CHAPTER 51

PHYSICS AND MATHEMATICS OF WAVES IN COASTAL ZONES

H. Lundgren*

ABSTRACT

This paper presents a discussion of four methods available for the computation of small amplitude periodic waves in coastal zones of arbitrary topography. With no reflections (or reflection from a single structure) the methods that proceed in space (Refraction and Propagation) or time (Timestep) seem the natural ones from a physical point of view. With repeated reflections, recourse must be taken to the solution of an Elliptic boundary value problem. - It is suggested that a P-method based on (energy) flux lines and energy fronts be developed for cases where the R-methods give crossing orthogonals.

1. WAVE EQUATION

In 1949 Lowell (Ref. 9) derived, for shallow water waves, the wave equation

$$(c^2 \eta_x)_x + (c^2 \eta_y)_y = \eta_{tt} \tag{1}$$

where $c = \sqrt{gh}$ is the local phase velocity, $\eta(x,y;t)$ the surface elevation, and $h(x,y)$ the local depth, while indices x,y,t denote differentiation. For shorter waves c is a function not only of h but also of the period T, and the (reduced) wave equation must be confined to a fixed T, for which $\eta_{tt} = -\omega^2 \eta$, where $\omega = 2\pi/T$. In 1951 Pierson (Ref. 11) asked whether (1) could be derived for an arbitrary period. Indeed, this was easily done from simplified considerations of continuity and equilibrium (Ref. 10).

In 1952 Biésel (Ref. 4) published the potential theory solution for the two-dimensional case correct to terms of order h_x (slope of bed). On this basis, Svendsen (Ref. 14) in 1967 derived a two-dimensional form of the wave equation that later (Ref. 8) was shown to agree with the three-dimensional form

$$(c\, c_g \eta_x)_x + (c\, c_g \eta_y)_y + \frac{c_g}{c} \omega^2 \eta = \nabla \cdot (c\, c_g \nabla\eta) + \frac{c_g}{c} \omega^2 \eta = 0 \tag{2}$$

where $\nabla = (\partial/\partial x, \partial/\partial y)$. In 1972 Berkhoff (Ref. 2) derived (2) using a surface potential φ_0 instead of η (and Schönfeld had obtained it in a different form). c_g is the group velocity.

With the local wave number $k = 2\pi/L = \omega/c$ it is easily seen that the classical shoaling formula for a progressive wave

*Professor of Marine Civil Engineering, Institute of Hydrodynamics and Hydraulic Engineering, Technical University of Denmark, DK-2800 Lyngby. Consultant, Danish Hydraulic Institute.

$$\eta = \frac{1}{2} H_0 \sqrt{c_{g,0}/c_g} \sin\left[\omega t - \int_{x_0}^{x} k \, dx\right] \qquad (3)$$

satisfies the mild-slope equation (2), including terms of the order c_x and $(c_g)_x$, both corresponding to the order h_x, but excluding terms of the order $(h_x)^2$ and h_{xx}. Assuming that (3) holds good also for steeper and curved slopes

$$\nabla \cdot (c \, c_g \nabla \eta) + \left[\sqrt{c_g} \, \nabla \cdot (c \, \nabla \sqrt{c_g}) + \frac{c_g}{c} \omega^2\right] \eta = 0 \qquad (4)$$

(which includes terms $(h_x)^2$ and h_{xx}) should be slightly better than (2). However, Eq. 4 (or the correct formula!) would have to be derived from potential theory and, the improvement is of limited value for most waves, which are not sinusoidal in small depths.

The important *conclusion* of (2) and (3) is that a mild slope does not reflect any energy of the order h_x but, at most, of the order h_x^2 (and hardly of the order h_{xx}). It should be noted that derivatives of η are normally an order of magnitude larger than derivatives of c.

2. R-METHODS (REFRACTION METHODS)

The development of R-methods can be said to have been completed with the establishment of fast, inexpensive computer programs that plot not only the orthogonals but also the wave heights along the orthogonals, taking account of bottom friction (Refs. 12 - 13). The R-methods will continue to be useful tools in most applications where it is not justified to mobilize better and more expensive methods. In principle, R-methods may be applied to an arbitrary directional spectrum.

The main deficiencies of the R-methods are due to converging and, particularly, crossing orthogonals, in which cases the solutions are unreliable or unusable.

