CRITERION FOR STABILITY OF SHORELINE PLANFORM

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INTRODUCTION

The problem of beach planform stability has been known for a long time: When does a small perturbation on a straight beach tend to grow with time and when does it tend to be flattened out? The interest in this problem arises from evidence of instabilities occurring in nature, but perhaps more importantly it is a problem that must be taken into account when formulating models for beach evolution and erosion. Existing mathematical models describing shoreline changes assume that the beach planform is stable and in equilibrium. It is therefore important to establish the range of wave conditions for which instabilities could occur, thereby invalidating such models. In the present case our interest is specifically directed towards determining conditions for which a model for shoreline evolution is intangible because of development of local instability.

Grijm (1960) gave an approximate mathematical analysis indicating that at the point where the longshore sediment transport Q as a function of wave angle is maximum the shoreline must either be straight or form a cusp. Under his assumption that Q is proportional to sin 2α the maximum occurs for $\alpha = 45^{\circ}$. Le Mehaute and Soldate (1977) summarizes other studies that essentially arrive at the same results, viz. when the deep water wave angle is greater than 45° the shoreline is unstable. This result did not seem to be substantiated by field or laboratory observations.

In this study of shoreline planform we first derive a criterion for instability of straight beaches. Then assuming that longshore sediment transport is proportional to the alongshore wave energy flux component at the point of breaking we determine the range of deep water wave characteristics and beach slopes which would cause unstable situations to occur.

We consider only the longshore transport and exclude effects of on-offshore transport.

PLANFORM STABILITY CRITERION

Consider a straight beach along which an x-axis is defined, see Fig. 1. We shall take the usual definition of instability as a condition under which an initial infinitesimal disturbance grows in time. Accordingly, we introduce an infinitesimal protuberance on a straight shoreline, which is initially in equilibrium with

$\partial Q/\partial x = 0$. A mass balance on the control volume also shown in Fig. 1 is performed. The seaward boundary of the control volume is chosen at a depth large enough so that the longshore transport is zero there.

Two situations may arise:

a) The longshore transport Q increases across the disturbance, i.e. $\partial Q/\partial x > 0$. To satisfy conservation of mass the increase in transport must come from erosion within the control volume and hence the protuberance is eroded until it disappears. The beach is stable for $\partial Q/\partial x > 0$.

b) In the case of decreasing longshore transport, sediment accumulates in the control volume and the protuberance grows. The beach is unstable for $\partial Q/\partial x < 0$.

The above analysis applies to the case of a protuberance from the beach, the opposite conclusions would be derived for a recession. It is therefore useful to generalize the results by considering Q as a function of the breaking wave angle, i.e. $Q = Q(\alpha_b)$. According to our earlier assumption Q is also a function of deep water wave height, H_o; wave period, T; and beach slope, S; however these can be assumed unaffected by the small disturbance and independent of x.

We can then write

$$\frac{\partial Q}{\partial x} = \frac{\partial Q}{\partial \alpha_b} \frac{\partial \alpha_b}{\partial x}$$

and considering the sign for $\partial \alpha_b/\partial x$, which is positive for the protuberance and negative for the recession the criterion for stability becomes

$$\frac{\partial Q}{\partial \alpha_b} > 0 \qquad \text{STABLE} \tag{1}$$

and similarly

$$\frac{\partial Q}{\partial \alpha_b} < 0 \qquad \text{UNSTABLE} \tag{2}$$

Since α_b is a function of H_o, T, S, and the deep water incident wave angle, α_o, the condition $\partial Q/\partial \alpha_b = 0$ can also be given as a critical value of α_o for given H_o, T, and S. Expressing the stability criterion in terms of deep water wave characteristics makes it easier to determine whether given wave conditions would cause instability. In the following we use linear and nonlinear wave theories to determine those critical values of α_o.

