The Dissipation of Wave Energy by Turbulence 1

by

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I. Abstract

This paper first gives a brief review of the existing research worke on the laws governing the dissipation of wave energy by turbulence. Starting from the general theory of turbulent motion and the writer's suggestion in regard to the mixing length of water particles in two-dimensional flow and making use of the principle of dimensional analysie and the trochidal wave theory, a formula has been derived to compute the mean diesipation per unit time and per unit horizontal area of wave energy due to turbulence. The formula takes the horizontal and vertical gradients of both the horizontal and vertical velocity fields into consideration. Coefficient in the formula has been determined through laboratory experimente.

2. A Brief Review of Former Research Works

It is not far since the presentation of the suggestion that fluid turbulence playe an important role also in wave motion. The scientific researches on the laws governing the dissipation of wave energy by turbulence were started in the late forties of this century, but only in and after the fifties of this century had more research works been gradually done.

There are generally three different methods to study the problem of turbulent dissipation of wave energy. The first one baees solely on the principle of dimensional analyeie, making use of the π-theorem. The advantages of this method lie in the simplioity of the process of derivstion, but the selection of the independent variables and the determination of the formula patterns are to a certain degree arbitrary. The typical example applying thie method of analysis can be found in literature (I).

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The second method utilizes the theoretical relationship of viscous dissipation of wave energy, replacing the coefficient of kinematic viecosity by means of the coefficient of kinematic eddy viscosity and then finds the functional relationship between the latter coefficient and the relevant physical quantities characterizing wave motion, applying the principle of dimensional analysis. The virtues and defects of this method are basically the same as that of the first method. Its typical example of application can be found in literature (2).

The third method makes use of the theory of turbulent flow. This method proceeds from the internal structure of the current, gives a deeper insight into the essence of the phenomenon and therefore has been widely used. Nevertheless, the existing theories take only the vertical gradients of the horizontal velocity field into consideration. But in wave motion, the magnitudes of the horizontal and vertical gradients of both the horizontal and vertical velocity fields are of the same order. They should be considered simultaneously. Literatures (3) and (4) can be referred to as the examples of application of this method.

The discrepancies among the results of the existing research works are very great. For instance, according to Жуковец (5), the average rate of energy dissipation of wave motion due to turbulence is 108 times as great as that computed by means of the formula suggested by Крылов (1). Similarly, if one uses the results of Шулейкин (6) and Доброклонский (3), the calculated values of this quantity will be several times to nearly one hundred times as great as that of Крылов for flat waves and steep waves respectively.

3. Theoretical Analysis

For two-dimensional turbulent flow, one may assume (7), (8)

$$\left.\begin{aligned} \tau_{xy} &= \rho\varepsilon\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\\ \tau_{xx} &= -\dot{C}_1^2+2\rho\varepsilon\frac{\partial u}{\partial x}\\ \tau_{yy} &= -\dot{C}_2^2+2\rho\varepsilon\frac{\partial v}{\partial y} \end{aligned}\right\} \qquad (1)$$

in which: τ_{xy} ----y component of the turbulent stress acting on a surface element, the outward normal of which is parallel to the X-axis;

τ_{xx} and τ_{yy} have a similar meaning;

ρ ---- mass density of liquid;

ε ---- coefficient of kinematic eddy viscosity;

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U, V ---- component velocities averaged over time in the X and Y directions;

C_1, C_2 ---- constants;

Referring to Prandtl's suggestion (8), in the case of two-dimensional flow, one may put

$$\varepsilon = \ell^2 |F| \tag{2}$$

Where

$$F^2 = 2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2 \tag{3}$$

ℓ is the mixing length of fluid particles.

