MASS TRANSPORT IN VOCOIDAL THEORY

by

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ABSTRACT

The concept of mass transport is theoretically discussed within the framework provided by Vocoidal theory. The Lagrangian mass transport is divided into two parts; firstly treating the fluid as being inviscid and secondly, incorporating viscosity by means of the free surface and bottom boundaries. Eulerian mass transport is defined and is shown to correspond, in deep water, to the net flow predicted by Stokes and others.

INTRODUCTION

The Lagrangian mass transport is defined as the mean velocity of a marked particle and results from the fact that the trajectories of the fluid particles under finite amplitude waves are not closed. Since the original discussion by Stokes (1847), this concept remained theoretically untouched until 1953 when Longuet-Higgins treated it from the point of view of a viscous fluid. Since then many authors, of which Huang (1970) is the most notable, have written on this subject. Eulerian mass transport has only relatively recently been defined in papers by Dalrymple (1976) and Tsuchiya and Yasuda (1981).

Experimentally the effect was observed as early as 1878 by Caligny, the US Beach Erosion Board (1941) and Bagnold (1947). The most comprehensive observations were carried out by Russell and Osorio (1957), whose results confirmed the Longuet-Higgins model. In 1980 Tsuchiya, Yasuda and Yamashita observed drift profiles in a flume incorporating a natural water recirculation process. Results from these tests agreed with both their and the Stokes drift profiles, the net drift being forward throughout the fluid.

VOCOIDAL THEORY

Vocoidal theory was developed to predict the behaviour of non-breaking waves on a horizontal bed (Swart, 1978) and

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applies to water of all depths. The theory is two-dimensional and is based on the equations of motion and continuity and adheres to the bed and free surface boundary conditions. The assumptions on which this theory is based can be summarised by the following three definitions.

Wave profile: $\eta/H = \{(\cos^2(\pi X))^P - \eta^*_t\}$ (1)

Wave celerity: $c^2/gd = \tanh(Nkd)/kd$ (2)

Horizontal orbital velocity:

$$u/c = \eta M(X) k \cosh(M(X)kz)/\sinh[M(X)k(d+\eta)] \quad (3)$$

where $X = (x-ct)/\lambda$; z = vertical coordinate (defined from the bed upwards); H = wave height; λ = wave length; k = wave number $(2\pi/\lambda)$; η^*_t is the dimensionless trough elevation and where P, N and $M(X)$ are parameters depending on the wave conditions (H/d, λ/d).

In deep water, when H/d and λ/d are small, Vocoidal theory reverts to Airy wave theory while in shallow water it becomes solitary wave-like. Curve-fitting techniques were used to allow the use of these numerically determined parameters in a predictive manner for an extensive range of H/d and λ/d values.

Because of an approximation to $M(X)$ during the curve-fitting process, a vorticity was introduced, that is, the theory is rotational whereas in principle it should have been irrotational. The vocoidal vorticity is defined by the equation $\omega = \nabla^2\phi$, where $\phi = -c\eta \sinh(M(X)kz)/\sinh[M(X)k(d+\eta)]$ is the vocoidal stream function.

In order to determine to what extent the theory is rotational the induced vorticity was compared with the vorticity generated by the straining in the irrotational motion (Phillips, 1966) and by the laminar bed boundary layer. The argument on which this comparison is based is that the dissipation of energy in a wave is accompanied by a decrease in wave momentum which, as Longuet-Higgins (1969) showed, is distributed throughout the fluid. This decrease in mean momentum must be accompanied by a mean stress across horizontal planes below the surface. A mean second order viscous stress is set up to balance this loss of momentum. Thus a mean second-order vorticity ω is generated below the free surface.

Comparisons of the average vorticity for various wave conditions are given in Table I, which shows that the average vorticity generated by Vocoidal theory is generally much less than that generated by the viscous and bed shear forces. Since the latter effects are regarded as

negligible in most wave theories Vocoidal theory can be regarded as essentially irrotational to second order.

