CHAPTER 28

PROBABILITY DENSITY FUNCTIONS OF BREAKING WAVES

By

Edward B. Thornton[1] and George Schaeffer[2]

ABSTRACT

Waves in the surf zone are a highly nonlinear process which is evident by the appearance of secondary waves. The secondary waves appear as strong peaks in the period PDFs corresponding to the first harmonic of the peak of the wave spectrum. The strong first harmonic period peak is also reflected in the highly correlated height and velocity PDFs. Due to the high probability of the secondary waves, the mean wave period for breakers is a poor descriptor of the average period of the offshore incident waves.

The joint probability density functions for periods and heights of the breaking waves show high correlation (0.60-0.80) which says that greater wave periods are associated with larger breaker heights. The joint PDFs of period and particle velocity, and velocity and height, suggest that the maximum onshore particle velocities are correlated with both the wave periods and wave heights.

I. INTRODUCTION

Studies of the PDFs of period, height, and wave lengths have been conducted over the last 20 years. The majority of these studies have been mainly concerned with theoretical or deep water aspects of the problem.

The theoretical distribution of wave amplitudes for a narrow-banded Gaussian surface elevation in deep water, was shown by Longuet-Higgins (1952) to be a Rayleigh distribution. Tayfun (1977), in studying the transformation of deep water waves to shallow water waves, showed that the Rayleigh distribution for wave amplitude was generally applicable to all bandwidths.

---

[1] Associate Professor, Naval Postgraduate School, Monterey, California, 93940

[2] Lieutenant, United States Navy

Longuet-Higgins (1975) extended his earlier work on narrow-banded Gaussian surface elevations in deep water and derived an expression for the joint distribution of wave periods and amplitudes

$$p(\xi,\eta) = \frac{\xi^2}{(2\pi)^{\frac{1}{2}}} \exp(-\xi^2(1+\eta^2)/2) \qquad (1)$$

where ξ and η are the normalized amplitudes and periods as defined by,

$$\xi = a/\mu_o^{\frac{1}{2}} \qquad (2)$$

and

$$\eta = (T-\langle T\rangle)/\nu\langle T\rangle \qquad (3)$$

where $\langle T\rangle$ is the mean period and ν is proportional to the spectral width where

$$\nu = (\mu_2/\mu_o)^{\frac{1}{2}}(\langle T\rangle/2\pi) \qquad (4)$$

and μ_o and μ_2 are the zero and second moments of the energy spectrum. The wave periods are defined by their zero up-crosses. It is noted that the marginal distribution of wave periods in deep water is bell shaped (but not Gaussian). Longuet-Higgins (1975) compared this theoretical distribution of ocean waves to that of the deep water data obtained by Bretschneider (1959) and found good agreement. He found that for the marginal wave height distribution there is a slight excess of waves with heights near the middle of the range and a deficit at the two extremes. Since much of the high frequency portions of the wave records are filtered out by the pressure transducer, he suggests that the narrow-band approximation may not be as applicable for the unfiltered records. In shallow water with much steeper waves, the distribution can again be expected to be less applicable due to the non-linearities which become increasingly important.

Chakrabarti and Cooley (1977) compared the theoretical joint distribution of Longuet-Higgins (1975) with wave data recorded during a 1961 storm in the North Atlantic. The spectrum of these waves did not fall in the category of narrow-band. Through the comparison, it was found that there was a considerable agreement at the higher wave heights, but a definite trend away from the theoretical curve at the lower wave heights.

Bretschneider (1959) investigated the PDFs of wave heights, H, wave lengths, L, and wave periods, T. He concluded from his empirical study that in deep water wave systems, the PDFs of wave heights and lengths can be approximated by a Rayleigh distribution. Using the relationship, wave length proportional to period squared for deep water linear waves, he concluded that the PDF of periods squared is a Rayleigh distribution.

