CHAPTER 50

Resonant Refraction by Round Islands

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Wave trapping by refraction can give rise to resonance of a kind unfamiliar in oceanography. Trapping over realistic seabed topographies is incomplete, but conversely, possesses a mechanism for direct, harmonic excitation from the open sea that is unknown in classical resonance. These phenomena have been studied for the simplest example of an axisymmetric island of typical shape with small seabed slope. Asymptotic analysis has led to simple formulae for resonant frequencies, energy leakage rates and resonant response coefficients. Resonances of extraordinarily large but narrow response have been found.

I. Introduction

Localized natural surface wave modes in the open sea had long been known to be impossible, because their energy could propagate away freely. Edge waves, however, are just such modes, and their practical significance has come to be appreciated rapidly since their mathematical discovery [Ursell 1952] . Meanwhile, much more unfamiliar resonances have been studied, which are also caused by refraction. The key example of a round island exhibits these novel phenomena in their simplest form, and the following summarizes some recent results.

The mechanism of trapping by refraction is simple. The propagation velocity of surface gravity waves on water increases with depth. Therefore, if parts of a wave crests lie over deeper water, those parts travel faster and in brief, all wave crests over water of uneven depth always turn towards the shallows, during propagation. The possibility thus arises that waves coming from shore might be turned back shoreward, by and by, before they can reach the open sea: such waves are trapped. (The invisible barrier beyond which they do not travel seaward is called a caustic by analogy with optics.) If the phase relationships are just right, moreover, then resonance becomes a plausible possibility.

Such a direct approach, however, proves inadequate because realistic trapping turns out to be much less straightforward. Over an axisymmetric seabed topography, for instance, the depth level lines themselves are circles and to be trapped, the crests must therefore be turned faster than the local level circle. That is impossible [Longuet-Higgins 1967, Shen, et al. 1968] at sufficiently large distances from the center. The far field is therefore always a field of progressive waves radiating energy from and to the open sea. This stands in striking contrast to edge waves which decay exponentially with distance from shore (a behavior now recognized as rather special and exceptional).

Figure 1 shows the most typical pattern of wave crests for a trapped wave mode around an island. An inner wave ring around the island is separated from the far field of radiation by a quiescent zone, in which the motion is damped exponentially rapidly with distance from the zone edges. All the same, since the damping zone has finite width, the two wave motions separated by it cannot be independent. For instance, if such a wave mode be set up and left to develop freely, without further energy supply, then a non-zero amount of wave energy will pass through the damping zone to the outer wave field, whence it will ultimately be radiated to infinity. Such "leakage" makes complete trapping impossible. There can be no resonance in the clear-cut classical sense. Rather, any resonance must be a matter of degree, depending on the leakage rate.

II. Natural Frequencies

A first question is what trapped wave modes are possible and at what frequencies? Shen, Meyer and Keller [1968] obtained estimates for round islands of quite realistic seabed topography by an approximation based on smallness of the seabed slope ε. If horizontal distances are measured in units of the island radius L and vertical distances, in units of εL, then the standard equations [e.g. Stoker 1957] of the classical, linear theory of surface waves for the velocity potential $\Phi = \phi(x, y, z) \exp(-i\omega t)$ read

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \varepsilon^{-2} \frac{\partial^2 \phi}{\partial z^2} = 0 \qquad \text{for } 0 > z > -h(x, y)$$

$$\partial \phi / \partial z = \varepsilon \omega^2 \phi \qquad \text{at } z = 0 \tag{1}$$

$$\varepsilon^{-2} \frac{\partial \phi}{\partial z} + \frac{\partial \phi}{\partial x} \frac{\partial h}{\partial x} + \frac{\partial \phi}{\partial y} \frac{\partial h}{\partial y} = 0 \quad \text{at } z = -h(x, y)$$

The first is Laplace's equation, the second combines the linearized, kinematical and dynamical surface conditions, the last expresses impermeability of the seabed; viscosity and surface tension are ignored. In the form (1), for $\varepsilon << 1$, they lead to a refraction approximation for gentle water depth variation, but not to any short- or long-wave approximation in a usual oceanographical sense.