MASS TRANSPORT

The treatment of mass transport by Vocoidal theory will be divided into three parts: the first dealing with the Lagrangian mass transport in an inviscid fluid, the second with the Lagrangian mass transport incorporating viscous and boundary layer effects, while the third part will deal with Eulerian mass transport.

(i) Lagrangian Mass Transport in an Inviscid Fluid

The mass transport in an inviscid fluid will be treated in a general manner.

The horizontal orbital excursion or displacement ξ_x is defined by $\frac{d\xi_x(t)}{dt} = U_\varepsilon(x,y,z,t)$, where $U_\varepsilon(x,y,z,t)$ is the horizontal velocity following the particle's path. Based on this definition the mass transport can be defined as:

$$U_m = \frac{1}{T}\int_0^T U_\varepsilon(x,y,z,t)dt \qquad (4)$$

For progressive waves of a permanent type the streamlines and particle paths coincide. Thus once the elevation of a specific streamline ψ^o is known, the particle Lagrangian velocity along this streamline can be obtained from $U_\varepsilon(x,t) = U(x,z_{ST})$, the Eulerian velocity at position (x,z_{ST}), where z_{ST}, the streamline elevation, is obtained by iteration of the expression:

$$z_{ST} = \frac{\psi^o}{c} + S(z_{ST}) \quad \text{and where the}$$

function S depends on the particular wave theory involved.

The mass transport is thus numerically obtained from the expression:

$$U_m(z_{ST}) = \sum_{i=1}^{n} U(x_i,z_{ST})\Delta t_i \Big/ \sum_{i=1}^{n} \Delta t_i, \qquad (5)$$

where the time interval Δt_i is defined as $\Delta t_i = \Delta x_i(u_i-c)$ and where n is such than $n.\Delta x_i = \lambda$. Results obtained by this method are similar to those of

Stokes, namely, forward at the free surface and backward near the bed.

TABLE I: THE RATIO OF THE VOCOIDAL VORTICITY TO VORTICITY GENERATED BY THE LAMINAR BED BOUNDARY LAYER AND FROM THE STRAINING IRROTATIONAL MOTION

T_c \ H/d	0.01	0.02	0.05	0.1	0.2	0.5	1.0
1.0	0.0071	0.0066					
2.0	0.0535	0.0482	0.0443				
5.0	0.0045	0.0035	0.0029	0.0027	0.0025	0.0015	
10.0	0.0006	0.0003	0.0003	0.0016	0.0012	0.0003	
20.0	0.0002	0.0032	0.0027	0.0027	0.0028	0.0019	0.0004
40.0	0.0034	0.0032	0.0029	0.0030	0.0033	0.0020	0.0006
60.0	0.0035	0.0032	0.0030	0.0030	0.0034	0.0025	0.0006

TABLE II: VALUES FOR THE NON-DIMENSIONALISED EULERIAN MASS FLOW $q_m/(H^2g/8c)$

T_c \ H/d	0.01	0.02	0.05	0.1	0.2	0.5	1.0
1.0	1.000	1.000					
2.0	1.000	1.000	0.999				
5.0	0.993	0.993	0.996	1.000	1.009	1.026	
10.0	1.031	1.033	1.040	1.074	1.093	1.039	
20.0	0.988	1.038	1.008	0.881	0.708	0.487	0.616
40.0	1.039	0.905	0.654	0.492	0.363	0.253	0.328
60.0	0.870	0.671	0.458	0.336	0.247	0.179	0.166

* Comment: $T_c = T\sqrt{g/d}$

(ii) Lagrangian Mass Transport in a Viscous Fluid

To incorporate the effects of viscosity within the framework of Vocoidal theory an approach similar to that given by Johns (1970) and Isaacson (1976) was used. The bottom boundary layer will be dealt with in some detail.

