SECOND ORDER THEORY OF MANOMETER WAVE MEASUREMENT

F. BIESEL

ABSTRACT

The paper refers to pressure gage wave measurements . First order transformation of the pressure spectrum into a surface level spectrum leads to hitherto unexplained discrepancies with prototype simultaneous pressure and level measurements . Use of second order gravity wave theory allows to draw the following conclusions . Second order effects appear to give a reasonable explanation of the observed discrepancies . A complete check would require specially made wave measurements and analyses . Second order corrections do not significantly affect mean values, such as significant height, if the manometer depth is not unduly large .

1 INTRODUCTION

When measurements of irregular sea waves are carried out with an immersed pressure gage, standard procedures, based on random oscillations theory, allow to compute an estimate of the "pressure" power spectrum . With the further help of linear gravity wave theory, this spectrum can be transformed into a first approximation of the corresponding free surface spectrum .

Simultaneous measurements of pressure and surface levels have shown that this first approximation is not always satisfactory. For instance, CAVALERI (1980) found discrepancies as large as 20 % between surface spectra computed in this way and spectra computed from direct surface measurements . As a rule, wave attenuation with depth appeared to be larger than the theoretical value for the low frequency parts of the spectra and lower for the high frequency parts .

This poor agreement was rather surprising because laboratory tests, made with regular waves, have consistently shown that first order wave theory gives a very reliable estimate of the ratio between wave pressure and wave height . Non linear corrections are only necessary for large wave length to depth ratios or close to breaking wave heights .

The present paper results from an attempt to reconcile fact and theory, using second order wave formulae . This line of research may seem rather doomed beforehand as second order terms have no influence on the pressure to height ratio of "regular" waves . However, this proved not

1 General Manager of AMTEC 8,rue Jean Goujon 75008 PARIS FRANCE

to be the case for irregular waves and, actually, the second order theoretical value of these ratios may widely differ from their first order approximations .

2 NOTATIONS

The space coordinates are x,y and z . The origin is at the mean water level . The x and y axes are horizontal and the z axis is vertical upwards . The other notations are :

t time	g acceleration of gravity	Z surface elevation
H water depth	p_o atmospheric pressure	p water pressure
ρ water unit mass	$Q=(p-p_o)/(\rho \cdot g) + z$	

Random spectra are discretized into I "lines" (regular unidirectional component waves) . The ith line has the following parameters :

a_i amplitude ω_i angular frequency k_i wave number θ_i random phase

α_i angle of wave direction of propagation with positive x axis

For the theoretical outline and numerical examples, only unidirectional waves are considered (i.e. $\alpha_i=0$), moreover, spectral lines are equidistant (i.e. $\omega_i=i \cdot d\omega$, k_i being given by $\omega_i^2=g \cdot k_i \cdot \tanh[k_i \cdot H]$) .

Other parameters are defined to simplify the formulae in the text . Those relating to only one line are defined with the index "i" . Those relating to two lines are defined with indices "i,j" . These latter parameters are either what we shall call "add" type quantities, marked with a prime, or "subtract" type quantities, marked with a double prime . For both these quantities, the indices are not written except when they are different from i,j . "Add" terms exist for i=j but "subtract" terms are not used in this case .

$u_i=k_i \cdot (x \cdot \cos\alpha_i + y \cdot \sin\alpha_i) - \omega_i \cdot t + \theta_i$	$u'=u_i + u_j$	$u''=u_i - u_j$
$c_i= \cos u_i$	$c'= \cos u'$	$c''= \cos u''$
$s_i= \sin u_i$	$s'= \sin u'$	$s''= \sin u''$
	$k' = k_i + k_j$	$k''= k_i - k_j$
	$\omega' = \omega_i + \omega_j$	$\omega''= \omega_i - \omega_j$
		$\gamma''= \cos (\alpha_i-\alpha_j)$
$C_i= \cosh(k_i \cdot H)$	$C'= \cosh(k' \cdot H)$	$C''= \cosh(k'' \cdot H)$
$S_i= \sinh(k_i \cdot H)$	$S'= \sinh(k' \cdot H)$	$S''= \sinh(k'' \cdot H)$
$CZ_i= \cosh[k_i \cdot (H+z)]$	$CZ'= \cosh[k' \cdot (H+z)]$	$CZ''= \cosh[k'' \cdot (H+z)]$
$SZ_i= \sinh[k_i \cdot (H+z)]$		

3 OUTLINE OF SECOND ORDER RANDOM WAVE THEORY

In this section, as mentioned above, we shall use a discretized form of random wave theory which allows numerical computations to be carried out for any spectral shape. We shall also only consider unidirectional waves, travelling in the direction of the positive x axis.