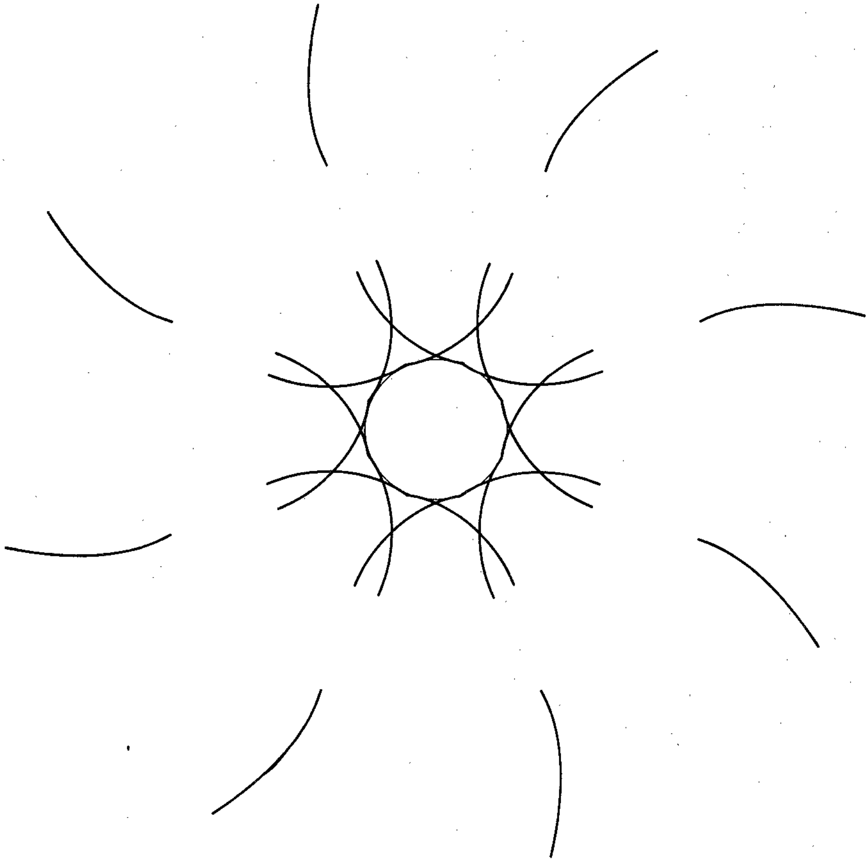

Figure 1

The manner in which ε appears in (1) suggests application of Keller's [1958] Geometrical Optics Approximation, a sophisticated and very powerful form of familiar ray methods. The fuss with rays and orthogonals, etc., is cut out by the simple idea that a natural mode must accommodate precisely an integer number of wave lengths both (i) around the island and (ii) between shore and the inner caustic circle bounding the trapped wave ring (Fig. 1. Only patterns with a single trapped wave ring will be considered here; more complicated patterns and spectra are possible, but rare for round islands [Shen et al. 1968]). Condition (i) is easy to apply at a caustic circle because the crests must be perpendicular to this refraction boundary. The wave length along the inner caustic $r = r_1$ is therefore $\lambda_1 = 2\pi/[\omega^2 k(r_1)]$, if $\omega^2 k(r)$ denotes the wave number magnitude (this notation reflects convenience of scaling for $\varepsilon << 1$), and (i) says simply

$$2\pi r_1/\lambda_1 = r_1 k(r_1) = n \qquad n = 1, 2, 3, \cdots \tag{2}$$

For (ii), it will be seen in Section V below that $\omega^2 \xi(r)/2\Pi$ with

$$\xi(r) = \int_1^r |k^2 - n^2/(\omega^4 r'^2)|^{1/2} \, dr' \tag{3}$$

counts radial distance in units of local wave length (which varies greatly with the radius r), if the wave number function $k(r)$ is determined from the full dispersion relation [e.g. Stoker 1957] .