The horizontal flow velocity outside the bottom boundary is expanded in a Fourier series:

$$U = \sum_{n=-\infty}^{\infty} A_n e^{ik_n X} \tag{6}$$

where $A_o = 0$; $A^*_{-n} = A_n$; $k_n = 2\pi n/\lambda$ and $X = x - ct$. Applying the usual boundary layer approximations we have that the motion within the bottom boundary layer is described by:

$$u_t + uu_x + wu_z = U_t + UU_x + \nu[K(z)\, u_z]_z \tag{7}$$

where u is the boundary layer velocity, U is the velocity above the boundary layer and K(z), the eddy coefficient.

Expanding the boundary layer velocity u by the method of successive approximations (Schlichting, 1968) the equation above reduces to the following two equations:

$$\text{First-order:} \quad u_{1t} = U_t + \nu[K(z)u_{1z}]_z \tag{8}$$

$$\text{Second-order:} \quad u_{2t} + u_1 u_{1x} + w_1 u_{1z} = UU_x + \nu[K(z)u_{2z}]_z \tag{9}$$

Introduce a non-dimensional vertical coordinate η, where $\eta = \sqrt{(\sigma/2\nu)}z$ with $\sigma = 2\pi/T$; T = wave period; ν = kinematic viscosity coefficient and assume laminar flow, K(z) = 1. Further, assuming the first-order boundary layer velocity to be given by:

$$u_1 = \sum_{n=-\infty}^{\infty} A_n[1-F(\eta_n)]e^{ik_n X}, \tag{10}$$

it is found after substitution that the function $F(\eta)$ must satisfy the equation:

$$\frac{d^2F(\eta)}{d\eta^2} + 2iF(\eta) = 0 \tag{11}$$

with boundary conditions $F(0) = 1$ and $F(\infty) = 0$.

Substitution into the second approximation, where the vertical velocity component w_1 is obtained from the continuity equation, and extracting only the real time - independent second-order term, results in an expression for u_2, given by:

$$\bar{u}_2 = \sum_{n=1}^{\infty} \frac{A_n^2}{c} \mathrm{Im}(G(\eta_n)) \qquad (12)$$

$\mathrm{Im}(G(\eta))$ is the imaginary part of $G(\eta)$, with $G(\eta)$ satisfying the equation:

$$\frac{d^2G(\eta)}{d\eta^2} = |F(\eta)|^2 - 2\mathrm{Re}(F(\eta)) + \frac{dF^*(\eta)}{d\eta} \int_0^{\eta} (1-F(\eta))d\eta$$

subject to the boundary condition $G(0) = 0$ and $G(\eta)$ finite as $\eta \to \infty$ (* implies the complex conjugate).

Substituting u_1 and $\bar{u}_2$ into second-order mass transport velocity, defined by Longuet-Higgins (1953), namely,

$$U_m = \bar{u}_2 + \overline{u_{1x} \int_0^t u_1 dt'} + \overline{u_{1y} \int_0^t w_1 dt'} \qquad (13)$$

gives an equation for the transport above the bottom boundary layer as:

$$U_m = \sum_{n=1}^{\infty} \frac{A_n^2}{c} \mathrm{Im}(H(\infty)) \qquad (14)$$

with

$$H(\eta) = G(\eta) + \frac{i}{2}\left[\frac{dF^*}{d\eta} \int_0^{\eta} (1-F(\eta))d\eta - 1 + 2\mathrm{Re}(F(\eta)) - |F(\eta)|^2\right]$$

Generalising the results of Huang (1970) to include both the free surface and interior regions, the mass transport throughout the fluid can be calculated. These results, however, depend on an important approximation, namely that the series solution be arbitrary but finite. If a zero net mass transport is assumed then the mass transport in the interior can be calculated from the expression:

$$U_m = \frac{1}{4} \sum_{n=1}^{m} \frac{A_n^2 \sigma_n k_n}{\sinh^2 k_n d} \left\{2 \cosh 2 k_n d\mu - \frac{3}{2}\left[1-(1-\mu)^2\right] \frac{\sinh 2k_n d}{k_n d} + \frac{9}{2}(1-\mu)^2 - \frac{3}{2}\right\} \qquad (15)$$

with $\mu = z/d$ and the bed defined at $\mu = 0$ (that is, $z = 0$).