Assuming that the marginal PDFs are Rayleigh, Bretschneider (1959) examined the extreme cases of 0 and +1 correlation. For the case of zero correlation, the marginal Rayleigh PDFs are given by

$$p(\zeta) = \frac{\pi}{2}\,\eta\,\exp(-\pi\zeta^2/4)\;;\quad \zeta = H/\bar{H} \tag{5}$$

$$p(\lambda) = \frac{\pi}{2}\,\lambda\,\exp(-\pi\lambda^2/4)\;;\quad \lambda = L/\bar{L} \tag{6}$$

For zero correlation, i.e. total independence, the joint PDF of heights and lengths is given by

$$p(\zeta,\lambda) = \frac{\pi^2}{4}\,\zeta\,\exp(-\pi\zeta^2/4)\cdot\lambda\exp(-\pi\lambda^2/4) \tag{7}$$

In terms of heights and periods,

$$p(\zeta,\tau) = 1.35\,\zeta\,\exp(-\pi\zeta^2/4)\cdot\tau^3\,\exp(-0.675\,\tau^4)$$

where $\tau = T/\bar{T}$. (8)

For the case of total dependence, the correlation coefficient is equal to one. For a correlation coefficient of one, all data points on a plot of joint Rayleigh PDFs fall on a 45 degree straight line passing through the origin.

Koele and de Bruyn (1964), Goda (1967), and Siefert (1970) have observed that the distribution of wave heights in shallow water does not correlate well with the Rayleigh distribution. Kuo and Kuo (1974) have suggested that this is due to: 1) the non-linear effects of wave interactions yielding more larger waves, 2) the effect of bottom friction yielding a reduction in the low frequency components, and 3) the effects of wave breaking, which would truncate the distribution and transfer some of the kinetic energy to the high frequency components. Therefore the PDFs of wave heights with a certain intensity, may be considered to approximate a conditional Rayleigh distribution truncated by the local breaking height.

A zero-up-cross analysis was conducted on waves measured on a reef by Black (1978), who then compared the observed distribution with the Rayleigh, truncated Rayleigh, and the Weibull distributions. He observed that the truncated Rayleigh distribution fit well for those waves in the breaker zone when the heights were depth limited. The truncated Rayleigh distribution did not fit well for those waves measured either inshore or offshore of the breaker zone. He has found that a Weibull distribution more closely approximated these measured distributions.

It would be expected that the Weibull distribution would yield the better fit. The Rayleigh distribution is a function of the variance of the data, whereas the Weibull distribution is a function of higher moments about the mean. Neither theoretical distribution adequately describes observed distributions which are multi-modal.

Black's (1978) observations of wave periods show distributions which are skewed to the right. This does not fit well with the symmetrical Longuet-Higgins (1975) distribution. It was found that the period distribution more closely followed the Rayleigh distribution than the period square as proposed by Bretschneider (1959). It was found that the Weibull distribution most closely fit the empirical PDF, but there is considerable variation in the peakedness parameter. His attempts in relating the period statistics to the position on the reef were unsuccessful. He feels that this is due to the non-linearities of the waves in shallow water.

In this paper, the empirical data collected for breaking waves are compared with the Rayleigh distribution.

II. EXPERIMENT AND ANALYSIS

Three different California experimental sites were used to obtain data on the three types of breaking waves--plunging, spilling, and collapsing. Plunging and spilling waves were measured on Del Monte Beach, Monterey, California; collapsing breakers were measured on the Carmel River Beach, which is located approximately five miles south of Monterey, California. Due to the steepness of the beach and the rapidity of breaking of the shoaling waves, reflected waves are found to be present. Spilling breakers on a gently sloping beach were measured at Torrey Pines Beach, a site just north of La Jolla, California.

Capacitance type gauges were used to measure the change in the sea surface elevation. The velocity of the water particles was measured with Marsh-McBirney Model 721 and 722 Electromagnetic Current meters. The meters were calibrated in the laboratory prior to use in the field. The instruments were placed on towers in the surf zone at low tide at the expected point of breaking of the waves for the subsequent high tide. Each tower held a capacitance wave gauge and an electromagnetic flowmeter. The flowmeter was placed below the wave gauge, approximately 0.5 m from the bottom, to insure constant submergence. Data were collected within an hour or two on both the ebb and flow sides of high tide. Details of the experiments are given in Thornton _et al_ (1976).

