Beach Renourishment, North Cronulla, New South Wales

PART I

THEORETICAL AND OBSERVED WAVE CHARACTERISTICS

Beach Erosion at Floreat Beach after Cyclone "Alby", Perth, Western Australia

SOME IMPLICATIONS OF RECENT ADVANCES IN WAVE THEORIES

by

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1. INTRODUCTION

Calculations using the 'Design Wave' approach in coastal and offshore engineering begin with the specification of wave conditions usually in terms of a wave height, wave period and mean water depth. These three dimensional parameters are sufficient to define all the wave characteristics uniquely if a number of assumptions are adopted, including those of periodicity in space and time, absence of viscosity, and two-dimensional flow. Although these assumptions result in an enormous simplification of waves in the ocean, the ultimate idealised conditions hold some attraction in providing a standard description of wave mechanics for design purposes. Once the design wave approach has been adopted, it follows that the engineer needs to ensure that the theory which he uses to analyse waves generates results equivalent to those of the most accurate theory currently available.

There are in engineering use several wave theories (e.g., Airy, Stokes 5th order, Cnoidal, Stream Function) by means of which required wave properties can be computed from an initial wave specification on the basis of ideal conditions. Unfortunately, none of these theories is exact and in general they disagree, most significantly in conditions of shallow water or high wave steepnesses. The differences between them have stimulated comparative studies based on boundary condition errors (Dean, 1970) and experimental data (Le Méhauté et al, 1968), leading to recommendations on the use of specific theories for given conditions. However, the value of these comparisons has been somewhat limited by the absence of an accurate solution to the problem valid over the whole range of interest.

Such a solution has now been brought nearer by the work of Cokelet (1977), and it is the purpose of this paper to study the implementation of Cokelet's theory and other recent theories in an engineering context. The paper does not make any contribution to the analysis itself but is concerned with applications, and the question whether such developments should influence current engineering practice.

Previously, the wave theory offering greatest analytical accuracy and widest range of application was the Stream Function Theory of Dean (1965).

Tables of functions (Dean, 1974) have facilitated the application of this wave theory, but some results for steep waves given in the tables disagree with Cokelet's results.

The use of Cokelet's theory in an engineering context is discussed in Section 2, and other relevant recent developments introduced in Section 3. In Section 4 numerical results are presented for comparison between different methods.

2. ENGINEERING APPLICATIONS OF THE EXTENDED STOKES THEORY

As Cokelet (1977) presented it, the extended Stokes theory is not in a very convenient form for engineering application. The two dimensionless independent parameters[1] d_* and ε_* are not written explicitly in terms of wave height H, period T and mean water depth d. The expansion parameter ε_* lies in the range zero (for zero wave height) to unity (for waves of limiting height), and $d_*L/2\pi$ (where L is the wave length) is slightly less than the mean water depth d, and differs from it by 4% at most. In order to relate d_* and ε_* to H, T and d, it is necessary to compute some of the series expansions given by Cokelet using Padé approximants. In particular:

$$d/L_0 = \frac{c_*^2}{2\pi}(d_* + \bar{\eta}_*) \qquad (1)$$

and

$$H/L_0 = \frac{c_*^2 a_*}{\pi} \qquad (2)$$

where $L_0 = gT^2/2\pi$

The problem of going in the opposite direction, i.e., finding from given d/L_0 and H/L_0 the corresponding d_* and ε_*, and thereby a complete solution for given wave conditions, is a more demanding one. Possibly the extended Stokes solution could be re-formulated with different independent parameters, but in the present work we adopt a more pragmatic approach which provides a satisfactory entry to Cokelet's theory from conditions specified in terms of d/L_0 and H/L_0. The corresponding values of d_* and ε_* are interpolated from Cokelet's (1977) tabulated results as described below.

At points on the d/L_0, H/L_0 plane corresponding to the intersections of lines at constant d_* and ε_*, some of which are shown in Fig. 1, Cokelet provided numerical results of a range of integral wave properties, including c_*^2, $\bar{\eta}_*$ and a_*. The co-ordinates of the intersections may easily be calculated from Eqs. (1) and (2). In order to interpolate d_* at a specified point in the plane, the procedure adopted in the present work was as follows.

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[1] Where Cokelet's parameters are introduced, they appear with a subscripted asterisk.

Conditions of limiting wave height H_B were identified on the axes of Fig. 1 by cubic splines fitted (Ahlberg et al, 1967) through the points at which $\varepsilon_* = 1$, to approximate the function H_B/L_0 (d/L_0). The co-ordinates of the intersections at which the data is tabulated could then be transferred to the d/L_0, H/H_B plane, in which lines of constant ε_* are predominantly in line with the d/L_0 axis. Cubic splines were then used, along each line of constant ε_* in turn, to approximate the functions H/H_B $(d/L_0, \varepsilon_*\text{=constant})$ and $d_*(d/L_0, \varepsilon_*\text{=constant})$. From each pair of splines H/H_B and d_* were interpolated at the required value of d/L_0 to give a series of (unequally spaced) points on the curve approximating $d_*(H/H_B, d/L_0\text{=constant})$. The final interpolation at the required H/H_B was again performed by means of cubic splines. Linear scales were used throughout, except for d/L_0 and H/L_0, which were represented in the curve fitting by their logarithms. Various end conditions for the splines were tested and the most accurate results were generally obtained by setting second derivatives equal over the first pair and last pair of data points. Any other of the parameters tabulated in Cokelet's tables (or functions of them) can be interpolated in the same way. Alternative end conditions were used for the limit $H/H_B = 0$ when the behaviour of the required parameter was known from Stokes wave theory.