3. E-METHODS (ELLIPTIC METHODS)

An elliptic equation pertains to a problem the solution of which is defined when the boundary conditions along a closed curve are given. (2) is an elliptic equation in (x,y), corresponding to the fact that wave energy might enter the region bounded by a closed curve from all sides. Particularly, if there are many successive reflections, such as inside harbours, E-methods are the natural tool.

The most common elliptic problem is Laplace equation for which various computational methods are available. The wave equation, however, is distinctly different from Laplace equation because of the last term in (2), giving rise to solutions with wavelengths that are small compared to the region considered. This circumstance eliminates several computational methods.

Finite elements were used by Berkhoff (Ref. 2 - 3) for the wave motions around an island, over a shoal, and in a harbour basin. Chen and Mei (Ref. 6) also used finite elements, correcting Berkhoff's functional. According to Ref. 3, 5 and 10 elements per wavelength give errors of 10% and 4%, respectively.

In their very nature, E-methods are confined to regions outside the breaker zone.

4. T-METHODS (TIMESTEP METHODS)

By a T-method is understood a method by which the development of the water surface and velocity field is followed over the entire coastal zone, one timestep after the other. Because of the large number of grid points and timesteps, the requirements to storage and machine time are heavy. Despite these difficulties a T-method is, in principle, the only one that - probably in a rather distant future - would be able to handle the general situation of irregular waves in a coastal zone, including breaking, surf and wave-induced currents.

Ito and Tanimoto (Ref. 7) presented examples of T-method calculations, applying an approximate version of the wave equation.

5. P-METHODS (PROPAGATION METHODS)

According to the conclusion in Sec. 1, the reflection from a gently sloping bottom of waves approaching the coast is definitely negligible. (Some swell-exposed beaches are so steep though that they reflect an essential part of the energy.)

In the case of shoals the R-methods give orthogonals that meet at a point (focus) or are tangential to certain curves (caustics), cf. Fig. 1. No reflection can be expected from foci or caustics because, by means of diffraction theory, it has been shown that the rays from a lens pass through the focus without reflection.

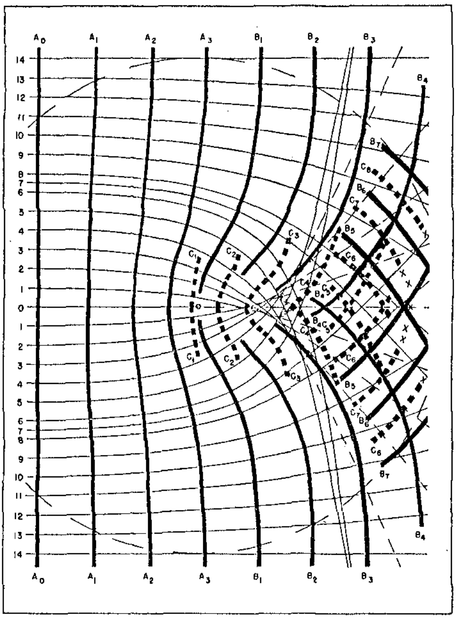

Fig. 1
Refraction orthogonals (thin lines) and actual fronts (heavy lines) for waves passing over a clock glass (from Ref. 11)

It is thus concluded that waves from the sea normally propagate without essential reflection onto the beach where their energy is dissipated. Hence, for periodic waves the elliptic character of the wave equation is rather a mathematical than an important physical property and, therefore, it is highly natural to look for methods where the propagation is followed from deep water into shallower areas, without consideration of the 'future' of the wave motion nearer the beach.

For small waves P-methods should also be applicable if there is a single reflecting structure such as a breakwater. With repeated reflections inside a harbour they are not fit.

The R-methods represent the simplest type of P-methods. A proper P-method should be able to account for the conditions around foci, caustics and crossing orthogonals (Fig. 1).

In principle, the deformation of waves over a gently sloping bottom may also be treated as a P-problem. Hence, in the future computations by P-methods might be continued through breaker and surf zones, though the interaction with wave-induced currents would require iteration.

6. HUYGENS' PRINCIPLE

Historically, Huygens' principle was the first P-method proposed, dealing with the progressive construction of wave fronts. However, because of the smoothing of the wave fronts, it could never produce the crossing orthogonals known to occur in nature (Ref. 11). An attempt at computing refraction with 'moderate diffraction' was made in Ref. 5, with the result that the interference behind the shoal was expressed by wavy orthogonals rather than crossing ones.