Record lengths of approximately 30 minutes from each data set were analyzed. It was desired to only examine the sea-swell band of frequencies lying between 0.03-1.0 Hz (30 sec-1 sec). Hence, the data were passed through both a high and low pass digital filter. The high pass filter had a cut-off frequency of 0.03 Hz (30 sec). The high pass filter used a Fast Fourier Transform (FFT) algorithm to obtain the Fourier spectrum of the entire 30 minute record. The Fourier coefficients corresponding to 0-0.03 Hz were then used to synthesize a low frequency time series which was subtracted from the wave record. The low pass filter was a 25 weight inverse transform filter. A cut-off frequency of 1.2 Hz was used with a terminal frequency of 1.8 Hz. The mean and variance of sea surface elevations were calculated for each record as well as the mean and variance of the heights, periods and maximum velocities.

Wave heights were determined from the surface elevation record by means of the zero-up-cross method. The zero-up-cross technique defines a wave period as the interval between adjacent upcrosses. The maximum and minimum of the surface elevation within the interval defines the crest and trough of a wave. Maximum onshore particle velocities were calculated from the flowmeter records. Maximum onshore velocity was defined as the maximum velocity amplitude occurring between successive zero upcrossings as defined by the sea surface elevation record. Joint PDFs of wave height and period, wave height and maximum velocity, and maximum velocity and period were calculated from the analyses.

Heights and maximum velocities were normalized by dividing by their respective standard deviations of the surface elevations and particle velocities respectively. The periods were normalized by dividing by their mean period. Histograms were calculated for the normalized heights, periods, and maximum velocities. The histograms were transformed into PDFs by dividing the number of values over an interval by the total number of values in the array and by dividing by the interval width.

Joint probability density functions of periods versus heights, velocities versus heights and periods versus velocities were calculated in a similar fashion. A 15 by 15 grid was used to determine the frequency of values which simultaneously fill each joint interval.

The energy density spectrum (power spectrum) was obtained by means of a Fast Fourier Transform (FFT) algorithm. The energy density spectrum tells how the variance is distributed with respect to frequency. The spectral width parameter was calculated using the equation:

$$\varepsilon = \sqrt{\frac{m_0 m_4 - m_2^2}{m_0 m_4}} \qquad (9)$$

where m_0, m_2, and m_4 are the spectral moments (Cartwright and Longuet-Higgins, 1956). For $\varepsilon = 0$ the waves have only one frequency and the energy spectrum would be represented by a spike. As $\varepsilon \to 1$

the sea surface can be described as a broad-band process for which the distribution would be Gaussian and the spectrum would approach a horizontal line. Swell can therefore be described as a narrow-band process and seas as broad-band. Due to the appearance of secondary waves, a narrow-banded deepwater spectrum of waves broadens in the surf region.

The correlation coefficients of the joint values were calculated to obtain a measure of the amount of linear relationship of random pairs of values of periods and heights, velocities and heights, and periods and velocities.

III. DISCUSSION

Data measured on 16 March 1977 for the case of spilling breakers at Torrey Pines Beach is chosen for subsequent discussion. The offshore waves on this date appeared as narrow-band swell. But spectral analysis of the surface elevation records in the surf zone yielded spectral width parameters ranging from 0.94-0.97, suggesting a broad-band spectrum in the surf zone. The broad spectral width can be attributed to the non-linear wave-wave interactions yielding harmonics in the breaking zone.

In order to determine the origin of the large spectral width parameter, ε was calculated in the ranges of 0-0.25 Hz, 0-0.50 Hz, 0-1.0 Hz and 0-nyquist frequency. The analysis revealed that large values of the spectral width parameter occurred in all cases. In the case of 0-0.25 Hz which contains the first three harmonics, ε was greater than 0.7, which was the case with the smallest ε. It is concluded that the change from a narrow deep water spectrum to a broad-band shoaling spectrum is primarily due to generation of secondary waves at harmonic frequencies.

Probability Density Functions of Period and Height

Bretschneider (1959) made the assumption that the wave lengths were Rayleigh distributed; using this assumption and the linear wave theory relationship

$$L = \frac{g}{2\pi} T^2 \tag{11}$$

the wave periods squared would be Rayleigh distributed in deep water. Bretschneider's analysis can be extended to the shallow water case using the shallow water approximation for wave length

$$L = \sqrt{gh}\ T \tag{12}$$

Assuming the wave lengths are Rayleigh distributed, the periods in shallow water would be expected to be Rayleigh distributed since period and wave length are directly proportional.