According to the trochoidal wave theory of deep water (7), the Cartesian coordinates of water particles are:

$$\left.\begin{aligned} x &= a + \frac{1}{2} h e^{Kb} \sin\varphi \\ y &= b - \frac{1}{2} h e^{Kb} \cos\varphi \end{aligned}\right\} \tag{4}$$

where: $\varphi = Ka + \sigma t$ $\quad K = \frac{2\pi}{\lambda}$ $\quad \sigma = \frac{2\pi}{T}$;

λ ---- wave length;

T ---- wave period;

t ---- time;

h ---- wave height;

a, b ---- the Lagrangian coordinates of a water particle, b=0 at free surface;

e ---- base of the natural logarithm;

X,Y ---- the horizontal and vertical coordinates of water particle (a, b) at time t; X-axis coincides with the central line of water surface and its positive direction is opposite to that of wave propagation; Y-axis is vertical and positive upwards. It passes wave trough at t=o.

Differentiating Eq.(4) with respect to t yields

$$\left.\begin{aligned} u &= \frac{\partial x}{\partial t} = \frac{1}{2} h\sigma e^{Kb} \cos\varphi = -\sigma(y-b) \\ v &= \frac{\partial y}{\partial t} = \frac{1}{2} h\sigma e^{Kb} \sin\varphi = \sigma(x-a) \end{aligned}\right\} \tag{5}$$

Differentiating Eqs.(4) and (5) with respect to X and Y successively and simplifying the resulting equations gives

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= \frac{-\sigma k_1 \sin\varphi}{1-k_1^2}, \quad & \frac{\partial u}{\partial y} &= \frac{\sigma k_1 (k_1 + \cos\varphi)}{1-k_1^2} \\ \frac{\partial v}{\partial x} &= \frac{-\sigma k_1 (k_1 - \cos\varphi)}{1-k_1^2}, \quad & \frac{\partial v}{\partial y} &= \frac{\sigma k_1 \sin\varphi}{1-k_1^2} \end{aligned}\right\} \tag{6}$$

in which 4

$$k_1 = \frac{\pi h}{\lambda} e^{Kb}$$

Hence

$$\left.\begin{aligned} F^2 &= \frac{4\sigma^2 k_1^2}{(1-k_1^2)^2} \\ |F| &= \frac{2\sigma k_1}{1-k_1^2} \end{aligned}\right\} \qquad (7)$$

Substituting Eqs.(2),(6) and (7) into Eq.(I) and rearranging the results, one can obtain

$$\left.\begin{aligned} \tau_{xy} &= \frac{4\rho l^2\sigma^2 k_1^2 \cos\varphi}{(1-k_1^2)^2} \\ \tau_{xx} &= -C_1^2 - \frac{4\rho l^2\sigma^2 k_1^2 \sin\varphi}{(1-k_1^2)^2} \\ \tau_{yy} &= -C_2^2 + \frac{4\rho l^2\sigma^2 k_1^2 \sin\varphi}{(1-k_1^2)^2} \end{aligned}\right\} \qquad (8)$$

Based on the theory of turbulent flow (9), the energy loss ψ of turbulent motion per unit time and per unit liquid volume is

$$\psi = \tau_{xx}\frac{\partial u}{\partial x} + \tau_{yy}\frac{\partial v}{\partial y} + \tau_{xy}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \qquad (9)$$

Substituting Eq. (6) and (8) into Eq. (9) yields

$$\psi = \frac{8\rho l^2\sigma^3 k_1^3}{(1-k_1^2)^3} + (C_1^2 - C_2^2)\frac{\sigma k_1 \sin\varphi}{1-k_1^2} \qquad (10)$$

If one imagines two vertical planes to be drawn at unit crest width apart, parallel to the direction of wave propagation and extended from water surface to bottom, the total turbulent dissipation E_λ per unit time and per wave length of the fluid between these planes is

$$E_\lambda = \iint \psi \, dx\,dy \qquad (11)$$

According to the rule of changing of variables (10), it follows immediately that

$$\iint \psi \, dx\,dy = \iint \psi \left|\frac{D(x,y)}{D(a,b)}\right| da\,db \qquad (12)$$

in which $\left|\frac{D(x,y)}{D(a,b)}\right|$ is the functional determinant of X,Y with respect to a,b.