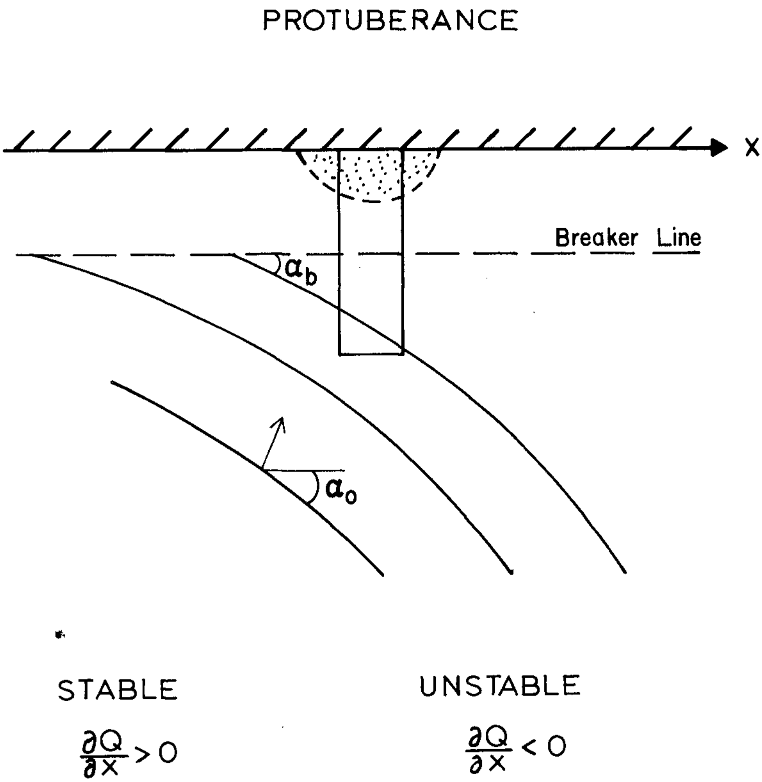

Figure 1. Local instability of a straight shoreline.

LINEAR THEORY

In the case of initially straight parallel contours, the deep water wave characteristics can easily be transformed into breaking wave characteristics by assuming conservation of energy flux between wave orthogonals.

If linear theory is used to express energy flux conservation, one obtains the following:

$$\frac{H_b}{H_o} = \sqrt{\frac{c_{go}}{c_{gb}}} \cdot \sqrt{\frac{\cos \alpha_o}{\cos \alpha_b}} \tag{3}$$

c_g = wave group velocity

Sub index o refers to deep water values and index b refers to breaking. By adding Snell's law; the dispersion relationship and the equation relating wave group velocity to wave phase velocity, c,

$$\frac{L_b}{L_o} = \frac{\sin \alpha_b}{\sin \alpha_o} \tag{4}$$

$$\frac{L_b}{L_o} = \tanh \frac{2\pi h_b}{L_b} \tag{5}$$

$$c_{gb} = \frac{1}{2} \frac{L_b}{T} \left(1 + \frac{\frac{4\pi h_b}{L_b}}{\sinh \frac{4\pi h_b}{L_b}}\right) \tag{6}$$

where h_b is the depth at breaking and L is the wave length, one has 4 equations to determine the 5 unknowns: H_b, h_b, L_b, c_{gb} and α_b. The fifth equation consists of a breaking criterion for which we use the empirical relation proposed by Le Mehaute and Koh (1967).

$$\frac{H_b}{H_o} = 0.76 \, .S^{1/7} \left(\frac{H_o}{L_o}\right)^{-1/4} \tag{7}$$

Equation (7), which was derived for normal incident waves is modified to make it applicable to obliquely incident waves. The deep water wave height, H_o, is replaced by its unrefracted value

$$H_o \rightarrow H_o \frac{\cos \alpha_o}{\cos \alpha_b}$$

and the bottom slope is replaced by the slope

$$S \rightarrow S \cos \alpha_b$$

leading to the result

$$\frac{H_b}{H_o} = 0.76 \left(\frac{H_o}{L_o S^{4/7}}\right)^{-1/4} \cos^{3/8} \alpha_o \cos^{-13/56} \alpha_b \tag{8}$$