The period PDF is shown in Figure 1 and it is noted to have a definite skewness. The Rayleigh PDF has been superimposed on the period PDFs for comparison. The period PDFs only loosely follow the Rayleigh PDF. The Rayleigh PDF overpredicts the wave periods greater than the Rayleigh mode value, and underpredicts the observed periods less than the peak Rayleigh value. The observed period PDF has the mode lying close to the maximum of the superimposed Rayleigh PDF.

The mode of the breaking wave distribution is located at a period (7 sec) corresponding to the first harmonic of the offshore peak wave period (14 sec). A peak corresponding to the peak offshore wave period is absent. The waves periods as determined by a zero-upcross of the surface elevation record for breaking conditions yields a poor indication of the peak of the energy spectrum which corresponds to the mean offshore wave period. The mean periods corresponded to the first harmonic periods or even to a valley between the primary and first harmonic periods. The appearance of strong peaks at one-half the offshore wave period is indicative of the importance of nonlinearities in the form of secondary waves in the surf zone.

The wave height PDF is shown plotted against the theoretical Rayleigh PDF at the top of Figure 1. The height PDF is loosely described by the Rayleigh PDF. The major height peak is centered at 2.2. The Rayleigh PDF overestimates the density of the small wave observations and tends to underestimate the density of the large waves. The heights are truncated at approximately six standard deviations. Since the wave gauge is stationary and therefore the depth of the water is essentially fixed, this truncation occurs due to the breaking of the large waves seaward of the wave gauge. The calculated mean height value on 16 March 1977 lies at 2.7, which actually corresponds to a dip in the PDF relative to the peak intervals on either side. The bimodality was very commonly observed and is attributed to the very narrow band incident wave system and the generation of secondary waves at the first harmonic.

The joint PDF of periods and heights is contoured in Figure 1. It is noted that the longest wave periods are associated with the greatest wave heights suggesting strong correlation between periods and heights. The correlation coefficient was calculated for all cases and varied between 0.60 and 0.80. The area of maximum joint density is found to be associated with the first harmonic of the offshore wave period, and the mode of the observed breaker heights.

Greater density was found at short periods and heights for all records than predicted by the Rayleigh PDF. A high frequency wave riding on a long period wave causes an additional zero crossing as the longer wave passes through zero. Most previous comparisons of heights or periods of waves have been done using pressure sensors which greatly filter the high frequencies due to hydrodynamic attenuation. Capacitance wave staffs and electromagnetic flowmeters have good response times and thus measure high frequencies. Therefore, even though the data were filtered at 1 Hz, all records showed peaks at the shortest periods and heights.