Calculations have shown that m need not exceed 50, even for the most highly non-linear case. Figures 1 to 4 show typical mass transport profiles for various values of T_c $(= T\sqrt{(g/d)})$ and H/d. For low values of T_c and H/d (Figure 1) we see that, with the exception of Longuet-Higgins, all profiles correspond to that given by Stokes. It should be noted that in order to distinguish between the

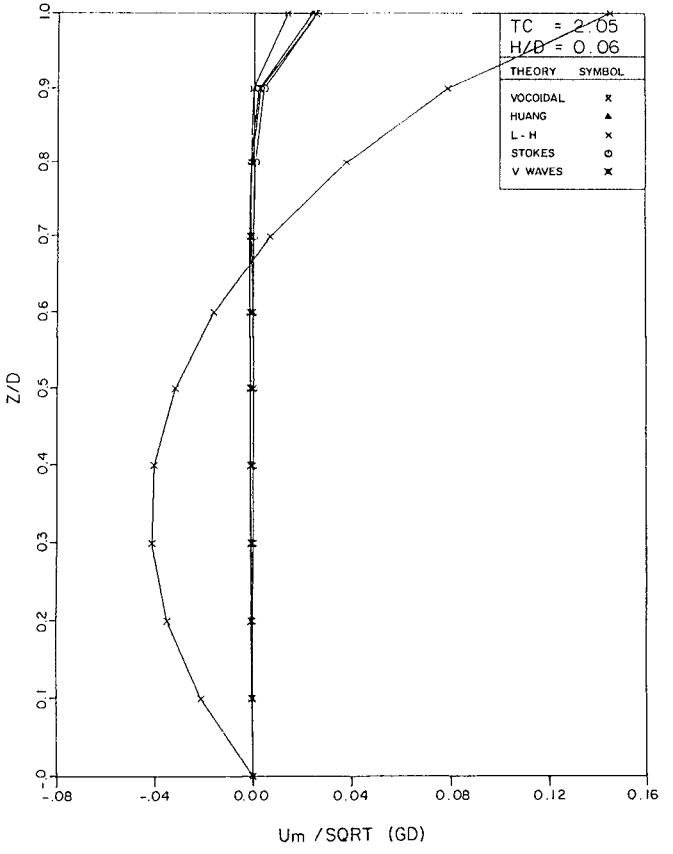

Figure 1 Comparison of the theoretical non-dimensional drift profiles for various wave theories

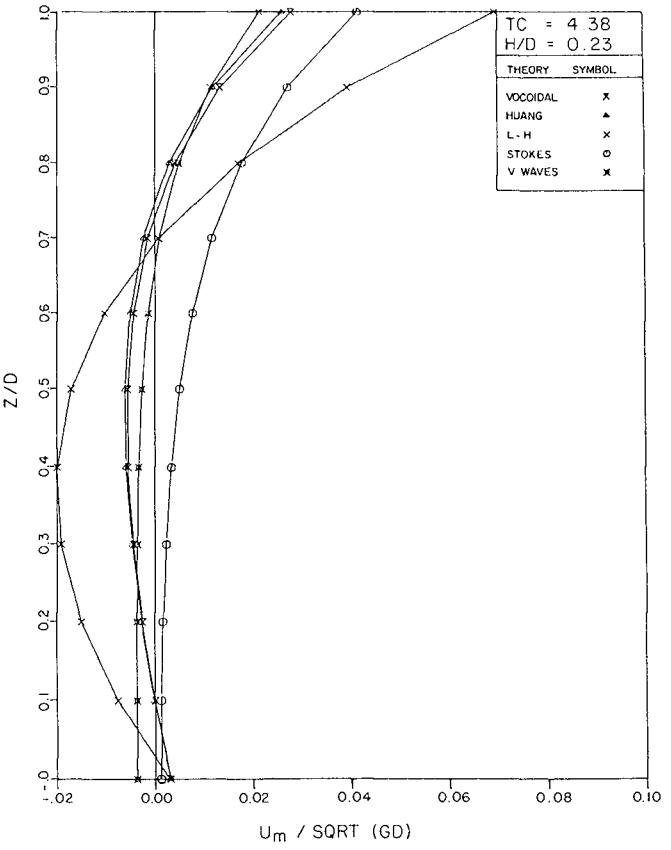

Figure 2 Comparison of the theoretical non-dimensional drift profiles for various wave theories