By straightforward manipulation of Eqs. (3) - (8), one obtains the following implicit equation for determining the breaking wave angle as a function of incident wave angle, α_o and the parameter $H_o/L_oS^{4/7}$

$$0.76 \left(\frac{H_o}{L_o S^{4/7}}\right)^{-1/4} \cos^{3/8} \alpha_o \cos^{-13/56} \alpha_b$$
$$= \left(\frac{\sin \alpha_o}{\sin \alpha_b}\right)^{1/2} \left(\frac{\cos \alpha_o}{\cos \alpha_b}\right)^{1/2} \left\{ 1 + 2 \ln \left(\frac{\sin \alpha_o + \sin \alpha_b}{\sin \alpha_o - \sin \alpha_o}\right) \left[\frac{4 \sin \alpha_o \sin \alpha_b}{\sin^{-2} \alpha_o - \sin^{-2} \alpha_b}\right]^{-1} \right\}^{-1/2} \tag{9}$$

The equation is solved numericallyand the results are shown in Fig. 2.

LINEAR STABILITY CRITERION

The usual assumption is that longshore transport is proportional to the longshore component of wave energy flux at the point of breaking:

$$Q \propto H_b^2 c_{gb} \sin 2\alpha_b \tag{10}$$

We can normalize this expression with the constant $H_o^2 c_{go}$ to give the remarkably simple result