$$k \tanh(kh) = 1. \tag{4}$$

Condition (ii), actually, is not quite correct as first stated. A full propagation loop goes, say, from shore through refraction at the caustic back to shore (Fig. 1), and the radial distance $\omega^2 \xi(r_1)/2\pi$ from shore to inner caustic in wave length units must therefore be a half-integer,

$$\frac{\omega^2}{\pi} \int_1^{r_1} |k^2 - n^2/(\omega^4 r^2)|^{1/2} \, dr = m - \frac{1}{2}, \qquad m = 1, 2, \cdots \tag{5}$$

The extra $-\frac{1}{2}$ stems from the phase shifts of shore reflection and caustic refraction [Shen and Keller 1975, Shen et al. 1968] and may serve as a reminder that the simplicity of the present account represents, of course, use of hindsight for shameless corner-cutting; for slightly longer, more tenable derivations see [Shen et al. 1968, Meyer 1971] .

Given n and m, (2) and (5) are two equations for the frequency ω and caustic radius r_1, which turn out [Shen et al. 1968] to have at most one solution for realistic topography $h(r)$. Those are the <u>discrete eigenfrequencies</u> and it is clear from (i), (ii) that n counts the number of crests around the island at fixed radius, while m counts (twice) the number of crests radially outward from shore to caustic and back. Of course, $r_1 < \infty$ for a pattern as in Fig. 1 and a closer look at (2)-(5) [Shen et al. 1968] shows this to set an upper bound to the values of m (and hence, also of ω) for which (2), (5) do have a solution, given n. That bound, moreover, usually rules out the lowest pairs (n, m), which general experience would have suggested to furnish the most prominent and important modes... For a conical island, for instance, there is no natural mode for any pair (n, m) with $n < 3$ [Shen et al. 1968].

For each n, on the other hand, (2) and (4) can be satisfied with $r_1 = \infty$ (which makes condition (ii) irrelevant) for any frequency ω exceeding a "cut-off" value, and those are the continuous spectra. In sum, there is a countably infinite discrete spectrum embedded in a continuous spectrum of countably infinite multiplicity. That is, as the frequency increases, the continuous spectrum gains more and more progressive wave modes coexisting at the same frequency; but always, further discrete frequencies are encountered at which trapped modes also exist. It is remarkable that so complicated a physical situation can be described by as simple and practical a set of formulae as (2), (4) and (5); Keller's Geometrical Optics Approximation can therefore cope with more complicated circumstances and holds promise of a practical approach to real topographies.

The simple version [Keller 1958, Shen et al. 1968] of this theory actually fails at shore and caustics, but Shen and Keller [1975] have constructed a uniform approximation which now offers a practical approach to the estimation of wave amplitudes at just those places where they are maximal and of most direct interest.

III. Leakage and Response

The theory so far outlined ignores, however, that every natural wave mode must involve leakage of energy from the trapped wave ring through the damping zone to the far field whence that energy is radiated away to the ocean. All eigenvalues are therefore complex. (The real part will continue to be called frequency and the imaginary part will be called "leakage rate.") The flaw in the use of this form of refraction theory for the present purpose is not only that it is logically based on reality of the eigenvalues, which its own results on natural modes show to be impossible. A more weighty objection is that resonance is now a matter of degree, and we need actual estimates of the leakage rate before we can distinguish effectively resonant eigenvalues from harmlessly damped ones.

Indeed, the organic connection between the waves trapped near shore and the associated waves in the far field implies an equal possibility of energy leakage outward and inward across the damping zone. It thereby creates a mechanism for direct excitation of resonant modes by plane waves of the same frequency incident from the ocean. This is a new mechanism, entirely absent in classical water wave resonance.