Surface elevation Z1 of such a discretized first order random wave is given by

$$Z1 = \sum_i a_i \cdot c_i$$

This means that we have replaced the continuous power spectrum by a line spectrum. We shall also assume that the spectrum lines are equidistant along the frequency axis. If the frequency step $d\omega$ tends to zero, with a proportional increase in the number of lines and decrease of the line intensities (a_i^2), the resulting motion tends to a truly random one with a spectrum density $S(\omega)$ such that

$$S(\omega_i) = a_i^2/(2 \cdot d\omega)$$

For each set of values given to the random phases θ_i we have different "realizations" of the random function Z1 but our real interest will be in the mean values, over the entire population of these realizations.

Wave elevation to the second order of approximation is given by

$$Z = \sum_i a_i \cdot c_i + \sum_i \sum_j a_i \cdot a_j \cdot B' \cdot c' + \sum_i \sum_j a_i \cdot a_j \cdot B'' \cdot c''$$

B' and B" are rather complex, therefore, their value is given in the appendix together with similar coefficients in the pressure formulae.

At any point (x,z), Z will be a sum of sine functions having frequencies of the form $n \cdot d\omega$. Each of these sine functions will itself be a sum of the nth first order term ($a_n \cdot c_n$) and of a number of second order components which will have one of the two following forms

$$a_i \cdot a_{n-i} \cdot B'_{i,n-i} \cdot c'_{i,n-i} \qquad (\text{"add" terms})$$

$$a_i \cdot a_{i-n} \cdot B''_{i,i-n} \cdot c''_{i,i-n} \qquad (\text{"subtract" terms})$$

Because of the random phase lags in these components, the average intensity A_n^2 of the nth line, in the second order power spectrum of Z, will be the sum of the squares of the components amplitudes

$$A_n^2 = a_n^2 + \sum_i (a_i \cdot a_{n-i} \cdot B'_{i,n-i})^2 + \sum_i (a_i \cdot a_{i-n} \cdot B''_{i,i-n})^2$$

Thus, starting from a "first order" power spectrum with line intensities a_n^2, we can compute a second order spectrum with line intensities A_n^2 for the wave elevation Z.

We can do the same for the pressure variation at a depth -z. However, rather than the pressure p, we shall use the difference between p and the static pressure, divided by $\rho \cdot g$, i.e. expressed in height of water.

We shall call this quantity Q . We have a similar formula

$$Q = \sum_i q_i \bullet c_i + \sum_i \sum_j a_i \bullet a_j \bullet E' \bullet c' + \sum_i \sum_j a_i \bullet a_j \bullet E'' \bullet c''$$

$$\text{where} \quad q_i = a_i \bullet (CZ_i / C_i)$$

(CZ_i/C_i) is the classical first order ratio between pressure and surface elevation for regular waves . As shown above for Z, these second order formulae allow us to compute the average intensity Q_n^2 of the nth line in the second order power spectrum of Q .

Finally, the apparent pressure reduction,for the frequency $i \bullet d\omega$,will be

$$Q_i/A_i \quad \text{instead of} \quad q_i/a_i = CZ_i/C_i$$

Following CAVALERI's example, we shall use the ratio α of these two ratios

$$\alpha = (Q_i/A_i)/(CZ_i/C_i)$$

to show the numerical results in a simple way .