The accuracy of this method was tested by computing with Padé approximants the parameters c_*^2, $\bar{\eta}_*$ and a_*, for values of d_* and ε_* generated by interpolation from Cokelet's data with specified d/L_0 and H/L_0. The errors in the final d/L_0 and H/L_0 calculated by Eqs. (1) and (2) were always much less than 1%, except in the shallow water conditions $d/L_0 < 0.008$ where some of the integral wave properties converge at best to only one or two significant figures.

Other wave properties which can be computed directly from Cokelet's data by the above method of interpolation (i.e., without the use of Padé approximants) include the wavelength:

$$L/L_0 = c_*^2 \tag{3}$$

the elevation of the crest above mean water level:

$$\frac{\eta_{crest}}{H} = \frac{K_* = 2\bar{\eta}_*}{4a_*} \tag{4}$$

the dimensionless particle velocity at the crest q_{crest*} on a frame of reference moving at the wave celerity:

$$q_{crest_*} = -\,2a_* + \sqrt{4a_*^2 - c_*^4(\varepsilon_*^2 - 1)} \tag{5}$$

and the horizontal particle velocity at the crest on a stationary frame of reference:

$$\frac{u_{crest}}{H/T} = \frac{\pi(c_* - q_{crest_*})}{c_*\, a_*} \tag{6}$$

besides the energies and energy flux and other terms presented in the tables.

Unfortunately, no information is available by this means on the profile of the free surface or on particle velocities other than at the crest and the trough. To obtain the free surface profile or general particle velocities and accelerations, it is necessary to solve, (Cokelet, 1977), the series expansions for the coefficients of the Fourier series of the complex potential, from independent parameters d_* and ε_*. Since the complex spatial co-ordinate Z_* is expressed in the final solution as a function of the complex potential W_*, rather than vice-versa, it is ultimately necessary to interpolate for Z_*, for instance by complex Newton-Raphson iterations, to obtain conditions at a given location within the wave. A greater handicap, however, is the failure of the series expressions for the high order Fourier coefficients to converge adequately for strongly non-linear cases. Cokelet discussed this problem and showed how the degree of convergence could be determined by comparing values of parameters derived from the surface profile and alternatively directly from a series by Padé approximants.

To define the conditions for which the velocity potential Fourier series would converge we computed a number of cases and compared results for crest elevation and crest particle velocity derived by the two methods. The maximum wave heights for which errors in crest elevation and particle velocity derived from the Fourier series did not exceed 2% are shown as the broken line in Fig. 1. For the most shallow water conditions tested, this occurs at about one half the limiting wave height. In deeper water, with $d/L_0 > 0.1$, satisfactory results were obtained up to within 1% of the limiting height. These computations were carried out in double precision (approximately 28 significant places) on a CDC 7600 to a maximum order of 120. Convergence limits were almost exactly the same for both elevation and particle velocity at the crest. In view of the series formulation of the velocity potential, it might be expected that particle velocity would fail to converge before the profile itself, since the differentiation involved causes the truncated higher spatial frequency components to have a greater relative contribution to the total. This is not the case, however, owing to the inverse formulation of the Stokes series. It is, nevertheless, reasonable to expect that the region of convergence for particle accelerations will be rather more restricted than these, although no numerical comparisons have been carried out.

We conclude that the extended Stokes theory can be used in the same framework as engineering wave theories by means of cubic spline interpolation for the independent parameters d_* and ε_*. The velocity potential and its derivatives do not converge in all cases, but much useful information can be computed by interpolation from Cokelet's tables. Comparison of some results with those of other methods follows in Section 4.