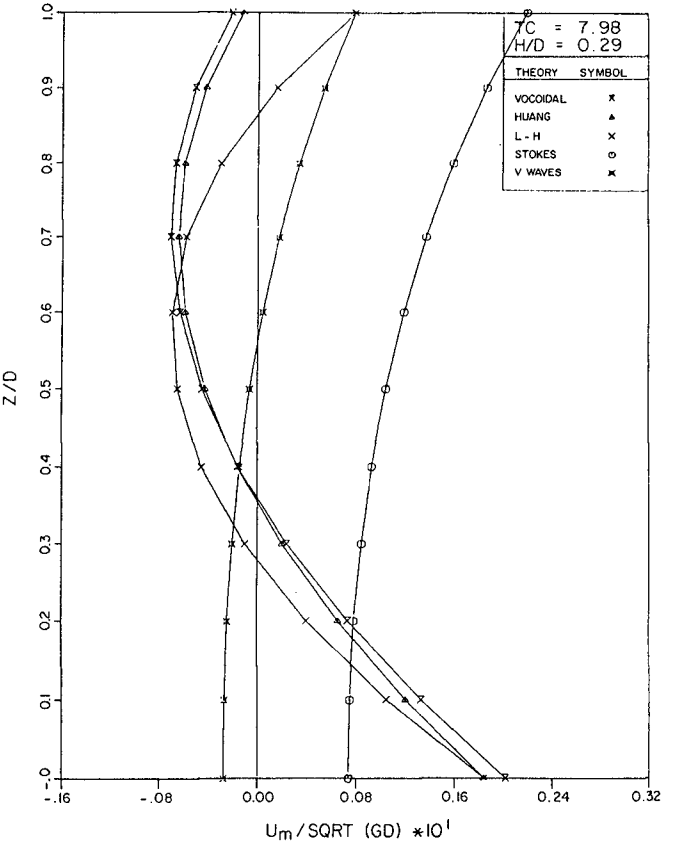

Figure 3 Comparison of the theoretical non-dimensional drift profiles for various wave theories