4 NUMERICAL EXAMPLES

The numerical examples concern an unidirectional storm wave having a JONSWAP type first order spectrum . The peak period is 4 seconds, the peak enhancement coefficient is 5 and the significant height is 2.3 m . This spectrum is approximated by nineteen lines defined as follows

f(Hertz)	a^2 (m^2)	f(Hertz)	a^2 (m^2)
0.025	0	0.275	0.217
0.05	0	0.3	0.05
0.075	0	0.325	0.0286
0.1	0	0.35	0.0221
0.125	1.08 E-08	0.375	0.0169
0.15	0.00014	0.4	0.0129
0.175	0.0054	0.425	0.0099
0.2	0.0237	0.45	0.0077
0.225	0.083	0.475	0.006
0.25	0.235		

The water depth H is 12 meters for all examples .

Figure 1 shows the computed values of α for a manometer resting on the bottom . These values are very close to unity throughout the frequency range of the strong spectrum lines but they drop very steeply at the low frequency end . In such a case, there seems to be practically no significant error due to the neglect of second order terms .

The above spectrum was then slightly modified to incorporate some low frequency agitation such as may be due to surf beats or distant storms. For this purpose, the three first lines of the spectrum were given the value 0.02 instead of zero . The effect on the values of α were quite significant as shown in figure 2 (thick lines) . This result was much more similar to CAVALERI's observations which are shown in thin lines on the same figure, as published in his paper, for manometer depth of the same order of magnitude . Values of α computed for manometer depth of 6 and 4 meters are shown on figures 3 and 4 together with CAVALERI's results for these depth ranges .

5 PHYSICAL EXPLANATION

A physical explanation can be briefly outlined here . We saw, in section 3, that there were two types of second order terms which we called "add" and "subtract" types . The first are chiefly higher frequency terms, because their frequency is the sum of two component frequencies. The corresponding pressure terms decrease less rapidly with depth than first order terms of the same frequency . This explains why decrease with depth is less for the higher frequencies . The reverse is true for the lower frequencies where subtract type terms decrease faster than their first order counterparts .

6 CONCLUSIONS

Second order theory of random waves appears to offer an explanation of observed discrepancies between attenuation of pressure fluctuations with depth and first order theory . More accurate checks would require to extend spectral analysis of wave records down to very low frequencies . Unfortunately, this raises difficult instrumentation problems .

Some computations were made with multi-directional waves, however, the results were not significantly different from those reported here . It would therefore seem that further investigations may be carried out with relatively "simple" unidirectional waves .

The second order formulae may be used to deduce the first order spectrum from the measured spectrum, neglecting terms of an order higher than two . Incidentally, this raises an interesting philosophical question : in which of the two surface spectra are we more interested, the first or the second order one ?

With moderate manometer depths, it would seem that, for most engineering purposes, second order corrections are not necessary because significant heights are practically unaffected . However, mean frequency of the first order spectrum may be over-estimated, chiefly when there is a sizable amount of low frequency agitation .

7 REFERENCES

CAVALERI, L. (1980) Wave Measurement Using Pressure Transducers - Oceanologica Acta Vol 3 No 3 July .

BIESEL, F. (1966) Les phénomènes du second ordre rayonnants dans les ondes de gravité - Houille Blanche No 4 .