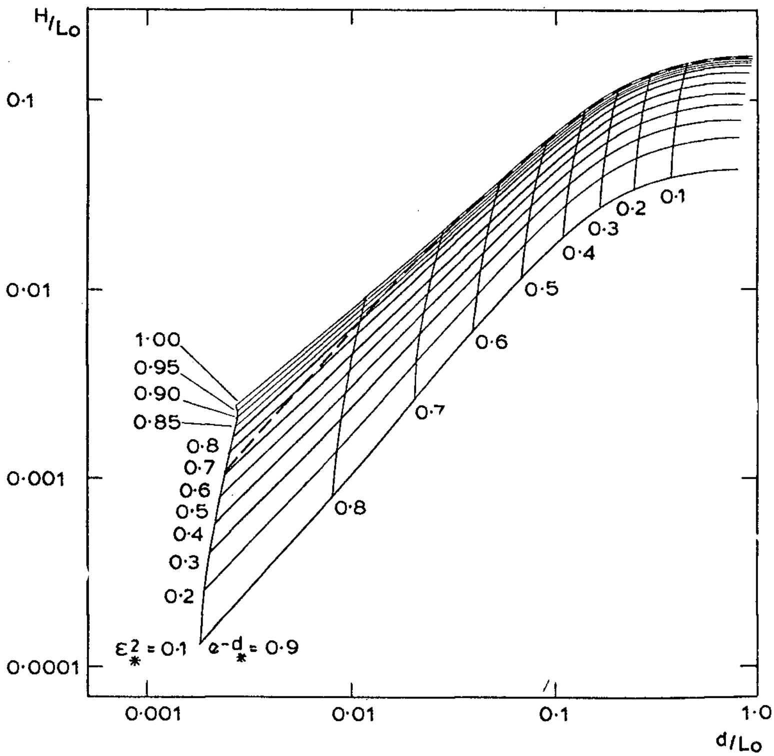

FIG. 1. LINES OF CONSTANT DIMENSIONLESS UNDISTURBED DEPTH d_* AND EXPANSION PARAMETER ε_* CALCULATED IN TERMS OF WAVE HEIGHT H AND MEAN WATER DEPTH d FROM Cokelet's TABLES.
L_o = SMALL AMPLITUDE DEEP WATER WAVELENGTH

3. FURTHER RECENT METHODS

Properties of steep waves can also be computed accurately by the stream function wave theory (Chaplin, 1980). While this method is more cumbersome than Cokelet's, and does not have in-built checks for convergence, it has the advantage of being formulated in conventional terms, and seems to converge better for steep waves in shallow water. The re-formulation necessary to achieve these results was carried out in an attempt to eliminate errors apparent in data for the steepest waves presented in stream function tables by Dean (1974). Some further comparisons using modified stream function results are presented in the next Section.

The analysis of steep waves by either extended Stokes theory or stream function theory demands much costly computer storage and time. Consequently, simple yet accurate approximations for the crest of the almost-highest wave and for the wave of limiting height in deep water by Longuet-Higgins (1979a and 1973) have great appeal. Results of both methods are presented in the next Section.

The almost-highest wave (AHW) approximation (Longuet-Higgins, 1979a) is expressed in terms of a velocity scale which must be derived by other means from given wave conditions. In the present work this has been achieved by means of interpolation by cubic splines, as described above, from Cokelet's (1977) data as follows. The required unit of velocity is $q/\sqrt{2}$, where q is the crest particle velocity in a reference frame moving with the wave. In terms of Cokelet's parameters:

$$\frac{q}{\sqrt{2}} = \frac{q_{crest_*}\ c_*\ C_0}{\sqrt{2}} \tag{7}$$

where C_0 is the small amplitude deep water celerity. The associated unit of length is:

$$\frac{q^2_{crest_*}\ c_*^2\ C_0^2}{2g}$$

and in order to fix the origin it is necessary also to compute in advance the elevation of the crest.

As shown in Eqs. (4) and (5) these parameters can be related to those presented in Cokelet's tables, and thus obtained for any specified wave conditions by interpolation. The computing effort necessary for the interpolation, followed by application of the AHW approximation is a small fraction of 1% of that required for a full solution by series methods. Furthermore, it applies to the crest where conditions for many purposes are likely to be most severe, and where conventional wave theories are at their weakest owing to poor convergence associated with the original Stokes type series formulation. The almost-highest wave approximation is related to more accurate solutions derived by Longuet-Higgins and Fox (1977, 1978). In the second of these papers the solution for the crest is matched to the rest of the wave for deep water conditions. Particle trajectories near the crest derived from the AHW approximation, the hexagon transformation and other methods are given in Longuet-Higgins (1979b).

4. COMPARISON OF NUMERICAL RESULTS

In comparing results from different wave theories, particular attention is paid to particle velocities in the region close to the crest for the reasons stated above. Also, we concentrate on steep waves since the earlier stream function tables (Dean, 1974) are accurate to within 5% for almost all cases up to 90% of limiting wave height (Chaplin, 1980). While integral parameters can be derived from any of the theories mentioned, they are now more easily and accurately computed from Cokelet's tables.

Fig. 2 shows the percentage errors in crest elevation above mean water level η as a proportion of wave height H for waves of limiting height. The results for cnoidal theory (for which U denotes the Ursell number L^2H/d^3) were computed in accordance with Keulegan and Patterson (1940) and those for Stokes 5th order theory with Skjelbreia and Hendrickson (1960). Clearly the stream function tables (in which nominally breaking waves are denoted as case D) are the most accurate, although they refer to slightly different wave heights.

Similarly, Fig. 3 shows crest particle velocities in a stationary reference frame u_{crest} as a proportion of the celerity for waves of limiting height. Again the stream function tables are the most accurate, giving results closest to unity for most water depths. In both comparisons, cnoidal theory is seriously in error, either at low Ursell numbers for crest elevations, or at high Ursell numbers for crest particle velocities. Since particle velocities vary only gradually through the water depth in shallow water conditions, it is reasonable to expect that the errors in cnoidal theory demonstrated in Fig. 3 are not confined to the region near the surface. Conversely in deep water conditions, the most significant errors in Stokes 5th order theory occur only close to the surface and predominantly at the crest. A comparison of horizontal particle velocities and vertical particle accelerations on a vertical line through the crest of the wave of limiting height in deep water is given in Fig. 4, in which y is measured upwards from mean water level. Since the velocity potential series in the extended Stokes theory does not converge for this case, we adopt the hexagon transformation (Longuet-Higgins, 1973) as an easily computed and accurate alternative. It gives inevitably the crest particle velocity u equal to the celerity and the upward vertical acceleration dv/dt at the crest level equal to - 1/2 g. Stokes 5th order theory and the stream function tables seriously underestimate both, but their errors diminish rapidly away from the surface.