Mathematically, any solution to (2) may be written $\eta(x,y;t) = A(x,y) \exp i\omega t$, where A is the complex amplitude satisfying

$$(c\, c_g A_x)_x + (c\, c_g A_y)_y + (c_g/c)\,\omega^2 A = 0 \qquad (5)$$

For constant depth (5) simplifies to $A_{xx} + A_{yy} + k^2 A = 0$, valid for all waves in a homogeneous medium. For this special case it has been derived from Green's formula, in 3 dimensions by Helmholtz and in 2 dimensions by Weber (Ref. 1), that A at a point Q_2 inside a closed surface (or curve), S, can be expressed as an integral over S. Weber's formula is

$$A(Q_2) = \frac{1}{4} \int \left[Y_0(kr)\ \partial A/\partial n_1 - A(Q_1)\ \partial Y_0/\partial n_1 \right] ds \qquad (6)$$

where Y_0 is the Bessel function of the second kind, $\partial/\partial n_1$ denotes differentiation along the inward normal at the point Q_1 on S, and r is the distance Q_2Q_1, cf. Fig. 2, where line 1 is only a part of S.

Though (6) has been called the 'mathematical theory of Huygens' Principle' in commemoration of this genius (who advocated the wave nature of light in opposition to Newton), it hardly represents what has been on his mind physically, for the following reason: (6) gives the solution A inside S in the general case where wave energy passes S both inwards and outwards. Since the elliptic wave equation is of the second order, the complete solution A inside S is defined if the values $A(Q_1)$ are given along S. In (6) it seems that $\partial A/\partial n_1$ must also be known on S; however, in principle, these derivatives can be found by solving a large number of equations established by applying (6) to points inside of and close to S. Thus (6) is the *mathematical* solution to an elliptic problem with *one* boundary condition at each point of the *closed* curve S. In contrast to this, Huygens' *physical* intuition said that the field of progressive waves would be defined if, along one *open* curve (a wave front) *two* boundary conditions were given, viz. elevation + phase (i.e. A) and phase velocity (i.e. $\partial A/\partial n_1$).

7. APPLICATIONS OF HUYGENS' PRINCIPLE

Since the author felt that the physical possibilities of Huygens' principle had not been fully explored, he made - in cooperation with

Dr. Uri Kroszynski - an attempt at calculating the wave motion at Q_2 (Fig. 2) from a knowledge of the wave conditions along line 1.

It was found that the basic formula should be

$$A(Q_2) = \frac{1}{2} \int \left[Y_0(k_2 r) \ \partial A/\partial n_1 - k_2 A(Q_1) \ Y_1(k_2 r) \cos q_1 \right] ds_1 \qquad (7)$$

where k_2 is the wave number at Q_2 and Y_1 the negative derivative of Y_0. The factor 1/2 in (7) indicates that, for constant depth, the integral along the infinite line 1 contributes one half to $A(Q_2)$ in (6), while the other half originates from the integral over an infinitely large semicircle.

It is assumed that the computation of the propagating wave has reached line 1, which is straight or curved. Normally, lines 1 and 2 will not be wave fronts but be chosen in a convenient manner in relation to the topography of the sea bed. According to (7), all points along line 1 have an influence on the wave motion at any point Q_2 on line 2.

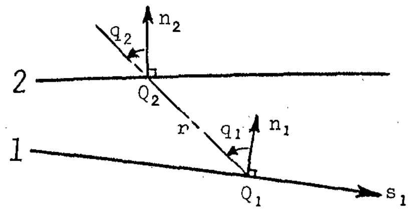

Fig. 2 Influence at Q_2 of wave motion at Q_1

The influence coefficients, Y_0 and $k_2 Y_1 = - dY_0/dr$, appearing in (7) correspond to the effects at Q_1 of a wave that spreads from Q_2 uniformly in all directions. These Bessel functions define the diverging wave exactly only when the depth is constant. When r is large, an essential depth variation may be expected between Q_2 and Q_1, introducing an error by the application of the Bessel functions. The error is reduced, however, partly by the oscillatory character of these functions (corresponding to the varying phase difference from Q_1 to Q_2) partly because the influence from Q_1 diminishes as $1/\sqrt{r}$ when r approaches infinity. Still, the assumption underlying (7) is that the distance between lines 1 and 2 is not too large.

Eq. 7 has been applied to the case of simple shoaling over a plane slope with a depth reduction of 10% in one wavelength. For a distance of L/8 from line 1 to line 2, $A(Q_2)$ was found to deviate 0.5% from the correct value, while the error was 2% for a distance of L/4.