Classical edge waves [Ursell 1952], for instance, have no far field and, therefore, linear theory has no mechanism by which they could possibly be excited (short of an earthquake, which Ursell [1952] had to use for his experiments). Accordingly, they remained academic until Galvin [1965] showed how to excite them subharmonically, a process dependent on small effects of the second order. That is typical of classical resonance. "Leaky modes," by contrast, possess an inherent, first-order, and therefore potentially highly effective, mechanism of direct harmonic excitation by the natural wave environment, which is new in oceanography although not unfamiliar in some other fields.

This first-order mechanism also implies an opportunity of calculating, from the classical linear theory, the amplitude response of the water surface around an island to a given wave input from the ocean. It turns out to depend greatly on the energy spectrum of the input which, in turn, depends a good deal on the circumstances. The extreme cases are a single, plane wave pulse and a steady monochromatic plane wave input. In the absence of fairly definite specification of the input spectrum, the latter, standing-wave extreme may be the most practical estimate, conservative in that it overestimates the response to be expected in a real situation. For that extreme, the "response coefficient," that is, the standing trapped wave amplitude for unit incident wave amplitude, turns out, plausibly enough, to be just the reciprocal of the leakage rate [Longuet-Higgins 1967, Lozano and Meyer 1976].

This reciprocal relation, on the other hand, creates the paradoxical situation that an estimate of the imaginary part of the eigenvalues is the more important, the smaller this imaginary part. The main engineering interest is precisely in the most negligibly small needles in the haystack, and this imposes a severe demand for precision on any approximate analysis.

These notions of leakage and response were first set out by Longuet-Higgins [1967] and Summerfield [1972] in their long wave approximation for wave trapping by a "hedge," that is, a steep slope terminating a pronounced shelf around an island. (They also offer helpful estimates and comments on the relation between input spectrum and trapping response in their approximation.) That somewhat exceptional topography lends itself, in the long wave limit, to idealization by a topography of piecewise constant depth, for which the natural modes can be represented classically in terms of Hankel functions. Detailed computation of many complex eigenvalues was made possible thereby, and among many others, also quite a number of modes of very large response were found. Their frequency band width of response is so narrow [Longuet-Higgins, 1967], however, that they are virtually inaccessible to a more directly numerical approach. Indeed, some brilliant computational efforts [Lautenbacher 1970] have been defeated by this feature of the response.

IV. Generalized Refraction Theory

The Geometrical Optics Approximation, on the other hand, is defeated by the complex eigenvalues, which are incompatible with its basic formulation [Keller 1958] as an asymptotic expansion in powers of the small parameter ε. Leakage introduces factors exponential in ε^{-1}, which transcend approximation to any order in powers of ε. Such inherent contradictions are, of course, less obvious in more vaguely based ray methods. In short, the better known refraction methods are designed for self-adjoint problems and fail decisively, like many well-known mathematical methods, for non-self-adjoint problems such as trapping.

A more general refraction approximation to (1) is therefore needed. Preferably, it should not be a long wave approximation because we have already seen in Section II that the usual, longest modes are absent, whence the longest actually present are doubtful candidates for strong resonance. For application to the most common topographies, it should be based on the gentleness of seabed variation expressed by the parameter ε, that is, on the assumption that the seabed topography varies only on a scale large compared with the local wave length. Such an approximation has been proposed independently by a number of authors who arrived by diverse reasoning at the hypothesis that, over such "gentle" topographies, the <u>vertical velocity profile depends to the first approximation only on the local water depth</u>, but not also on the horizontal derivatives of the water depth.

To express this, write the exact potential

$$\Phi(x, y, z, t) = \phi_S(x, y; \varepsilon) F(z; x, y, \varepsilon) \exp(-i\omega t)$$

with $F(0; x, y, \varepsilon) \equiv 1$ so that ϕ_S is the surface value of the potential. Then F expresses the vertical structure and its dependence on z must, to first approximation, be that predicted by the classical linear theory for waves over water of uniform depth equal to the local value of $h(x, y)$. But that is well known [Stoker, 1957] and implies that $\alpha(h)F$ is to be approximated by

$$F_0 = e^{-kh} \cosh[k(z+h)], \qquad k \tanh(kh) = \varepsilon\omega^2 \tag{6}$$