The surface profile for the wave of limiting height in deep water is plotted without distortion in Fig. 5 for comparison with Stokes 5th order and stream function tables (case 10D) results. Although the latter is only a third order solution, this refers to the number of terms in the stream function series rather than in the Fourier series for the surface profile as in Stokes 5th order theory. The surface

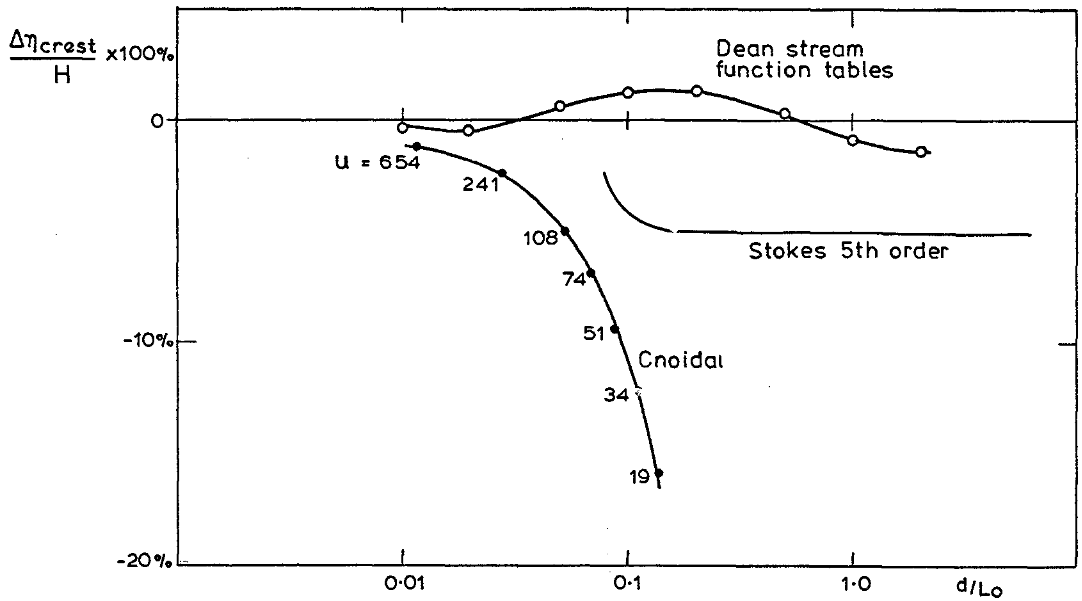

FIG. 2. CREST ELEVATIONS ABOVE MEAN WATER LEVEL IN WAVES OF LIMITING HEIGHT: ERRORS IN CNOIDAL, Stokes 5th ORDER THEORIES AND Dean STREAM FUNCTION TABLES AS A PERCENTAGE OF WAVE HEIGHT

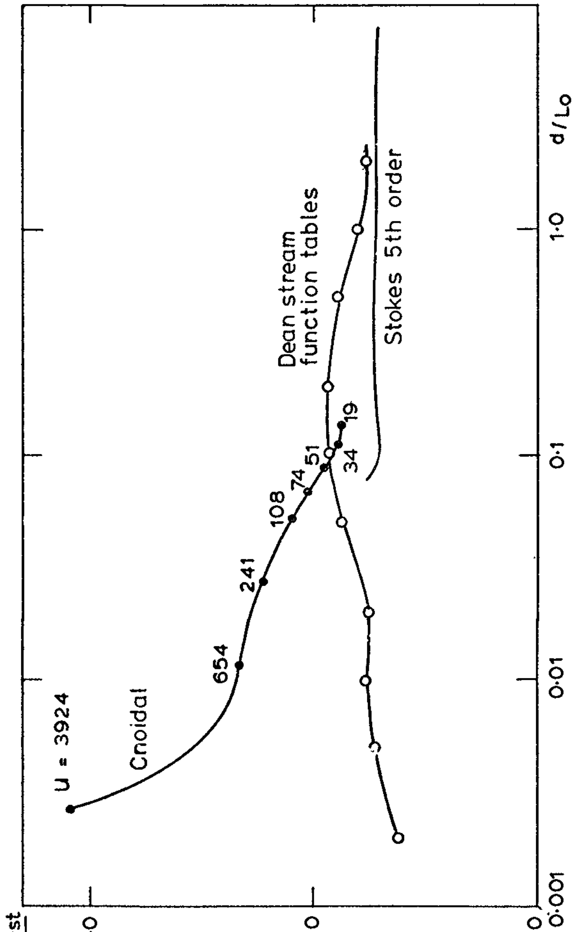

FIG. 3. CREST PARTICLE VELOCITIES. OF WAVES OF LIMITING HEIGHT CALCULATED FROM Keulegan and Patterson's CNOIDAL WAVE THEORY, Stokes 5th ORDER WAVE THEORY AND Dean STREAM FUNCTION TABLES

