NEW EQUATION OF SURACE ELEVATION IN WAVE MOTION

by

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ABSTRACT

A nonlinear solution of wave profile equation directly derived from stream function is submitted. In stead of expressing by trigonometric series to approach the real solution, implicit function is adopted. In the era of electronic computer, Such an expression will be convenient for practical utilization.

Equations in either deep water or in finite water depth are worked out. They are proved more reasonable in graphical shape of wave profile and consistant in the continuity of wave celerity in various depth in comparison with Stokes theory.

INTRODUCTION

For more than 100 years, Stokes' wave theory has been applied to various engineering problems. However, the theory is an approximate solution strictly. Furthermore, we all have the experience that the (n+1) th solution is not guaranteed to be better

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than nth. The only merit is that the equation is expressed explicitly. In the eva of electronic computer, implicit equation can be solved promptly. It is not necessary to be stuck on explicit equation. Reasonable solutions of wave profiles in deep and intermediate water area are worked out directly from stream function in following sections. Their validity and consistancy are examined closely.

FUNDAMENTAL EQUATIONS

(A) *Governing equations*

Water is supposed to be incompressible and irrotational, then the governing equations of wave motion are as follows :

$$\nabla^2\psi = 0 \quad , \quad \nabla^2\phi = 0 \tag{1}$$

$$\nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

x : *abscissa along the water surface, being positive toward the wave direction.*

y : *ordinate vertical to the water surface, being positive upward*

$\psi = \psi\,(x, y, t)$: *stream function*

$\phi = \phi\,(x, y, t)$: *velocity potential*

t : *time*

(B) *Boundary condition equations on free surface*

(a) *Dynamical equation*

$$\left(\frac{\partial \phi}{\partial t} \right)_{y=\eta} + g\eta + \frac{1}{2}\left[\left(\frac{\partial \psi}{\partial x} \right)^2 + \left(\frac{\partial \psi}{\partial y} \right)^2 \right]_{y=\eta} = Q(t) \tag{2}$$

g : *gravitational acceleration*

$\eta = \eta(x, t)$: *fluctuation of water surface elevation with respect to x − axis*

$$\frac{\partial\psi}{\partial x} = -\frac{\partial\phi}{\partial y} = v \;,\; \frac{\partial\psi}{\partial y} = \frac{\partial\phi}{\partial x} = -u$$

u , v : *horizontal and vertical velocity of water particle*

$Q(t)$: *Bernoulli's number varys with time only*

(b) *Kinematic equation*

$$\left(\frac{\partial\phi}{\partial y}\right)_{y=\eta} = \frac{\partial\eta}{\partial t} + \frac{\partial\eta}{\partial x}\left(\frac{\partial\phi}{\partial x}\right)_{y=\eta} \tag{3}$$

(C) *Boundary condition in bottom*

$$\left(\frac{\partial\phi}{\partial y}\right)_{y=-d} = \left(\frac{\partial\psi}{\partial x}\right)_{y=-d} = 0 \tag{4}$$

d : *vertical distance from x − axis to the bottom*

(D) *Assumptions of solution*

The solutions of above mentioned Laplace equation i.e. equation (1) are to be assumed as follows :

$$\psi(x, y, t) = \sum_{n=1}^{\infty} C_n e^{-nkd} \left[e^{nk(d+y)} - e^{-nk(d+y)} \right] \cos nk(x - ct)$$

$$\phi(x, y, t) = \sum_{n=1}^{\infty} C_n e^{-nkd} \left[e^{nk(d+y)} + e^{-nk(d+y)} \right] \sin nk(x - ct)$$

$$\tag{5}$$

The stream function of free surface is :

$$\psi(x, \eta, t) = 0$$

So that

$$\eta(x, t) = \sum_{n=1}^{1} \frac{C_n}{c} e^{-nkd} \left[e^{nk(d+\eta)} - e^{-nk(d+\eta)} \right] \cos nk(x-ct)$$

C_n : *constants to be worked out*

n : 1 , 2 , 3 ……

e : *exponential*

k : *wave number,* $k = 2\pi / L$

L : *wave length*

c : *wave celerity*

The motion is altered to be steady flow by adding an opposite velocity c, $(x-ct)$ *in above equations will be replaced by x, then*

$$\psi(x, y) = cy - \sum_{n=1}^{\infty} C_n e^{-nkd} \left[e^{nk(d+y)} - e^{-nk(d+y)} \right] \cos nkx$$

$$\phi(x, y) = cx + \sum_{n=1}^{\infty} C_n e^{-nkd} \left[e^{nk(d+y)} + e^{-nk(d+y)} \right] \sin nkx$$

$$\eta(x) = \sum_{n=1}^{\infty} \frac{C_n}{c} e^{-nkd} \left[e^{nk(d+\eta)} - e^{-nk(d+\eta)} \right] \cos nkx \qquad (6)$$