(The convenience factor $\alpha = \frac{1}{2}[1 + \exp(-2kh)]$ is used to avoid formal degeneracy as $h \to \infty$.) Vertical averaging of (1) [Lozano and Meyer 1976] then shows $\Psi = \alpha^{-1}\phi_S$ to be governed by the short-scale Helmholtz equation

$$G^{-1}\nabla(G\Psi) + \varepsilon^{-2}k^2\Psi = 0, \tag{7}$$

with

$$G = \int_{-h}^{0} F_0^2\,dz = [\sinh(2kh) + 2kh]/[4k\exp(2kh)] .$$

Equations of the form (7) have been proposed by Battjes [1968] and derived by Berkhoff [1973] from an assumed asymptotic expansion of the potential which, however, is known [Shen and Keller 1975] to be incorrect, especially in the present context. To avoid such pitfalls, Lozano has given a vertical averaging argument [Lozano and Meyer 1976] supporting (6), (7), but a mathematical proof is not claimed. On the other hand, (6), (7) are there shown to include as special cases all known exact solutions of (1), and all the best known approximations (including Geometrical Optics and other ray approximations).

From an oceanographers viewpoint, it is noteworthy that the generalized refraction equations (6), (7) encompass long and short waves impartially: the full, exact dispersion relation $k \tanh(kh) = \varepsilon\omega^2$ is used.

V. Leakage Rates

The Generalized Refraction Equations (6), (7) have been applied to trapping by axisymmetric islands, since those are the simplest topographies exhibiting the new physical effects. Apart from that symmetry, a general, realistic topography is envisaged: the water depth $\varepsilon h(r)$ increases monotonically and (analytically) smoothly from shore to ocean. To restrict attention to the simplest and most common trapping pattern (Fig. 1), exceptionally pronounced shelves are excluded (which makes the study complementary to Longuet-Higgins' [1967] and Summerfield's [1972], who specialized on the effect of such shelves). Attention has also been restricted to natural modes harmonic in the angular variable because the geometrical optics approach [Shen et al. 1968] has not revealed any others and the new study does not, in any case, aim at an exhaustive calculation of all possible such modes. The surface potential can then be written as

$$\phi_s = e^{in\theta}\,[\alpha(h)/g(r)]\;w(r), \qquad n = 1, 2, \cdots \tag{8}$$

in polar coordinates r, θ based on the island center, with factor

$$g(r) = (rG)^{1/2} = \tfrac{1}{2}\, e^{-kh}[2\,rh + (r/k)\sinh(2\,kh)]^{1/2}$$

introduced to simplify the differential equation for the unknown function $w(r)$.

Substitution in (7) shows that equation to be

$$\frac{d^2w}{dr^2} = \left(\frac{f(r)}{\varepsilon^2} + \frac{g''}{g}\right)w, \tag{9}$$

$$f(r) = (n\varepsilon/r)^2 - k^2, \tag{10}$$

$$k \tanh(kh) = \varepsilon\omega^2. \tag{11}$$

It is formidable, not only because of the complexity of the coefficient g''/g, but because even the main coefficient, $f(r)/\varepsilon^2$, can be defined only implicitly through the dispersion relation (11) for $k(r)$. The general form of $f(r)$ is determined by the assumption of a simple trapping pattern (Fig. 1), which implies [Shen et al. 1968] that $f(r)$ has precisely one maximum for $1 < r < \infty$. With $n\varepsilon$ chosen to exceed a certain cut-off value--which implies $n = 0(\varepsilon^{-1})$, i.e., a rather large number of wave crests around the island--$f(r)$ then has precisely two roots, r_1 and $r_2 > r_1$ (Fig. 2). They mark the caustic radii because the basic form of (9) is

$$\varepsilon^2 w'' - f\, w = 0(\varepsilon^2) \tag{12}$$

so that the solution are oscillatory for $f < 0$, i.e., in the trapped wave ring $1 < r < r_1$ (Fig. 2) and the far field $r > r_2$. For $r_1 < r < r_2$, $f > 0$ and the solutions are non-oscillatory; this is the damping ring.