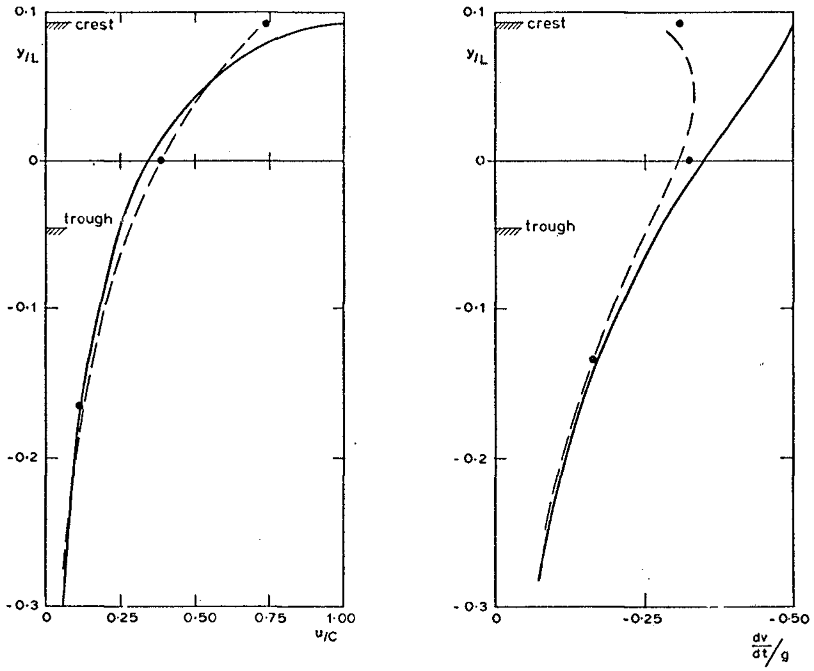

FIG. 4. PROPERTIES OF WAVES OF LIMITING HEIGHT IN DEEP WATER. PARTICLE VELOCITIES AND ACCELERATIONS ON A VERTICAL LINE THROUGH THE CREST PREDICTED BY THE HEXAGON TRANSFORMATION (Longuet - Higgins, ———) Stokes 5th ORDER WAVE THEORY(— — —) AND Dean STREAM FUNCTION TABLES (•)

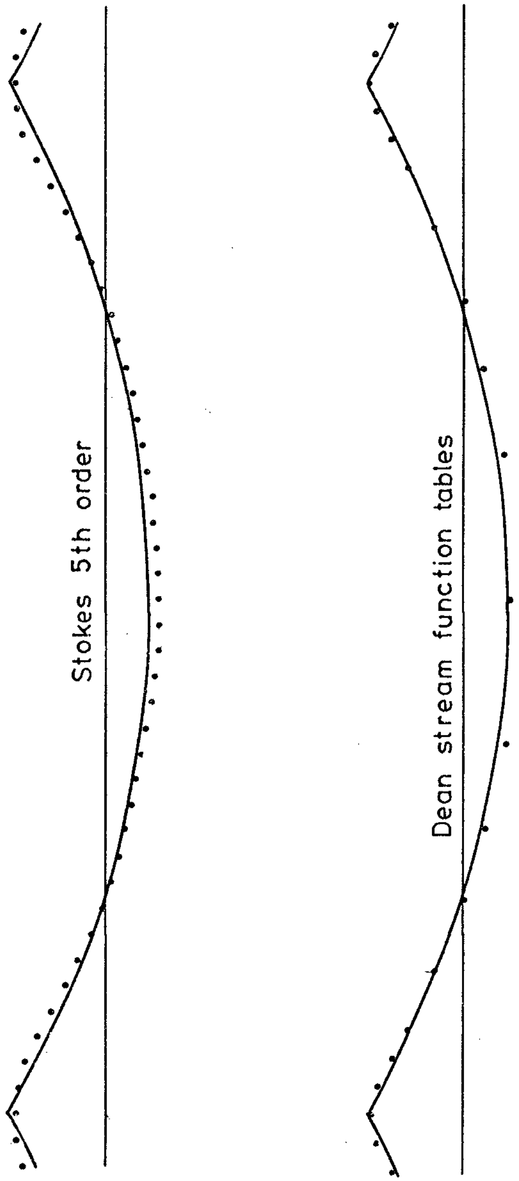

FIG. 5. THE WAVE OF MAXIMUM HEIGHT IN DEEP WATER. COMPARISON OF PROFILES WITH THE HEXAGON TRANSFORMATION (Longuet-Higgins)

profile in the stream function theory is derived from the stream function series, has therefore higher frequency terms and approaches more closely the discontinuity of surface slope at the crest.

The surface particle velocities predicted by the same three methods for the wave of limiting height in deep water are shown in Fig. 6 over the half wave length ahead of the crest. The errors in u at the crest in Stokes 5th order and stream function theories are again not representative of errors elsewhere.

It must be noted that the above comparison of results for limiting deep water waves serves only to demonstrate the performance of Stokes and stream function theories generally in deeper water. For this particular case the hexagon transformation is both more accurate and more convenient to compute than either of them.