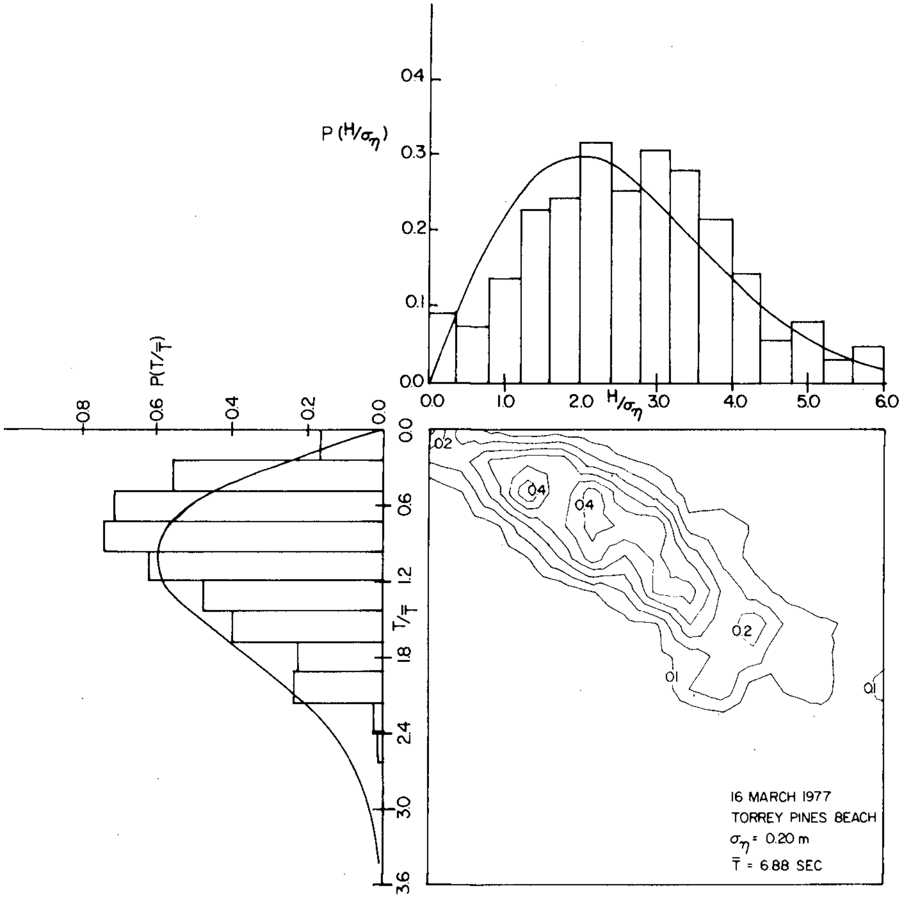

Figure 1. PDFs of Wave Heights and Periods, 16 March 1977

Joint Probability Density Functions of Velocities

The velocity PDFs were calculated as the maximum onshore velocities within the period segments determined by the zero-up-crosses of the surface elevation record shown in Figure 2. Using this technique can result in both positive and negative maximas for onshore particle velocities.

The theoretical Rayleigh PDF is superimposed on the same graphs for comparison and it is apparent that the velocity PDF is not adequately described by the Rayleigh PDF. The velocity PDF for spilling breakers appears to be symmetrical, with the mode value falling at the bin with center velocity of 1.5. There also appears to be a small peak value at 0.3.

The joint probability distributions of velocities and heights shown in Figure 2 show high correlation between velocities and heights. The high positive correlation (0.7) suggests that the larger maximum onshore velocities are correlated with the larger wave heights.

Figure 3 shows the joint PDF of periods and velocities in which the periods were determined from the zero-up-crosses of the wave gauge record and the coincident maximum onshore velocities are determined from the flowmeter record. The results are similar to the joint PDF of period and heights. The maximum spilling probability is found at the intersection of period 0.84 and velocity 1.5.

The negative values of the maximum velocity indicate that, during a wave period defined by the zero-up-crosses of the surface elevation, the maximum velocity can be negative. The joint PDF of wave height and velocity shows that the negative velocities are associated with the low wave heights. The joint PDF of velocity and period shows that small negative velocities are correlated with the short periods. Therefore, it is concluded the negative velocity maxima are associated with short periods and low wave heights. The secondardy waves occur after the passage of the primary wave crest and have a peak near the trough of the primary wave; the velocities associated with the primary wave would be negative maximum in this region so that the measured velocity, which is a sum of the primary and secondary waves, can be small or negative.

IV. CONCLUSIONS

Waves in the surf zone are a highly non-linear process which is evidenced by the appearance of secondary waves. The secondary waves appeared in the PDFs at the first harmonic of the primary period.

The joint probability density functions of the breaking wave properties show high correlations. The correlation factor ranged between 0.60-0.80 for periods and heights and suggests that the greater the wave period, the higher the breaker height will be. High correlation between the periods and velocities and the heights