Thus there is hope that Huygens' principle will eventually materialize in a numerical method applicable to general situations with a reasonably large space propagation step. With the accumulation of errors in a purely explicit method, it is believed that (7) should be somewhat modified. Because of the step size, asymptotic expressions requiring little computation may be used for the Bessel functions Y_0, Y_1 and Y_2, of which Y_2 occurs in the expression for the derivative $\partial A/\partial n_2$ (Fig. 2).

It is probable that (7) can also be applied to the combined diffraction and refraction around a breakwater, including reflection from it.

8. FLUX LINE METHOD

In analogy to the R-methods it seems possible to construct diagrams of energy flux lines, transcribing (5) into orthogonal, curvilinear coordinates consisting of flux lines and energy fronts. The computation progresses simultaneously along all flux lines. The crucial point of the method is, in analogy to (3), to utilize the energy velocity c_e for direct integration along the flux lines, thus reducing the difficulty of the spatial phase variation, which requires much machine time for E- and T-methods.

The variation of A along the energy fronts influences c_e, as can be seen from the interference of two similar wave trains propagating at right angles to each other. The phase velocity is then $c\sqrt{2}$ with maximum flux along the lines where the 'double crests' propagate.

REFERENCES

1. Baker, B.B. and E.T. Copson: The mathematical theory of Huygens' principle. (Oxford Clarendon Press) 2nd edit. 1953.
2. Berkhoff, J.C.W.: Computation of combined refraction-diffraction. Proc. 13th Coastal Engrg. Conf., Vancouver (1972). Am. Soc. Civ. Engrs., 1, pp. 471-490, 1973.
3. Berkhoff, J.C.W.: Linear wave propagation problems and the finite element method. In: Finite Elements in Fluids, Eds. R.H. Gallagher et al. (Wiley), 1, pp. 251-264, 1975.
4. Biésel, F.: Study of wave propagation in water of gradually varying depth. Gravity Waves. Proc. NBS Semicentennial Symp. on Gravity Waves, Wash.,D.C. (1951). Nat. Bur. Stds. Circ. 521, pp. 243-253, (U.S. Gov. Print. Off.) 1952.
5. Biésel, F.: Refraction de la houle avec diffraction modérée. Proc. 13th Coastal Engrg. Conf., Vancouver (1972). Am. Soc. Civ. Engrs., 1, pp. 491-501, 1973.
6. Chen, H.S. and C.C. Mei: Oscillations and wave forces in an offshore harbor. Applications of hybrid finite element method to water-wave scattering. Ralph M. Parsons Lab. for Water Resources and Hydrodyn., Mass. Inst. Tech., Rep. 190, 215 pp., 1974.
7. Ito, Y. and K. Tanimoto: A method of numerical analysis of wave propagation. Application to wave diffraction and refraction. Proc. 13th Coastal Engrg. Conf., Vancouver (1972). Am. Soc. Civ. Engrs., 1, pp. 503-522, 1973.
8. Jonsson, I.G. and O. Brink-Kjær: A comparison between two reduced wave equations for gradually varying depth. Inst. Hydrodyn. and Hydraulic Engrg., Tech. Univ. Denmark, Progr. Rep. 31, pp. 13-18, 1973.
9. Lowell, S.C.: The propagation of waves in shallow water. Comm. Pure and Appl. Math., 2, No. 2/3, pp. 275-291, 1949.
10. Lundgren, H.: The wave equation for gravity waves in water of varying depth. Coastal Engrg. Lab. and Hydraulic Lab., Tech. Univ. Denmark, Basic Res. - Progr. Rep. 7, pp. 2-5, 1964.
11. Pierson, W.J.: The interpretation of crossed orthogonals in wave refraction phenomena. Beach Erosion Board, Tech. Memo. 21, 1951.
12. Skovgaard, O. and J.A. Bertelsen: Refraction computations for practical applications. Proc. Int. Symp. Ocean Wave Meas. and Anal., New Orleans. Am. Soc. Civ. Engrs., 1, pp. 761-773, 1974.
13. Skovgaard, O., I.G. Jonsson, and J.A. Bertelsen: Computation of wave heights due to refraction and friction. Proc. Am. Soc. Civ. Engrs., 101, WW1, pp. 15-32, 1975.
14. Svendsen, I.A.: The wave equation for gravity waves in water of gradually varying depth. Coastal Engrg. Lab. and Hydraulic Lab., Tech. Univ. Denmark, Basic Res. - Progr. Rep. 15, pp. 2-7, 1967.