NEW EQUATION IN DEEP WATER

In above equations , $d \to \infty$, $n = 1$ *, we get*

$$\psi(x, y) = cy - C_1 e^{ky} \cos kx \qquad (7)$$

$$\eta(x) = \frac{C_1}{c} e^{k\eta} \cos kx \qquad (8)$$

η_{max} *and* η_{min} *are the elevation of crest and trough of wave profile.*

$$\eta_{max} = \eta_1 = \frac{C_1}{c} e^{k\eta_1} \qquad (9)$$

$$\eta_{min} = \eta_2 = \frac{C_1}{c} e^{k\eta_1} \tag{10}$$

Invoke a parameter ω which is defined as the ratio of maximan particle velocity at the wave crest, q_c, to wave celerity, c, namely

$$\omega = q_c / c = -\left(\frac{\partial \phi}{\partial y}\right)_{y=\eta_1} / c \tag{11}$$

ω is to be 0 when wave is breaking, and will be -1 on calm water surface i.e.

$$-1 \leq \omega \leq 0 \quad or \quad 0 \leq 1+\omega \leq 1$$

Substitute (7) to (11)

$$(1+\omega)/k = \frac{C_1}{c} e^{k\eta_1} \tag{12}$$

From equation (9)

$$(1+\omega)/k = \eta_1 = \frac{C_1}{c} e^{k\eta_1} \tag{13}$$

$$C_1 = \frac{c(1+\omega)}{k e^{(1+\omega)}} \tag{14}$$

is obtained.

Consequently :

$$\phi(x, y) = cy - \frac{c(1+\omega)}{k e^{(1+\omega)}} e^{ky} \cos kx \tag{15}$$

$$\eta(x) = \frac{1+\omega}{k e^{(1+\omega)}} e^{k\eta} \cos kx \tag{16}$$

From equations (9), (10) and (13)

$$-\frac{e^{k\eta_2}}{\eta_2}=\frac{e^{k\eta_1}}{\eta_1}=\frac{k\,e^{(1+\omega)}}{1+\omega} \tag{17}$$

H denotes the wave height $H=\eta_1-\eta_2$ 。

Accordingly

$$\frac{k\,e^{(1+\omega)}}{1+\omega}=\frac{e^{k(\eta_1-H)}}{H-\eta_1}=e^{(1+\omega)}e^{-kH}\,/\,(H-\frac{1+\omega}{k})$$

$$i.e.\quad kH=(1+\omega)(1+e^{-kH}) \tag{18}$$

Let $\delta=H/L$ *namely the wave steepness, equation* (18) *is altered to be*

$$1+\omega=\frac{2\pi\delta}{1+e^{-2\pi\delta}}=\theta \tag{19}$$

and

$$\phi(x,y)=c\,y-\frac{c\,\theta}{k\,e^{\theta}}\,e^{ky}\cos kx \tag{20}$$

$$\eta(x)=\frac{\theta}{k\,e^{\theta}}\,e^{k\eta}\cos kx \tag{21}$$

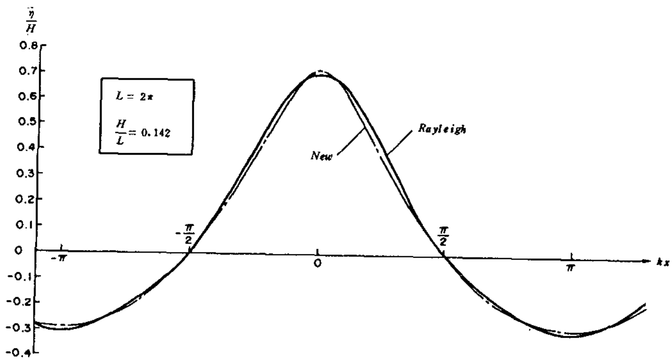
are oblained. Equation (21) *is an implicit equation, however, it is to be solved by Newton – Raphson's method promptly through computer.*

The curve is to be depicted for a critical case i.e. $\delta=0.142$ *and compared with Rayleigh's solution.*

Wave celerity c is worked out by following procedure.