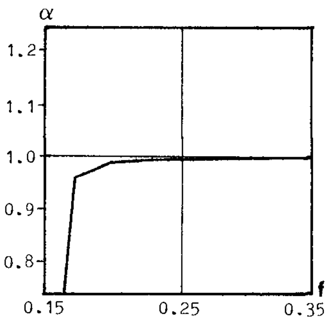

Figure 1

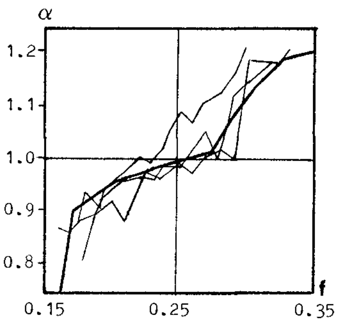

Figure 2

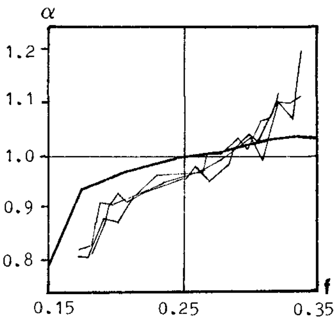

Figure 3

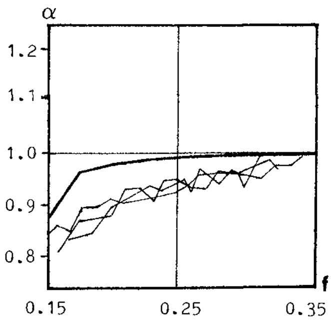

Figure 4

APPENDIX GENERAL SECOND ORDER THREE DIMENSIONAL FORMULAE

In sections 2 and 3, we restricted ourselves to unidirectional waves and to discretization with equidistant lines, but the formulae remain valid for multidirectional waves and for any other spectrum discretization. Thus, each line may represent a component wave having an amplitude, a random phase, a frequency and a direction of propagation independently defined, the sole restriction being that no two lines may have both the same frequency and the same direction. To write out the formulae, we still assume that the lines are identified by a single index varying from 1 to I.

The only information that is needed to use the formulae is the value of the second order coefficients, B' and B" for the water level Z, E' and E" for the pressure Q. We shall also define F' and F" for the velocity potential

$$\varphi = \sum_i F_i \cdot CZ_i \cdot s_i + \sum_i \sum_j a_i \cdot a_j \cdot F' \cdot CZ' \cdot s' + \sum_i \sum_j a_i \cdot a_j \cdot F'' \cdot CZ'' \cdot s''$$

where $$F_i = a_i \cdot \omega_i / (k_i \cdot S_i)$$

The formulae given for F' and F" are taken from BIESEL (1966), with a few changes in notation. Formulae for B', B", E' and E" may be deduced from the latter in a classical way.

$$F' = [\omega_i^3 \cdot S_j / S_i + \omega_j^3 \cdot S_i / S_j + 2 \cdot \omega_i \cdot \omega_j \cdot \omega' \cdot (C_i \cdot C_j \cdot \gamma - S_i \cdot S_j)] / D'$$

where $$D' = 2 \cdot S_i \cdot S_j \cdot (\omega'^2 \cdot C' - g \cdot k' \cdot S')$$

$$F'' = [\omega_i^3 \cdot S_j / S_i - \omega_j^3 \cdot S_i / S_j + 2 \cdot \omega_i \cdot \omega_j \cdot \omega'' \cdot (C_i \cdot C_j \cdot \gamma + S_i \cdot S_j)] / D''$$

where $$D'' = 2 \cdot S_i \cdot S_j \cdot (\omega''^2 \cdot C'' - g \cdot k'' \cdot S'')$$

$$B' = F' \cdot k' \cdot S' / \omega' + [k_i \cdot \omega_i \cdot C_i / S_i + k_j \cdot \omega_j \cdot C_j / S_j + (k_i \cdot \omega_j \cdot C_j / S_j + k_j \cdot \omega_i \cdot C_i / S_i) \cdot \gamma] / 2\omega'$$

$$B'' = F'' \cdot k'' \cdot S'' / \omega'' + [k_i \cdot \omega_i \cdot C_i / S_i - k_j \cdot \omega_j \cdot C_j / S_j + (k_i \cdot \omega_j \cdot C_j / S_j - k_j \cdot \omega_i \cdot C_i / S_i) \cdot \gamma] / 2\omega''$$

$$E' = F' \cdot \omega' \cdot CZ' / g - \omega_i \cdot \omega_j \cdot (CZ_i \cdot CZ_j \cdot \gamma - SZ_i \cdot SZ_j) / (2 \cdot g \cdot S_i \cdot S_j)$$

$$E'' = F'' \cdot \omega'' \cdot CZ'' / g - \omega_i \cdot \omega_j \cdot (CZ_i \cdot CZ_j \cdot \gamma + SZ_i \cdot SZ_j) / (2 \cdot g \cdot S_i \cdot S_j)$$

Note : In the double sums, the terms having indices i and j are equal to those having indices j and i. The above coefficients are doubled so that each pair of values i and j must be taken into account only once and, for "add" terms, when i=j, these coefficients must be divided by two.