Since $f(r)$ varies so much (Fig. 2), comparison of (12) with the standard equation $y'' + y = 0$ is assisted by the variable

$$\xi(r) = \int^{r} |-f(r')|^{1/2} dr'$$

which brings (11) into the form

$$\varepsilon^2 \frac{d^2 w}{d\xi^2} + \nu(\xi)\frac{dw}{d\xi} + w = 0$$

This shows $\xi/(2\pi\varepsilon)$ to measure radial distance in local wave lengths and explains (3) (with the normalization $\varepsilon\omega^2 = 1$).

Actually, all this holds only for real ω, which has been seen to be impossible for natural modes! With leakage, $k(r)$ is complex for real r, by (11), and so are $f(r)$ and its caustic roots r_1, r_2, which Fig. 3 shows in the complex plane of the radius r.

That figure also shows the Stokes lines of the differential equation (9) with each of which is associated an abstract pair of exact solutions of (9) of progressive wave character. The pair of L_0 (Fig. 3) represents the fundamental, incoming and outgoing, wave solutions near shore, the pair of L_∞ represents those in the open sea. The main object of the analysis is to calculate the matrix T which relates these pairs of fundamental solutions of (9), or at least, to estimate T adequately. The difficulty is that adequacy, in view of the importance of modes of the very smallest leakage rate, implies a precision unprecedented for differential equations of the generality and inexplicitness of (9) - (11). Adequate precision, however, has now been achieved [Lozano and Meyer 1976] in respect of the leakage rate.