If a full solution is needed for very steep waves in transitional water depth, the modified stream function theory (Chaplin, 1980) can provide converged results for cases which are beyond the limits we have found in the extended Stokes theory, and which are shown in Fig. 1. As noted above, the elevation and particle velocity at the crest can be found for all cases from Cokelet's tables by interpolation, and in Fig. 7 are plotted these results for very steep waves for two water depths. The agreement with results taken from full modified stream function solutions is within the accuracy of the interpolation procedure and of Cokelet's tables. Since conditions at the crest are more sensitive than those elsewhere to truncation errors this is taken as some confirmation of the overall accuracy of the full stream function solutions for the profiles and particle velocities of these steep waves.

The flow in a small region near the crest can also be computed, as described in the previous section, by the AHW approximation. For the two water depths referred to in Fig. 7, Figs. 8 and 9, in which x is measured horizontally forwards from the crest, compare elevations and velocity components of surface particles near the crests of steep waves computed by this method and by the modified stream function theory. The extent of agreement between the two methods naturally increases with increasing wave height as the upper part of the surface profile resembles more and more closely the Stokes corner flow which is the asymptote of the AHW approximation away from the crest. Also the two methods agree, not surprisingly, over a greater proportion of the wavelength in the deeper water case because the crest itself is relatively longer. Discrepancies at $x = 0$ are associated with differences between stream function results and those interpolated from Cokelet's tables to provide the scale and the origin of the AHW approximation. Away from the crest the latter tends towards a uniform surface slope and thus increasingly underestimates the actual surface profile elevation. Nevertheless, surface particle horizontal velocity components are accurately predicted by the approximation as far out as $x/L = 0.1$ for the steepest wave with $d/L_0 = 0.2$.

Sub-surface particle velocities are compared in Fig. 10 along a vertical line through the crest. Reasonable agreement is found over a region of similar dimensions as for surface elevations, namely about L/80 and L/20

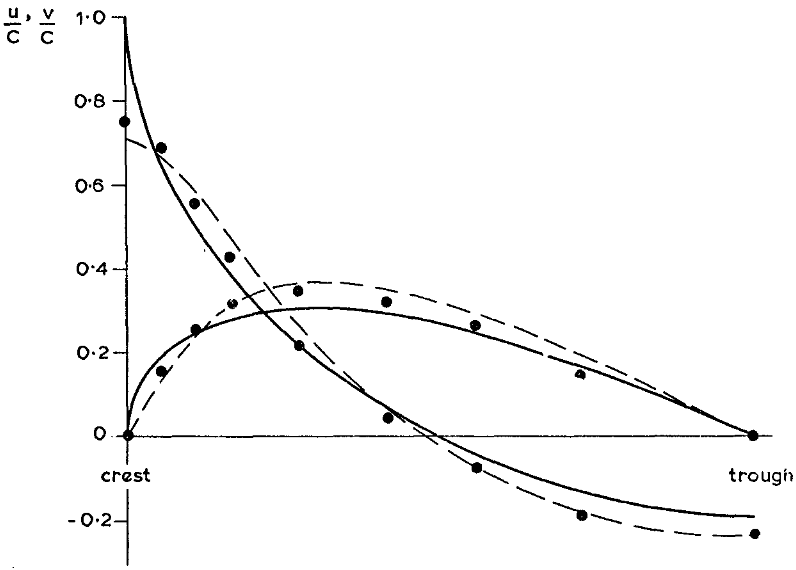

FIG. 6. THE WAVE OF MAXIMUM HEIGHT IN DEEP WATER. COMPARISON OF VELOCITY COMPONENTS OF SURFACE PARTICLES WITH RESULTS OF THE HEXAGON TRANSFORMATION (Longuet-Higgins) — — — Stokes 5th ORDER, • Dean STREAM FUNCTION TABLES, CASE 10D