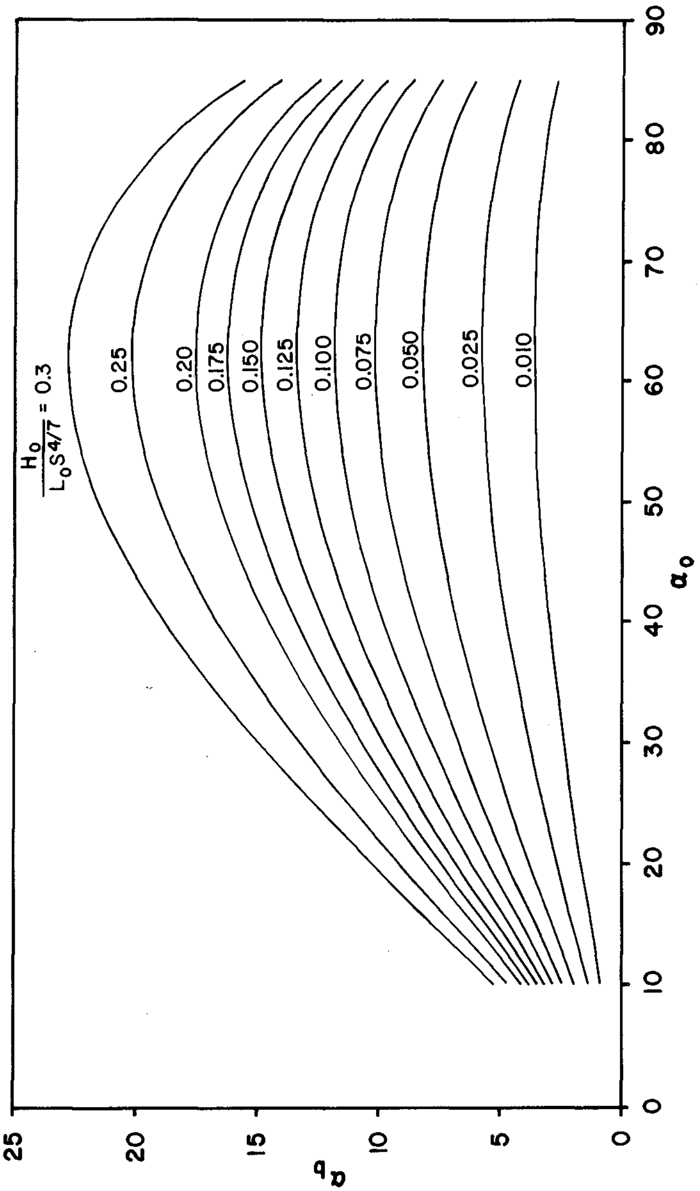

Figure 2. Breaking wave angle as a function of α_o and $\frac{H_o}{L_o S_o^{4/7}}$ linear theory.

$$Q \propto \left(\frac{H_b}{H_o}\right)^2 \frac{c_{gb}}{c_{go}} \sin 2\alpha_b$$

$$= \frac{c_{go}}{c_{gb}} \frac{\cos \alpha_o}{\cos \alpha_b} \cdot \frac{c_{gb}}{c_{go}} 2 \sin \alpha_b \cos \alpha_b$$

$$= 2 \cos \alpha_o \sin \alpha_b \tag{11}$$

Using Eq. (11) and (9) we can determine the critical value of α_o for which $\partial Q/\partial\alpha_b = 0$. This is again calculated numerically and the results are shown in Table 1. For each value of $H_o/L_oS^{4/7}$, two critical values of α_o are found. For incident wave angles between these two values the beach is unstable, otherwise it is stable.

The lower limit of the unstable region is almost invariant to the parameter $H_o/L_oS^{4/7}$ with a value of $\alpha_o = 42°$ while the upper limit decreases with increasing $H_o/L_o{}^{4/7}$. The upper limit in terms of α_o varies between $59°.8 \leq \alpha_o \leq 63°.5$.

NON-LINEAR WAVE THEORY

It is well-known that linear wave theory underestimates the wave height near breaking. Since we are using an empirical breaking criterion derived from observations of real waves, we would expect that the predicted breaking depth is smaller than the real depth and thus the predicted limits for α_o and α_b are also smaller than the real waves. Wang and Le Mehaute (1980) have shown that better results can be obtained for large deep water wave steepness using a non-linear hybrid wave theory which uses cnoidal theory to predict wave height and linear theory for wave length. A detailed description of the rationale and verification of this model is given in that paper. Here, we use the hybrid theory with the breaking criterion (8) to determine critical values of α_o.

A problem arises because wave energy flux is not properly defined for the hybrid theory. We carried out the stability analysis using both the linear expression (11) and the normalized cnoidal energy flux given by

$$Q \propto \left(\frac{H_b}{H_o}\right)^2 \frac{L}{L_o} B \sin 2\alpha_b \tag{12}$$

B is a function of elliptic integrals:

$\frac{H_o}{L_o S^{4/7}}$	Lower Limit α_o	Lower Limit α_b	Upper Limit α_o	Upper Limit α_b
.005	41.9	2.18	63.5	2.57
.01	41.9	3.09	63.5	3.64
.05	41.9	7.00	63.4	8.26
.1	42.0	10.1	63.1	11.9
.2	42.0	15.0	62.5	17.6
.3	42.1	19.5	61.6	22.8
.4	42.3	24.8	59.8	28.7

TABLE 1

α_o limits for shoreline instability
linear theory.