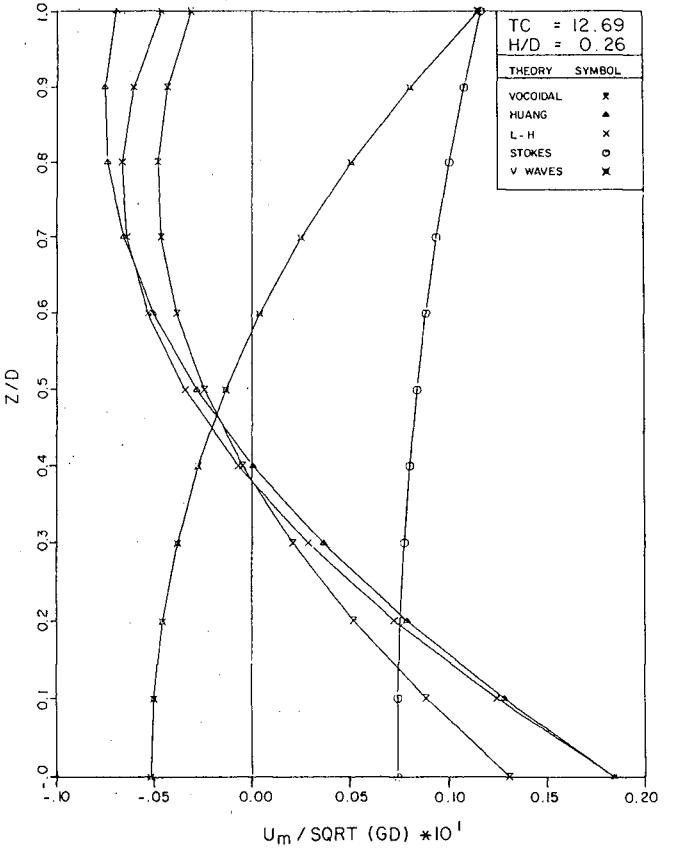

Figure 4 Comparison of the theoretical non-dimensional drift profiles for various wave theories

Stokes mass transport and the Vocoidal inviscid Lagrangian transport (V wave) the Stokes solution was not corrected for zero net mass transport. In order to do this a quantity $- H^2\sigma \coth kd/8d$ must be added to the Stokes results, thus shifting the profile to the left.

In the deep water region (Figures 1, 2 and 3) the mass transport velocity above the bottom boundary layer, as predicted by Vocoidal theory, is greater or equal to the values predicted by Longuet-Higgins and Huang. As the water becomes shallower the opposite is true (Figures 4 and 5). This latter result corresponds to the observations made by Brebner and Collins (1961). Figure 5 gives a comparison of the theoretical profiles with a data set observed by Russell and Osorio (1957).

(iii) Eulerian Mass Transport

The Eulerian mass transport will be determined using Dalrymple's (1976) approach and be defined as the net or average flow past any fixed point in the fluid:

$$M = \frac{\rho}{T} \int_0^T \int_0^{d+\eta} u(z,t)\,dz\,dt \tag{16}$$

Dalrymple obtained, for Airy theory, the well-known solution $M = \rho g H^2/8c$, while for Dean's stream theory he found that $M = -\rho\psi(x,\eta)$, the value of the stream function on the free surface.

Integrating the continuity equation over depth and applying the bed and free surface boundary conditions the net flux in Vocoidal theory is determined to within an integration constant:

$$\int_0^{d+\eta} u\,dz - c\eta = q_m \tag{17}$$

The choice $q_m = 0$ corresponds to a reference frame in which the net mass flux is zero (Tsuchiya and Yasuda, 1981). Assume $q_m \neq 0$, then in the light of Vocoidal theory's second order irrotationality the expression relating kinetic energy density to the momentum flux, namely,

$$\frac{\rho}{\lambda} \int_0^{\lambda} \int_0^{\eta+d} (u^2+w^2)\,dz\,dx = c\rho \int_0^{\lambda} \int_0^{\eta+d} u\,dz\,dx \tag{18}$$

(Starr, 1947; Longuet-Higgins, 1976) can be used to determine q_m. Since $q_m \neq 0$ the expressions for the orbital velocities change so as to include q_m. The resulting quadratic expression in q_m can be numerically

