{-4/5}$
2500 -	$\frac{32}{15\sqrt{3}} \left(1 - \frac{5\sqrt{3}}{U} \right) \frac{1}{U}$

Table 4. Values of I

Experimental results in case of horizontal bottom

The friction coefficients are determined from the experimental results, converted, by means of Eq.(41), to Kajiura's definition of the friction coefficient, and plotted in Fig.2. In the same figure, Kajiura's theories are given by two oblique lines, and three short horizontal lines correspond to the friction coefficients in steady flow.

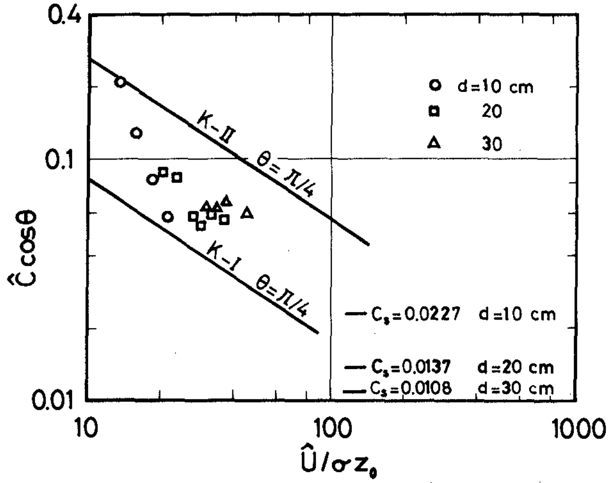

Fig. 2. Friction coefficients compared with Kajiura's theories and Adachi's formula

The friction coefficients in steady flow show very small values and are unable to be used for practical estimation of decay of water waves in shallow sea. The experimental results fall between Kajiura's theories. This suggests that the conversion formula given above is practically valid.

Taking into account the fact that the size of the roughness in the experiments is very big compared to the water depth and this condition does not frequently occur under natural conditions, it is concluded that the present theory with the friction coefficient converted from Kajiura's first theory can be used to predict the change in height of tsunamis and storm surges, while if combined with Kajiura's second theory it can be used to compute the case of wind waves and swells in shallow water.

Change in wave height on a slope

Since the method of evaluating the friction coefficient from given wave and bottom conditions is established, the present theory is compared with the second series of experiments. Figure 3 shows an example. Although values of friction coefficient varies with water depth on the slope, average value, 0.1 for this case, is used in calculation. Broken line in the figure shows the shoaling of cnoidal waves when no friction is taken into account. Solid line is with friction and agrees farely well with experimental results. Sctteting of the experimental data is considered due mainly to the reflection from the slope and the wave absorber installed at the end of the flume.

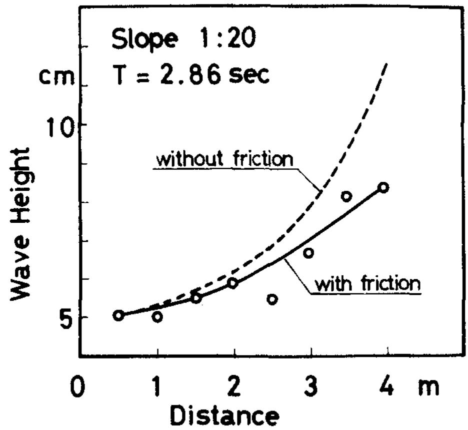

Fig.3. Change in wave height on a slope.

CONCLUSIONS

A fundamental equation for long waves is derived by using Kakutani's and Johnson's methods. The equation includes the effects of nonlinearity, dispersion, water depth, width of the channel and bottom friction.

In addition to several solutions obtained in analytical form, the change in wave height of cnoidal waves is given and listed in Tables 2 and 3. In order to obtain as simple and convenient formulas as possible, imposed, during the derivation, were restrictions which should be remembered at application. The length of an interval,x, should be well chosen so as that the wave height H at the end of the interval remains larger than 0.8 times H_0 , the wave height at the beginning of the interval. If this restriction is not welcome, Eqs.(25) and (27) should be used in place of formulas in Table 2, or H_0 in the right-hand sides of the equations in Table 3 should be replaced by H.

The friction coefficient C_l is estimated from known quantities of wave and bottom conditions. In the present experiments, artificial rectangular ribs are used, the roughness length, z_0 , of which is evaluated from Adachi's empirical formuls. This roughness length and wave characteristics gives the friction coefficient for sinusoidal waves, if one follows Kajiura's theories. Kajiura assumed sinusoidal waves and the present theory cnoidal waves. Conversion of the friction coefficient between the two different motions is possible through Eq.(41), provided that the mean energy dissipation is the same for the two motions.

From the first series of experiments, the method of estimation of the friction coefficient is confirmed. Results of the second series of the experiments show that the theoretical prediction of the change in wave height on a slope agrees very well with experiments.

It is concluded that the present theory combined with Kajiura's first thoery which assumes that the flow is fully turbulent gives good estimation of the change in wave height of tsunamis and storm surges, and that with Kajiura's second theory which assumes a thin bottom fricional layer the theory predicts the change in height of swells and wind waves in shallow water.

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