Differentiating Eq. (4) with respect to a and b successively, substituting the results into the expression of $\left|\frac{D(x,y)}{D(a,b)}\right|$ and simplifying leads to

$$\left|\frac{D(x,y)}{D(a,b)}\right| = 1 - k_1^2 \qquad (13)$$

From Eqs. (10)-(13) it may be seen that

$$E_\lambda = 8\rho\sigma^3 \int_{-\infty}^{0} \frac{k_1^3}{(1-k_1^2)^2}\left[\int_{a_x=0}^{a_x=\lambda} l^2 da\right] db \qquad (14)$$

In two-dimensional flow, the writer suggests that

$$l = l\left(F, \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}\right) \qquad (15)$$

In accordance with the principle of dimensional analysis, putting

$$l = \left|\alpha F^{\beta_1}\left(\frac{\partial F}{\partial x}\right)^{\beta_2}\left(\frac{\partial F}{\partial y}\right)^{\beta_3}\right|$$

where α is a dimensionless constant, one obtains

$$\beta_1 = 1 \;, \quad \beta_2 + \beta_3 = -1$$

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Thus

$$l = 2\left|\frac{\alpha F^2}{\frac{\partial F^2}{\partial y}}\left[\frac{\frac{\partial F^2}{\partial x}}{\frac{\partial F^2}{\partial y}}\right]^{\beta_2}\right| \tag{16}$$

Differentiating Eq. (7) with respect to X and Y respectively and substituting the raaulte into Eq. (I6) gives

$$l = \left|\frac{\alpha(1-k_1^2)^2}{K(1+k_1^2)(1+k_1\cos\varphi)}\left[\frac{-k_1\sin\varphi}{1+k_1\cos\varphi}\right]^{\beta_2}\right| \tag{17}$$

β_2 is determined from the following conditions: at bottom ($k_1=0$), $l=0$; at water surface (b=0), $l \approx h$. Based on these conditions, one may put $\beta_2=1$. Hence,

$$l^2 = \frac{\alpha^2 k_1^2(1-k_1^2)^4\sin^2\varphi}{K^2(1+k_1^2)^2(1+k_1\cos\varphi)^4} \tag{18}$$

Setting X equal to λ and 0 successively in the first part of Eq. (4) and subtracting yields

$$\lambda = a_{x=\lambda} - a_{x=0} + \frac{1}{2}he^{Kb}\left[\sin(Ka_{x=\lambda}+\sigma t) - \sin(Ka_{x=0}+\sigma t)\right] \tag{19}$$

Evidently, $a_{x=\lambda} - a_{x=0} = \lambda$ is a solution satisfying Eq. (I9). Utilizing this relationship and resolving rational fraction into simpler partial fractions, one gets from Eq. (I8) by integration

$$\int_{a_{x=0}}^{a_{x=\lambda}} l^2 da = \frac{\pi\alpha^2 k_1^2(1-k_1^2)^{3/2}}{K^3(1+k_1^2)^2} \tag{20}$$

Substituting Eq. (20) into Eq. (I4) and then integrating, neglecting the minor terms and applying the relationship $\sigma = CK$ (c is wave celerity), one finds the average dissipation E_T of wave energy due to turbulence per unit time and per unit horizontal area as follows:

$$E_T = \frac{E_\lambda}{\lambda} = \frac{4}{5}\rho\alpha^2c^3\left(\frac{\pi h}{\lambda}\right)^5\left[1-\frac{15}{14}\left(\frac{\pi h}{\lambda}\right)^2\right] \tag{21}$$

4. Experimental Results

In order to find the value of the constant α in Eq. (2I), experiments were conducted in hydraulic laboratory. The ratios of water depths to wave lengths were controlled in these experiments in such a manner that the condition of deep water wave was fulfilled and thus the dissipation due to bottom friction may not enter.