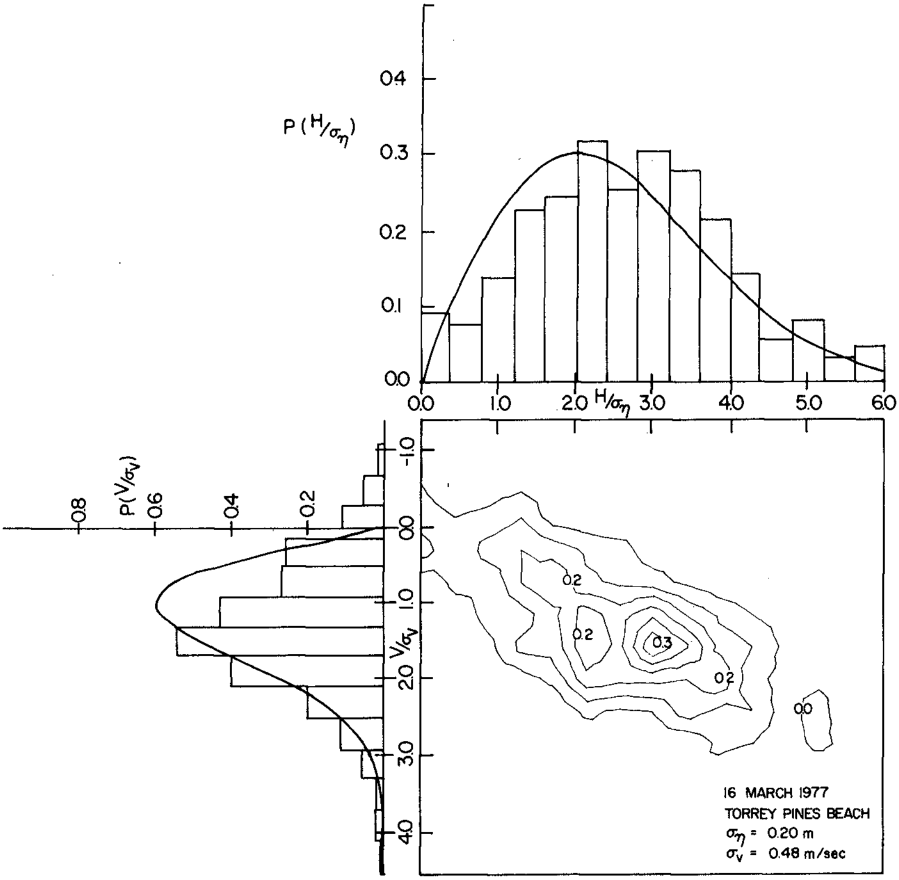

Figure 2. PDFs of Wave Heights and Velocities, 16 March 1977

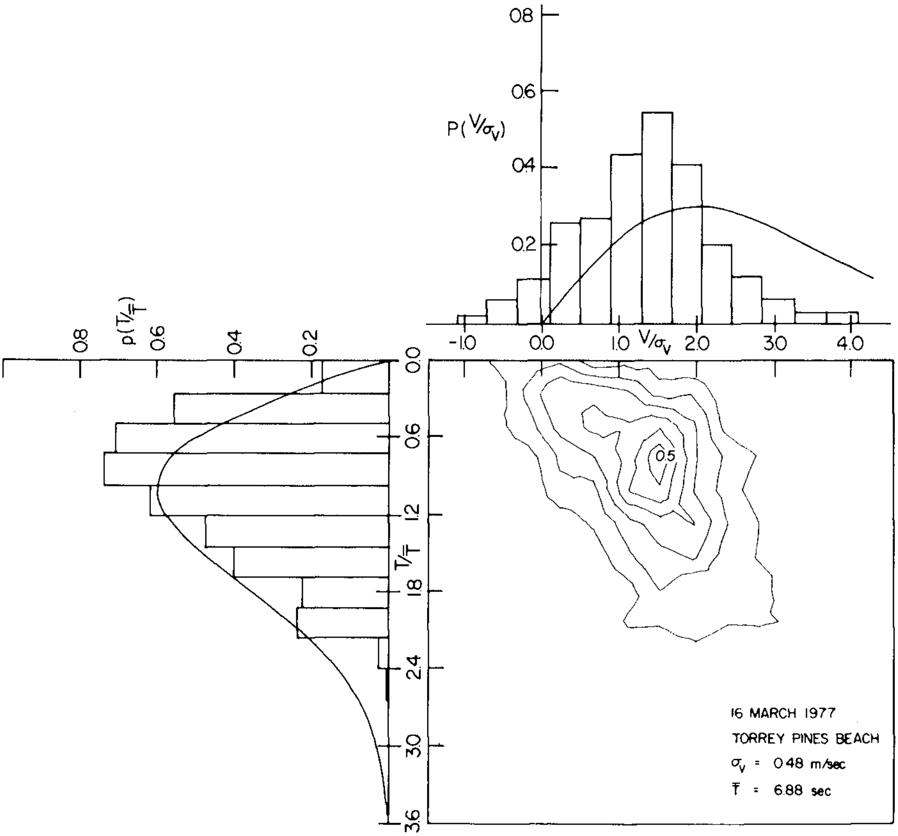

Figure 3. PDFs of Velocities and Periods, 16 March 1977

and velocities suggests that the maximum onshore velocities are a function of wave periods and breaker heights.