S	H_o/L_o	I Lower Limit α_o	I Lower Limit α_b	I and II Upper Limit α_o	I and II Upper Limit α_b	II Lower Limit α_o	II Lower Limit α_b
.01	0.005	46.7	13.0	77.6	16.7	48.3	13.4
	0.01	46.7	16.4	79.5	21.4	48.3	16.8
	0.03	47.3	21.1	85.0	28.8	48.4	21.6
	0.03	47.7	25.4	>85.		49.2	26.1
.02	0.005	46.7	12.2	77.3	15.5	48.1	12.4
	0.01	46.7	15.1	78.5	19.4	48.2	12.4
	0.02	46.7	18.9	80.2	24.9	48.4	19.4
	0.03	46.9	21.8	83.2	29.4	48.5	22.4
	0.04	47.0	24.2	>85.		48.6	24.9
	0.05	47.1	26.6	>85.		49.0	27.6
0.03	0.005	46.4	11.7	77.7	14.9	48.2	12.0
	0.01	46.6	14.4	78.0	18.6	48.3	14.8
	0.03	46.7	20.5	80.0	27.1	48.4	21.1
	0.05	46.7	24.4	>85.		48.5	25.1
	0.07	46.4	27.5	>85.		49.2	29.0
0.05	0.005	46.5	11.1	77.3	14.2	48.1	11.4
	0.01	46.7	13.8	78.2	17.6	48.3	14.1
	0.03	46.4	19.2	79.2	25.2	48.3	19.8
	0.05	46.3	22.5	80.0	30.2	48.3	23.3
	0.07	46.0	25.0	>85.		48.5	26.1
	0.09	45.5	27.0	>85.		48.6	28.5
	0.11	45.0	28.7	>85.		49.0	31.0
0.1	0.005	46.2	10.4	77.2	13.3	48.2	10.8
	0.01	46.2	12.8	77.3	16.4	48.2	13.2
	0.03	46.3	17.9	77.9	23.2	48.2	18.5
	0.05	46.3	20.8	79.2	27.5	48.2	21.5
	0.07	45.6	22.8	80.2	30.8	48.2	23.8
	0.09	45.4	24.6	80.8	33.8	48.2	25.8
	0.11	45.3	26.0	82.5	36.6	48.2	27.4
	0.13	45.2	27.3	>85.		48.2	28.9
	0.15	44.8	28.4	>85.		48.3	30.0

Table 2

Lower and upper limits for unstable shoreline regimes. I is computed using linear wave energy transport Eq. 11. II is computed using cnoidal wave energy transport Eq. 12.

$$B = B(m) \equiv \frac{1}{m^2} \left[\frac{1}{3} (3\ m^2 - 5\ m + 2 + (4\ m - 2)\ \frac{E(m)}{K(m)} \right.$$
$$\left. - (1 - m - \frac{E(m)}{K(m)})^2 \right] \tag{13}$$

K(m) and E(m) are complete elliptic integrals of the first and second kind respectively. The parameter m is given by, see e.g. Svendsen and Brink-Kjaer (1973):

$$m\ K^2(m) = \frac{3}{16} \frac{HL_c^2}{h^3} \tag{14}$$

where L_c is the cnoidal wave length parameter.

The critical values of α_o and α_b corresponding to $\partial Q/\partial\alpha_b = 0$ are presented in Table 2. Again two values for α_o are found indicating a bounded region of instability.

It is seen that as expected the critical values are somewhat greater than those computed using linear theory. The difference between using linear wave or cnoidal wave energy flux is small and only influences the lower limit. The upper limit is not affected by the choice of energy flux expression because it is determined practically by the maximum value of α_b for given S and H_o/L_o.

Of greater importance is the fact that the non-linear theory predicts a larger zone of instability since the upper limit is significantly higher than predicted by linear theory. For large deep water wave steepness the upper limit for α_o is greater than 85^o, however, this may be due to the fact that we are outside of the region for which the breaking criterion and wave theory are verified. Another source of uncertainty derives from the empirical relationship between longshore transport and energy flux used in this study. When better parameterization of transport rates becomes available the same methodology can however be used to investigate for instabilities.

CONCLUSION

As part of our effort to develop useful models for the prediction of long term shoreline evolution we have investigated the practical limitations imposed on the models by the external conditions. The local instability phenomenon considered in this study is an example of such a limitation. Other areas that need further research involves the definition of an equivalent monochromatic wave from a multi-directional spectrum and the temporal discretization of an observed wave record.

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