The wave tank is 62.40 m long, 0.80m wide and I.80m deep and has glass panels on both sides throughout its length. It is provided with an end slag mound of slope I:IO to avoid wave reflection. Two wave gauges of resistance type were used. They were placed along the center line of the tank in measuring sections I and 2 which were I8.00m apart. The equation of balance of wave energy between sections I and 2 is

$$E_1 - E_2 = L(E_T + E_\mu + E_a + E_w) \tag{22}$$

where

E_1, E_2---- the average quantity of energy transmitted by waves per unit time and per unit crest width in the direction of wave propagation through sections I and 2 respectively;

E_μ---- rate of dissipation due to fluid viscosity, negligible in comparison with that of turbulent dissipation;

E_a---- the average dissipation per unit time and per unit horizontal area in the boundary surface between air and liquid when wave propagates in calm air;

E_W---- the average dissipation per unit time per unit horizontal area caused by the friction of the sides walls of the wave tank;

L ----- horizontal distance between sections I and 2, L=I8.00m;

In accordance with the trochoidal wave theory, it is well known that

$$\left.\begin{aligned} E_1 &= \frac{\rho g h_1^2 c}{16}\left(1-\frac{\pi^2 h_1^2}{2\lambda^2}\right) \\ E_2 &= \frac{\rho g h_2^2 c}{16}\left(1-\frac{\pi^2 h_2^2}{2\lambda^2}\right) \end{aligned}\right\} \qquad (23)$$

in which h_1 and h_2 are the wave heights in sections I and 2 respectively, and g is the acceleration of gravity. E_a is calculated by means of Шулейкинѣ formula(6), which was derived on the basis of wind tunnel tests.

$$E_a = \bar{\chi}\rho'\frac{hc^2}{T} \qquad (24)$$

where ρ' is the density of air and $\bar{\chi}$ is a dimensionless coefficient. As for the value of E_W, Hunt's result (II) is applied.

$$E_W = \frac{\rho g}{4B}\sqrt{\frac{\pi\mu}{\rho T}}\,h^2 \qquad (25)$$

where B is the width of the wave tank and μ is the dynamic viscosity of water.

From Eqs. (2I)-(25), the value of α^2 can be computed with the help of the measuring data. The result of computation is shown in Table I.

From Table I, the mean value of α^2 can be calculated to be $\overline{\alpha^2} = .0376$, its standard deviation $\sigma = \sqrt{\frac{\Sigma(\alpha^2-\overline{\alpha^2})^2}{8}} = .0209$ and its coefficient of variation $C_V = 55.5\,\%$

Substituting $\overline{\alpha^2}$ into Eq.(2I) yields finally

$$E_T = \frac{3}{100}\rho C^3\left(\frac{\pi h}{\lambda}\right)^5\left[1-\frac{15}{14}\left(\frac{\pi h}{\lambda}\right)^2\right] \qquad (26)$$

5. Conclusions

Three methods of studying the problem of energy dissipation of wave motion due to turbulence have been reviewed and their advantages and disadvantages briefly discussed.

Table I

Experimental Results of Energy Losses due to Wave Motion

Run No.	Water Depth H (cm)	Wave Period T (sec)	Wave Celerity C (cm/sec)	Wave Length λ (cm)	H/λ	Wave Height (cm) h_1	h_2	$\frac{h_1+h_2}{2}$	Water Temperature (℃)
I	90	I.05	I70	I78	0.50	I3.64	I2.70	I3.I7	27.0
2	I05	0.96	I56	I50	0.70	I4.60	I3.I0	I3.85	I7.0
3	I05	I.07	I80	I93	0.54	I5.20	I3.80	I4.50	I7.0
4	I20	I.I0	I63	I79	0.67	I7.I3	I5.I6	I6.I4	II.2
5	I20	I.23	I96	24I	0.50	I7.90	I6.33	I7.I2	25.0
6	I30	I.28	208	266	0.49	I9.33	I7.49	I8.4I	II.2
7	I40	I.24	20I	249	0.56	20.23	I8.47	I9.35	2I.5
8	I40	I.36	22I	300	0.47	36.44	32.47	34.46	22.0

Run No.	Atmospheric Temperature (℃)	Atmospheric Pressure (mm Mercury Column)	Density of Air $\rho' \times 10^2$ (g/cm³)	Dynamic Viscosity of Water $\mu \times 10^2$ (Poise = dyne-Sec/cm²)	$E_1 \times I0^{-3}$ (g-cm/sec²)	$E_2 \times I0^{-3}$ (g-cm/sec²)
I	29.5	75I.4	0.II5	0.854	I,882	I,638
2	20.0	765.9	0.I2I	I.08	I,942	I,578
3	20.0	765.9	0.I2I	I.08	2,470	2,048
4	I2.0	759.7	0.I24	I.26	2,800	2,2I5
5	24.5	758.4	0.II8	0.89I	3,740	3,I30
6	I2.0	759.7	0.I24	I.26	4,645	3,8I8
7	22.0	759.2	0.I20	0.969	4,890	4,090
8	23.0	759.4	0.II9	0.958	I6,700	I3,460