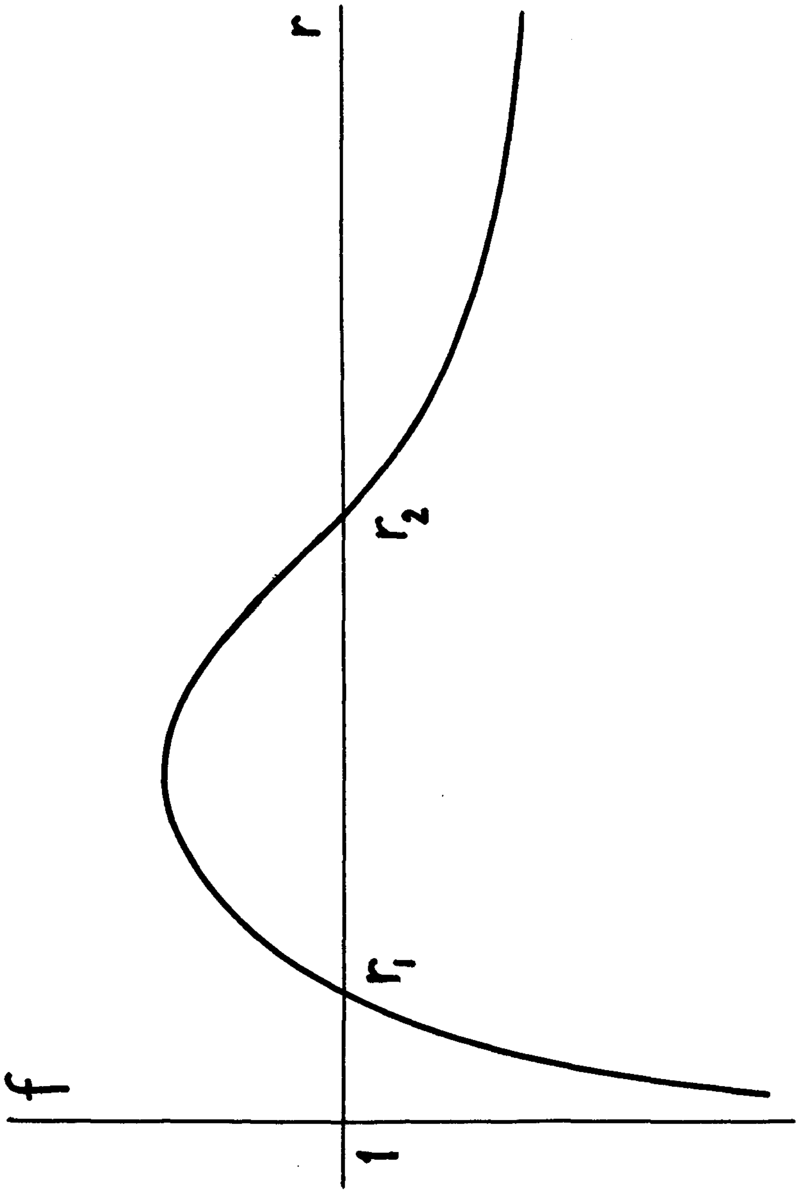

Figure 2

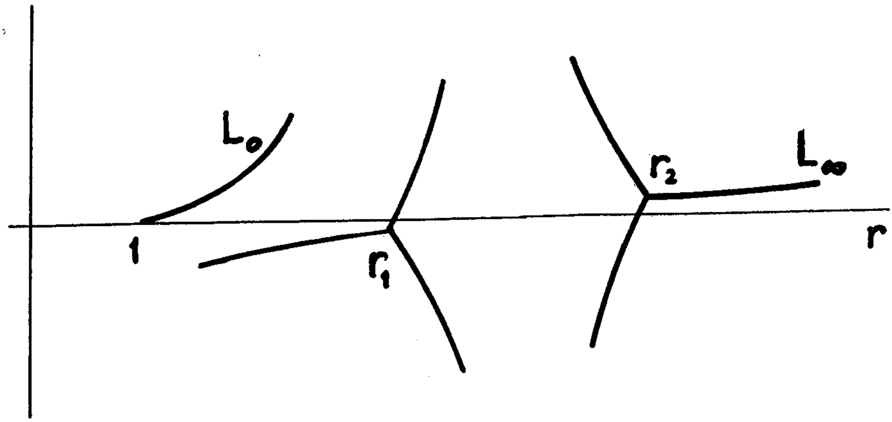

Figure 3

The exact wave solution pairs of the Stokes lines L_0 and L_∞ are of interest because physical boundary conditions are most naturally phrased in their terms. To-date, the radiation condition adopted for natural modes has been that no energy be supplied by radiation from infinity: we envisage a trapped wave mode to be set up, and watch it decay slowly in time by radiation damping. The shore boundary condition adopted to-date has been boundedness of the water velocity at the shore. The reason for it is two-fold, despite its implication of perfect reflection. First, it is successful in edge wave theory [Ursell 1952] and secondly, while a better shore condition could be incorporated into the analysis just as readily, none appears to be known...

The matrix T, which depends on ω, connects these boundary conditions into a characteristic equation [Lozano and Meyer 1976] which can be solved abstractly to deduce rigorous estimates for the eigenvalues. The result is that the frequencies ω_{mn} of Shen, Meyer and Keller [1968] (cf. Section II) are confirmed, with a correction factor of only $1 + 0(\omega_{mn}^{-4}) = 1 + 0(\varepsilon^2)$. In addition, the imaginary part of the eigenvalues is found to be

$$\text{im}(\omega^2) \sim -(2q)^{-1} \exp(-4\pi d) \quad \text{as } \omega_{mn} = \varepsilon^{-1/2} \to \infty \tag{13}$$

where

$$d = (2\pi)^{-1}\omega_{mn}^{2}\int_{r_1}^{r_2} |k^2 - n^2/(\omega_{mn}^4 r^2)|^{1/2} dr \tag{14}$$

would be the width of the damping zone in local "wave lengths," if there were waves in the damping zone... The leakage rate (13) is therefore seen to be exponentially small in ε and reciprocally, the response coefficient is exponentially large! In (14), ω_{mn} is found from (2), (5), with $k(r)$ found from the dispersion relation (4). Similarly,

$$q = \int_1^{r_1} |k^2 - n^2/(\omega_{mn}^4 r^2)|^{-1/2} \frac{k^2 dr}{1 + (k^2 - 1)h}$$

is readily computed by the help of (4).