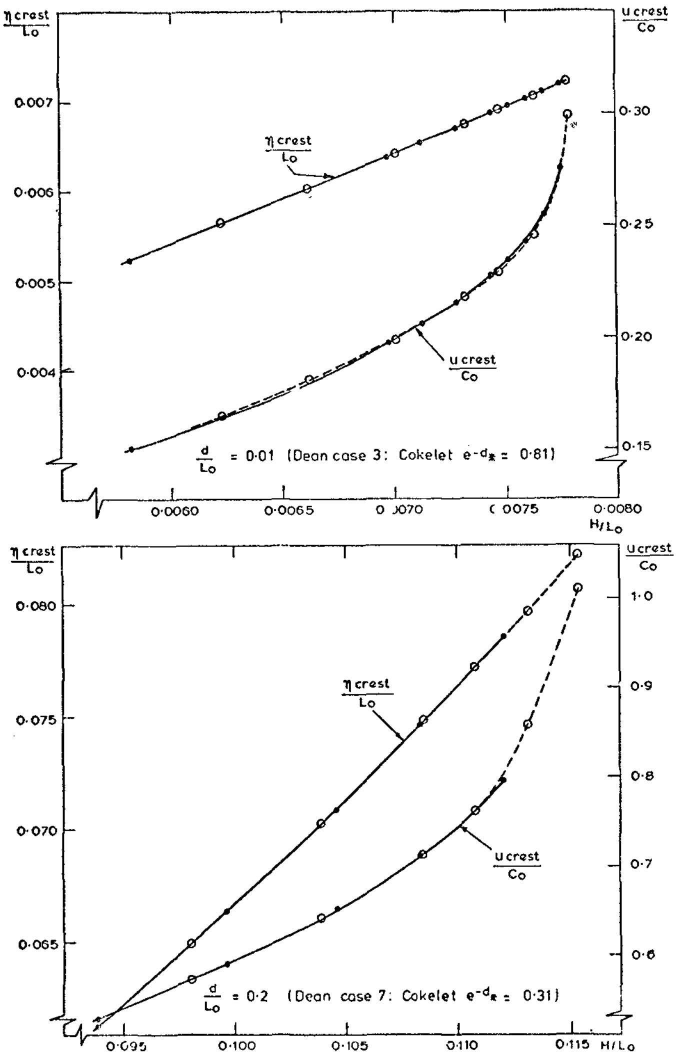

FIG. 7 CREST ELEVATION AND PARTICLE VELOCITY AT THE CREST FOR STEEP WAVES WITH CONSTANT PERIOD AND MEAN WATER DEPTH: ● - MODIFIED STREAM FUNCTION WAVE THEORY. ○ - INTERPOLATED BY SPLINES FROM INTEGRAL PROPERTIES TABULATED BY COKELE:

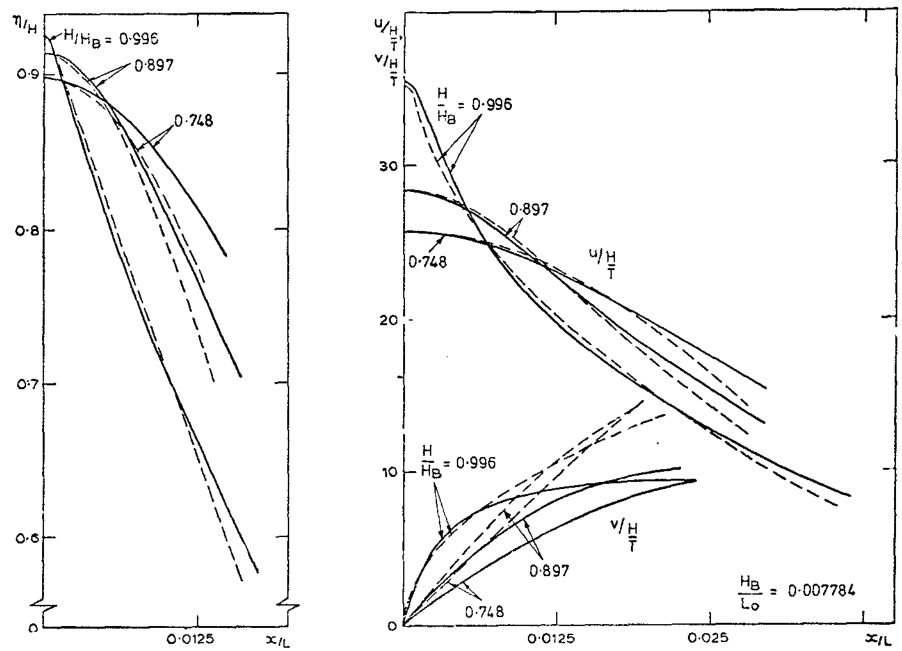

FIG. 8. ELEVATIONS ABOVE MEAN WATER LEVEL (η) AND HORIZONTAL AND VERTICAL VELOCITY COMPONENTS OF SURFACE PARTICLES NEAR THE CREST OF STEEP WAVES WITH $d/L_o = 0{\cdot}01$. COMPARISON OF RESULTS FROM STREAM FUNCTION THEORY ———, AND THE ALMOST HIGHEST WAVE APPROXIMATION - - - - -