In steady flow, $(\partial\phi/\partial t)$ *vanishes and* $Q(t)$ *becomes constant Q in dynamic boundary condition of free surface. Subslitute the new wave profile equation* (21) *to equation* (2)

$$\frac{\theta}{k\,e^{\theta}}\,e^{k\eta}\cos kx+\frac{1}{2g}\{\,c^2-2c^2\frac{\theta}{e^{\theta}}\,e^{k\eta}\cos kx+\frac{c^2\theta^2}{e^{2\theta}}\,e^{2k\eta}\,\}=\frac{Q}{g}=h$$



Wave profile in deep water

Put $e^{2k\eta} \doteqdot 1 + 2k\eta = 1 + \frac{2\theta}{e^{\theta}} e^{k\eta} \cos kx$, this equation becomes

$$\left(c^2 + \frac{c^2\theta^2}{e^{2\theta}} \right) + \frac{2\theta}{e^{\theta}} \left(\frac{g}{k} - c^2 + \frac{c^2\theta^2}{e^{2\theta}} \right) e^{k\eta} \cos kx = Q \tag{22}$$

Q is a constant, the coefficient of variable term $e^{k\eta} \cos kx$ must be zero. So we can get

$$C^2 = \frac{g}{k} \left(1 - \frac{\theta^2}{e^{2\theta}} \right)^{-1} = \frac{g}{k} \left[1 - \left(\frac{2\pi\delta}{1 + e^{-2\pi\delta}} \right)^2 / \exp \left(\frac{4\pi\delta}{1 + e^{-2\pi\delta}} \right) \right]^{-1} \tag{23}$$

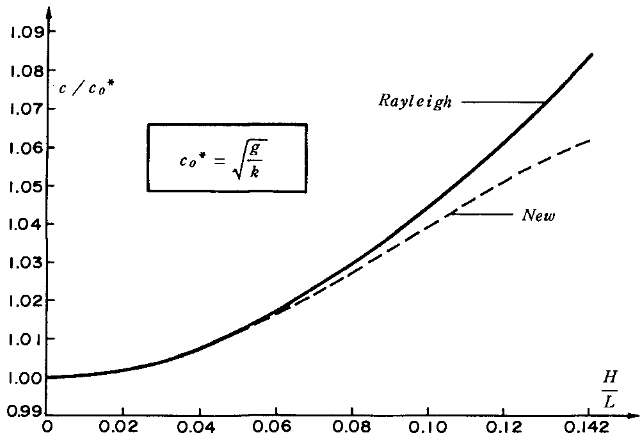
According to this equation, wave celerity increases with wave steepness exponentially in stead of linearly with the square of wave steepness in Rayleigh's theory, however, while δ is small

$$(c^2)_{\delta \to 0} = \frac{g}{k}\{1+(\frac{2\pi\delta}{1+e^{-2\pi\delta}})^2 / exp(\frac{4\pi\delta}{1+e^{-2\pi\delta}})\}$$

$$= \frac{g}{k}\{1 + k^2(\frac{H}{2})^2\} \tag{24}$$

Wave celerity calculated by the two theories are almost identical.

Following diagram shows the comparison of the wave celerities calculated by these two theories.



Comparison of wave celerity in deep water

The mean level of wave profile is worked out by the following equation

$$\xi = \frac{1}{L}\int_0^L \eta\, dx$$

Let $\eta \doteqdot \frac{\theta}{ke^{\theta}}(1+k\eta)\cos kx$

$$\eta = \frac{A\cos kx}{1 - Ak\cos kx} \qquad A = \frac{\theta}{ke^{\theta}}$$

Substitute this expression to the integration

$$\xi = \frac{1}{k}\left[\frac{1}{\sqrt{1-A^2k^2}} - 1\right]$$

$$\xi = \frac{1}{k}\left[\frac{1}{\sqrt{1-\theta^2/e^{2\theta}}} - 1\right] \doteqdot \frac{1}{2k}\left[\frac{\left(\dfrac{2\pi\delta}{1+e^{-2\pi\delta}}\right)^2}{exp\left(\dfrac{4\pi\delta}{1+e^{-2\pi\delta}}\right)}\right] \tag{25}$$

While the steepness is small i.e. $\delta \to 0$

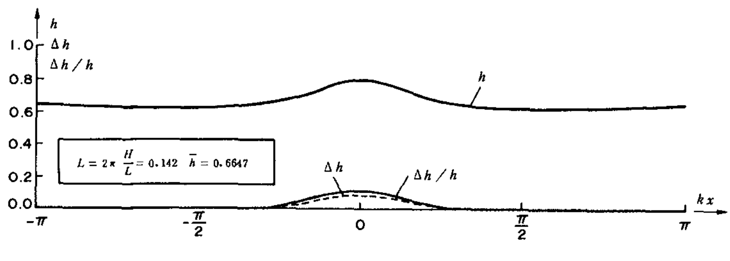
$$\xi = \frac{4\pi^2\delta^2}{2k} = \frac{\pi H^2}{L}$$

coinciding to Rayleigh's theory.