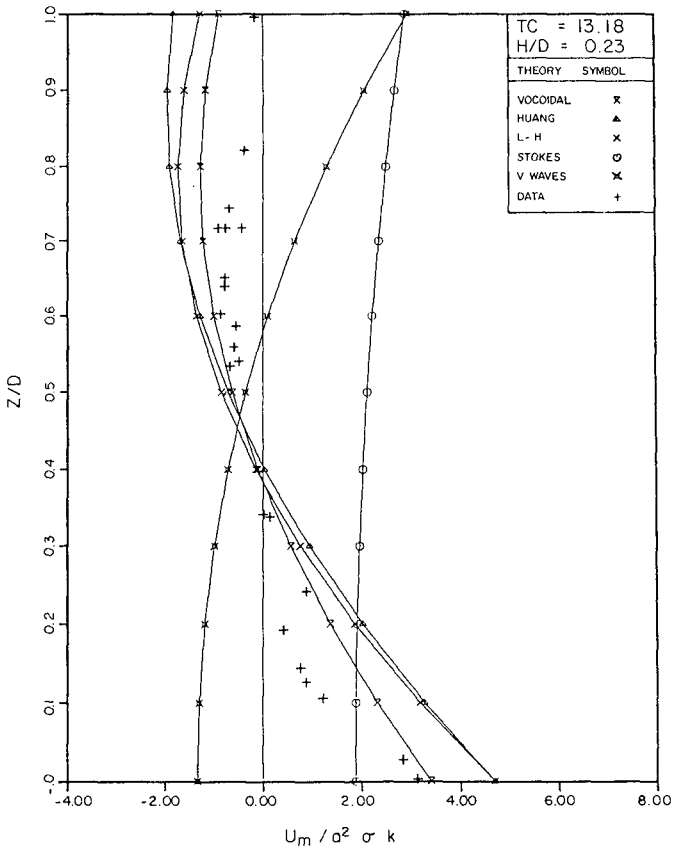

Figure 5 Comparison of the theoretical non-dimensional drift profiles and data observed by Russell and Osorio (1957, Figure 7)

solved, and values for various wave conditions can be seen in Table II. In the deep-water limit the expressions involving q_m reduces to:

$$q_m - q_m^2/cd = H^2g/8c.$$

CONCLUSIONS

(i) If viscosity is neglected the Lagrangian mass transport corresponds in profile to that given by Stokes.

(ii) Vocoidal mass transport including viscous effects is such that in deep water the velocities above the bed are generally greater than those predicted by Longuet-Higgins (1953) and Huang (1970) while for shallow water the reverse is true.

(iii) In deep water both the viscous and inviscid solutions coincide and reduce to the Stokes profile.

(iv) An Eulerian mass transport can be defined which approaches the Stokesian results in deep water but diverge as H/d and λ/d increase.

REFERENCES

BREBNER, A and COLLINS, J I (1961). Onset of turbulence at the bed under periodic gravity waves. Trans. Eng. Inst. Cam., 5 pp 55-62.

DALRYMPLE, R A (1976). Wave-induced mass transport in water waves. ASCE J. Waterways, Harbors Coastal Eng. Div., 102 (WW2) pp 255-264.

HUANG, N W (1970). Mass transport induced by wave motion. J. Mar. Res., 28 pp 35-50.

ISAACSON, M DE ST Q (1976). Mass transport in the bottom boundary layer of Cnoidal waves. J. Fluid Mech., 74 pp 401-413.

JOHNS, B (1970). On the mass transport induced by oscillatory flow in a turbulent boundary layer. J. Fluid Mech., 43 pp 177-185.

LONGUET-HIGGINS, M S (1953). Mass transport in water waves. Phil. Trans. Royal Soc., London, A 245 pp 535-581.

LONGUET-HIGGINS, M S (1969). On the transport of mass by time-varying ocean currents. Deep-Sea Res., 16 pp 431-447.

LONGUET-HIGGINS, M S (1975). Integral properties of periodic gravity waves of finite amplitude. Proc. Royal Soc., London, A 342 pp 157-174.

PHILLIPS, O M (1966). The dynamics of the upper ocean. Cambridge University Press.

RUSSELL, R C H and OSORIO, J D C (1957). An experimental investigation of drift profiles in a closed channel. Proc. 6th Conf. Coastal Eng., pp 171-193.

TSUCHIYA, Y, YASUDA, T and YAMASHITA (1980). Mass transport in progressive waves of permanent type. Proc. 17th Conf. Coastal Eng., pp 70-81.

TSUCHIYA, Y and YASUDA, T (1981). A new approach to Stokes wave theory. Bull. Disas. Prev. Res. Inst., Kyoto Univ., 31 pp 17-34.

SCHLICHTING, H (1968). Boundary layer theory. 6th edition (translated from German by J Kestin), McGraw-Hill.

STARR, V P (1947). A momentum integral for surface waves in deep water. J. Mar. Res., 6 pp 126-135.

SWART, D H (1978). Vocoidal water wave theory, Volume 1: derivation. CSIR Research Report 357, Stellenbosch, South Africa.