A truncated Rayleigh PDF was found to loosely approximate the empirical period and the height PDFs of spilling breakers. This truncation seems to be the result of long period waves with their associated larger breaker heights breaking prior to the arrival at the wave gauge.

The velocities PDFs of breaking waves, where the velocity maxima are associated with the periods defined by the surface elevation, yields a symmetrical distribution with some negative onshore velocities present. Since the Rayleigh distribution is undefined for negative values, the resulting fit of observed data with the theoretical Rayleigh distribution is poor.

The mean wave period as obtained by the zero-up-cross method in the surf zone is a poor indicator of the mean offshore wave period. Spectral analysis of the breaking waves shows an energy peak corresponding to the primary offshore wave period, but the period PDFs of breaking waves have a strong peak at the first harmonic of the wave spectra peak. The joint PDF (p(T,H)) of breaker waves showed a maximum density at a wave period equal to one half the peak period of the wave energy density spectrum, i.e. the first harmonic. The corresponding velocities (p(v,T)) or heights (p(H,T)) likewise were found to have a peak density associated with the first harmonic wave period. The mean period of breaking waves therefore is a poor estimation of the mean period of the incident waves offshore.

ACKNOWLEDGEMENTS

The data collected at Torrey Pines Beach was measured in cooperation with Robert T. Guza and the Shore Processes Laboratory at Scripps Institution of Oceanography. This work was supported by the Office of Naval Research, Geography Branch, under contract NR 388-114.

BIBLIOGRAPHY

Black, K. P., "Wave Transformation over Shallow Reef, (Field Measurements and Data Analysis)," James K. K. Look Laboratory of Oceanographic Engineering, Department of Ocean Engineering, University of Hawaii, Technical Report No. 42, March 1978.

Bretschneider, C. L., "Wave Variability and Wave Spectra for Wind Generated Gravity Waves," U. S. Army Corps of Engineers, Tech. Memo 118, 1959.

Cartwright, D. E., and M. S. Longuet-Higgins, "The Statistical Distribution of the Maxima of a Random Function," Proc. Roy. Soc., Ser A, 237, pp. 212-232, 1956.

Chakrabarti, S. K. and R. P. Cooley, "Statistical Distributions of Periods and Heights of Ocean Waves," Jour. of Geophysical Research, Vol. 82, p. 9, 20 March 1977.

Goda, Y., "Some Results of Observations on Wave Statistical Properties," Rept. of Port and Harbor Res. Inst., No. 39, 1967.

Koele, L. A. and P. A. de Bruyn, "Statistical Distribution of Wave Heights in Correlation with Energy Spectrum and Wave Depth," Proc. of 9th Conf. on Coastal Eng., A.S.C.E., 1964.

Kuo, C. T. and S. J. Kuo, "Effect of Wave Breaking on Statistical Distribution of Wave Heights," Proc. Conf. on Coastal Engineering (Copenhagen, Denmark, June 1974), pp. 377-389, 1974.

Longuet-Higgins, M. S., "On the Statistical Distribution of the Heights of Sea Waves," Journal of Marine Resources, Vol. 11:3, pp. 245-266, 1952.

Longuet-Higgins, M. S., "On the Joint Distribution of the Periods and Amplitudes of Sea Waves," Journal of Geophysical Research, Vol. 80:18, pp. 2688-2694, 20 June 1975.

Siefert, W., "Wave Investigation in Shallow Water," Proc. of 12th Conf. on Coastal Engineering, A.S.C.E., 1970.

Tayfun, M. A., "Linear Random Waves on Water of Nonuniform Depth," Ocean Engineering Report No. 16, Part II, Department of Civil Engineering, University of Delaware, Newark, DE, August 1977.

Thornton, E. B., J. J. Galvin, F. L. Bub and D. P. Richardson, "Kinematics of Breaking Waves Within the Surf Zone," Proc. of the 15th Conf. on Coastal Engineering, pp. 461-476, A.S.C.E., Jul 1976.