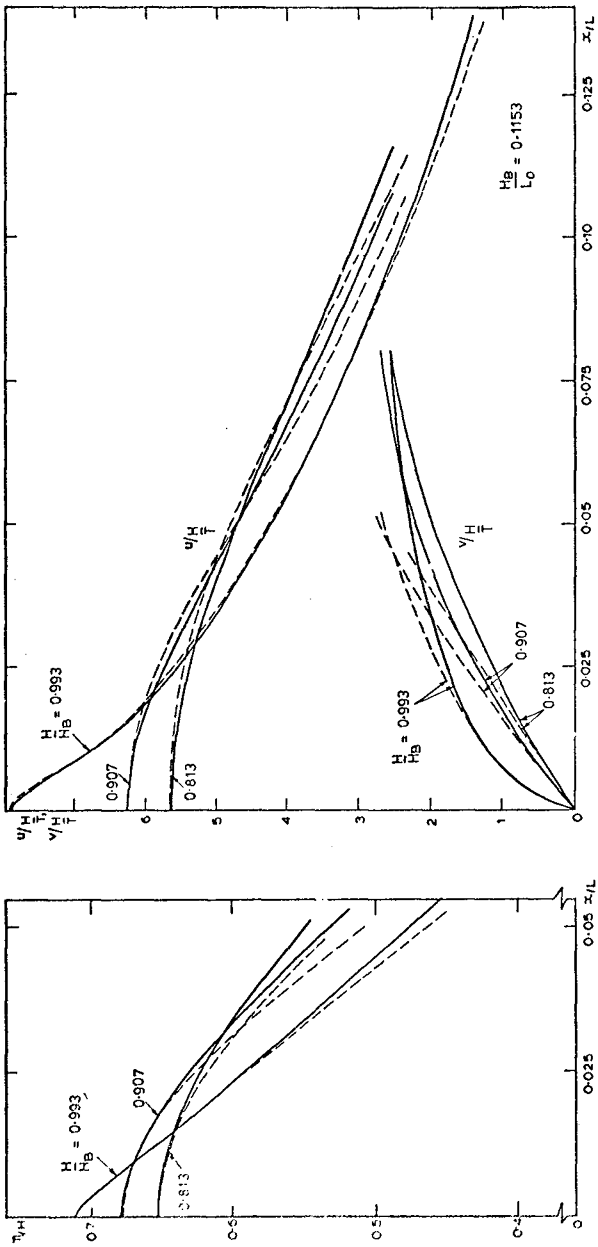

FIG. 9. ELEVATION ABOVE MEAN WATER LEVEL (η) AND HORIZONTAL AND VERTICAL VELOCITY COMPONENTS (u, v) OF SURFACE PARTICLES NEAR THE CREST OF STEEP WAVES WITH $d/L_o = 0{\cdot}2$. COMPARISON OF RESULTS FROM STREAM FUNCTION THEORY ———, AND THE ALMOST HIGHEST WAVE APPROXIMATION - - - - -

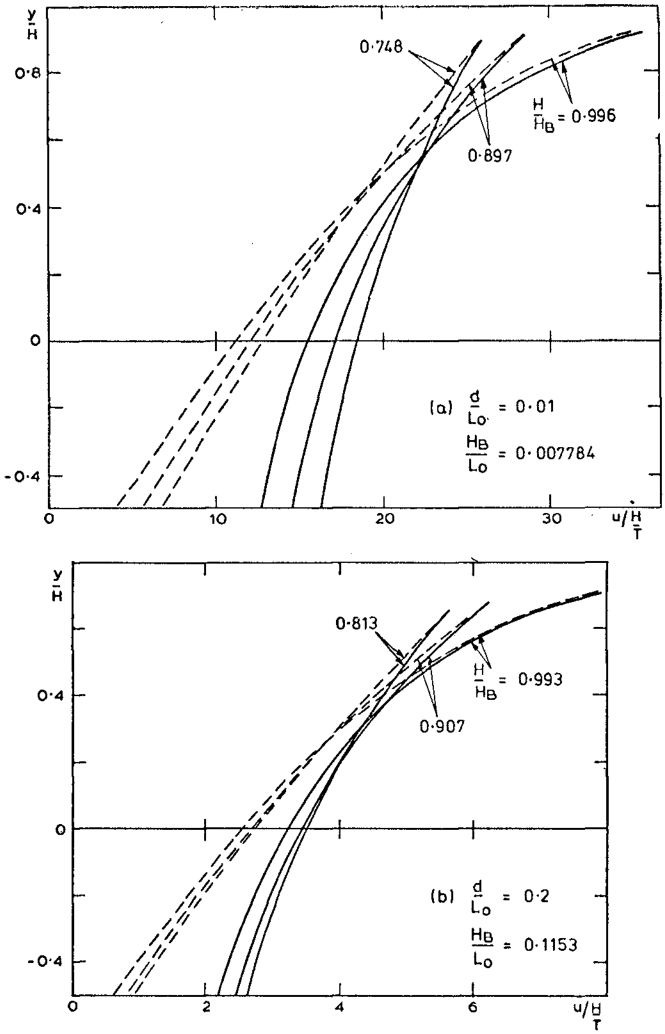

FIG. 10. HORIZONTAL PARTICLE VELOCITIES ON A VERTICAL LINE THROUGH THE CREST COMPUTED FOR STEEP WAVES BY MODIFIED STREAM FUNCTION THEORY ———— AND THE ALMOST HIGHEST WAVE APPROXIMATION — — — — : (a) $d/L_0 = 0{\cdot}01$, (b) $d/L_0 = 0{\cdot}2$

for d/L_0 = 0.01 and 0.2, respectively. For deep water conditions the approximation may be extended beyond this region by the method of Longuet-Higgins and Fox (1978).

5. CONCLUSIONS

The extended Stokes theory (Cokelet, 1977) may be used in an engineering context in the same framework as conventional wave theories. From the data which Cokelet tabulated suitable input parameters can be interpolated from known water depth, wave height and period. Integral wave parameters can then be computed for any conditions specified in this way, but the velocity potential series, from which surface profiles and particle velocities and accelerations are derived, does not converge adequately in the region above the broken line in Fig. 1. Full solutions in this region have, however, been obtained for wave heights up to within 1% of limiting wave height by the modified stream function theory (Chaplin, 1980).

Surface elevations and particle velocities near the crest, computed by these methods differ significantly from those calculated by conventional wave theories. However, accurate partial solutions can also be computed much more rapidly by approximations for limiting and near-limiting wave heights (Longuet-Higgins, 1973, 1979a). The almost highest wave approximation (Longuet-Higgins, 1979a) can be applied to specified conditions on the basis of data derived by interpolation from Cokelet's tables. A method for interpolation is described above, and it has been found to yield accurate results with very little computational effort. A subsequent paper will present results for shoaling and refraction computed in this way from Cokelet's tables.

6. ACKNOWLEDGEMENT

This work is supported by the Science Research Council through the North West Universities Consortium for Marine Technology.

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