Table I

Experimental Results of Energy Losses due to Wave Motion

Run No.	$\frac{E_1 - E_2}{L}$ (g/sec³)	E_a (g/sec³)	E_w (g/sec³)	$E_T = \frac{E_1 - E_2}{L} - E_a - E_w$ (g/sec³)	α^2	$(\alpha^2 - \overline{\alpha^2})^2 \times 10^4$
1	135.6	9.9	85.2	40.5	0.0161	4.62
2	202.2	13.2	111.2	77.8	0.0138	5.67
3	234.4	12.8	115.2	106.4	0.0332	0.19
4	325.0	14.5	152.2	158.3	0.0274	1.04
5	339.0	14.2	136.3	188.5	0.0593	4.71
6	459.4	16.7	183.5	259.2	0.0783	16.56
7	444.4	18.8	180.5	245.1	0.0465	0.79
8	1800	59.0	544.5	1196.5	0.0262	1.30

$\overline{\alpha^2} = 0.0376$

It seems more properly to approach the subject by making use of the theory of turbulent flow.

The importance of considering simultaneously the horizontal and vertical gradients of both the horizontal and vertical velocity fields have been pointed out. An analytical formula, i.e., the Eq. (26), has been derived theoretically, with the coefficient in it determined experimentally, which can be used to compute the rate of turbulent dissipation of wave energy.

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Appendix I References

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Appendix 2 Notation

a, b ---- Lagrangian coordinates of liquid particles, $b = 0$ at free surface

B ---- width of wave tank

c ---- wave celerity

c_1, c_2 ---- constants

c_v ---- coefficient of variation

D ---- symbol, $\left|\frac{D(X,Y)}{D(a,b)}\right|$ is the functional determinant of X,Y with respect to a, b

e ---- base of the natural logarithm

E_1, E_2 ---- average quantity of energy transmitted by waves per unit time and per unit crest width in the direction of wave propagation through sections I and 2 respectively

E_a ---- average dissipation per unit time and per unit horizontal area in the boundary surface between air and liquid when wave propagates in calm air

E_λ ---- turbulent dissipation of energy per unit time, per unit crest width and per wave length of the liquid volume extended from water surface to bottom

$E_T = E_\lambda / \lambda$

E_μ ---- rate of dissipation due to fluid viscosity

E_w ---- average dissipation per unit time and per unit horizontal area caused by the friction of the side walls of a wave tank

F ---- quantity defined by Eq.(3)

g ---- acceleration of gravity

h, h_1, h_2 ---- wave heights

$K = \frac{2\pi}{\lambda}$ ---- wave number

$k_1 = \frac{\pi h}{\lambda} e^{Kb}$

l ---- mixing length of fluid particles

L ---- horizontal distance between wave gauge stations

t ---- time

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T ---- wave period

u ---- component velocity in X direction

V ---- component velocity in Y direction

X, Y ---- Cartesian coordinates

$\bar{X}$ ---- dimensionless coefficient

$\alpha, \beta_1, \beta_2, \beta_3$ ---- dimensionless constants

ε ---- coefficient of kinematic eddy viscosity

λ ---- wave length

μ ---- coefficient of dynamic viscosity

$\pi = 3.142$

ρ ---- mass density of liquid

ρ' ---- mass density of air

σ ---- radian frequency ($= \frac{2\pi}{T}$); standard deviation

τ_{xy} ---- Y component of turbulent stress acting on a surface element, the outward normal of which is parallel to X-axis

τ_{xx}, τ_{yy} ---- similar in meaning to τ_{xy}

$\varphi = Ka + \sigma t$, Eq.(4)

ψ ---- energy loss of turbulent motion per unit time and per unit liquid volume