VI. Quasiresonance

One of the immediate applications of this result is to the standing waves around an island, i.e. monochromatic waves of real frequency ω and surface potential (8). For real frequency, (1) exhibits energy conservation in fixed space-domains [Stoker 1957] and (6), (7) preserve this property. A ratio of respective energy flux levels in the trapped wave ring and in the far field in the open sea is therefore definable [Lozano and Meyer 1976] and its square root represents an amplitude ratio $\rho(\omega)$ characteristic of wave amplification by trapping. This amplification may be computed from the matrix T, given the shore condition of bounded velocity, and is found extremely frequency-dependent. It varies from an exponentially small minimum

$$\min \rho = \frac{1}{2} \exp(-2\pi d)[1 + 0(\varepsilon)]$$

at a frequency midway between successive eigenfrequencies to an exponentially large maximum

$$\max \rho = \exp(2\pi d)\,[1 + 0(\varepsilon)] \tag{15}$$

close to an eigenfrequency [Lozano and Meyer 1976]. The occurrence of such large standing wave amplifications is called quasiresonance in quantum mechanics.

An advantage of the analytical approach is that it promotes an understanding of cause-and-effect relations. The amplification (15) is expressed directly in terms of the intrinsic width (14) of the damping zone that separates the trapped waves from the far field of progressive waves extending to the open sea. This confirms the plausible idea that the wider (in an appropriate sense) the zone of damping, the more it inhibits leakage and promotes amplification. Closer scrutiny permits us to distinguish two effects which promote amplification. First, natural modes with a large number m of half-wavelengths counted radially across the trapped wave ring have usually a correspondingly wide damping zone. (There are exceptions when n/ω_{mn}^2 is close to the cut-off value so that the maximum of $f(r)$ in Fig. 2 is small and the interval (r_1, r_2) of positive f-values is narrow.)

Secondly, natural modes with $n >> m$, i.e., with many more crests around the island than radially across the trapped wave ring can occur, and such crests have local character akin to edge waves on a straight coast, which have no leakage. Indeed, as $n \to \infty$, the analysis can be shown to cover the modes with m not large, even $m = 1$, and the trapped wave ring then becomes narrow by comparison with the island radius. For the spectral conditions (i), (ii) of Section II, however, the relevant scale is the width of the trapped wave ring, and on that scale, the island radius tends to ∞, the shore appears straight, the waves approach ordinary edge waves, the leakage almost disappears. An indication of this second effect is numerically apparent in Longuet-Higgins' [1967] and Summerfield's [1972] results for long wave trapping by idealized "hedges." Both effects can combine to produce exceptional amplifications due to leakage rates as tiny as [Lozano and Meyer 1976]

$$\mathrm{im}\, \omega^2 \sim - \pi^{-1} \varepsilon^{2n-1}, \qquad n \to \infty$$

where ε denotes the small beach angle.

The asymptotic estimates for the eigenvalues also permit estimation of the half widths of the resonant peaks of the $\rho(\omega)$ curve of amplification vs. frequency as

$$(4q\omega_{mn}^2 \sqrt{3})^{-1} \exp(-4\pi d)$$

[Lozano and Meyer 1976] . The frequency band width of resonance is therefore seen to be exponentially narrow!

This feature balances the large amplification to some extent, but the resonant modes should still be prominent under many circumstances [Longuet-Higgins 1967] --much in contrast to oceanographical experience and intuition (based on the classical prominence of the lowest modes). Of course, a balanced judgment of such unfamiliar phenomena is not easy at this early stage of their study.

The narrow band width of resonance also suggests experimental and observational difficulties. Scale effects, however, dominated the first experiments [Pite 1973] , which indicated the resonant modes to be suppressed by viscous damping in a table-top tank. Laboratory tests on a larger-scale [Barnard and Pritchard 1976] have now detected some of the resonant modes of a sill [Longuet-Higgins 1967] .

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