To check the relative error, substitute the value of c i.e. equation (23) *to the following equation to calculate the difference of total water head due to adopting* $1+2k\eta$ *in stead of* $e^{2k\eta}$

$$\Delta h = \frac{1}{2g}\frac{e^2\theta^2}{e^{2\theta}}\left(e^{2k\eta} - 1 - 2k\eta\right) \tag{26}$$

The result is shown in following figure, the relative error of total water head in the case of extreme heigh wave is about 10 % .



Relative accuracy in deep water wave

ENW EQUATION IN FINITE WATER DEPTH

Set $n=1$ *in equations of number* (6)

$$\phi(x,y)=cy-C_1e^{-kd}\left[e^{k(d+y)}-e^{-k(d+y)}\right]\cos kx \tag{27}$$

$$\eta(x)=\frac{C_1}{c}e^{-kd}\left[e^{k(d+\eta)}-e^{-k(d+\eta)}\right]\cos kx \tag{28}$$

$$\eta_1=\eta_{max}=\frac{C_1}{c}e^{-kd}\left[e^{k(d+\eta_1)}-e^{-k(d+\eta_1)}\right] \tag{29}$$

$$\eta_2=\eta_{min}=\frac{-C_1}{c}e^{-kd}\left[e^{k(d+\eta_2)}-e^{-k(d+\eta_2)}\right] \tag{30}$$

Parameter ω *is also invoked*

$$1+\omega=\frac{C_1}{c}e^{-kd}\cdot k\left[e^{k(d+\eta_1)}+e^{-k(d+\eta_1)}\right] \tag{31}$$

$$0\leq 1+\omega\leq 1$$

From equation (29)

$$(1+\omega)/k\left[e^{k(d+\eta_1)}+e^{-k(d+\eta_1)}\right]=\frac{C_1}{c}e^{-kd}$$

$$=\eta_1/\left[e^{k(d+\eta_1)}-e^{-k(d+\eta_1)}\right]$$

$$1+\omega=k\eta_1\coth(d+\eta_1) \tag{32}$$

is abtained.

From the following relationship and equation (31)

$$k(\eta_1+d)=\tanh^{-1}\left(\frac{\eta_1}{1+\omega}\right)=\frac{1}{2}\ell n\frac{(1+\omega)+k\eta_1}{(1+\omega)-k\eta_1} \tag{33}$$

$$(1+\omega)c/k=C_1e^{-kd}\left[\sqrt{\frac{1+\omega+k\eta_1}{1+\omega-k\eta_1}}+\sqrt{\frac{1+\omega-k\eta_1}{1+\omega+k\eta_1}}\right]$$

C_1 *is found to be*

$$C_1=\frac{c}{k}e^{kd}\frac{\sqrt{(1+\omega)^2-k^2\eta^2}}{2} \tag{34}$$

From equation (32)

$$1-(k^2\eta_1^2)/(1+\omega)^2=1-\tanh^2 k(d+\eta_1)=\operatorname{seeh}^2 k(d+\eta_1)$$

$$\sqrt{(1+\omega)^2-k^2\eta_1{}^2}=(1+\omega)\operatorname{sech} k(d+\eta_1) \qquad (35)$$

Conseqently

$$\phi(x,y)=cy-c\eta_1\frac{\sinh k(d+y)}{\sinh k(d+\eta_1)}\cos kx \qquad (36)$$

$$\eta(x)=\eta_1\frac{\sinh k(d+\eta)}{\sinh k(d+\eta_1)}\cos kx \qquad (37)$$

η_1 *should be expressed by wave height, length and water depth. From equations* (30) (34) *and* (35)

$$-\eta_2=H-\eta_1=\frac{\sqrt{(1+\omega)^2-k^2\eta_1{}^2}}{2k}\left[e^{k(d+\eta_1-H)}-e^{-k(d+\eta_1-H)}\right]$$
$$=\frac{(1+\omega)}{2k}\operatorname{sech} k(d+\eta_1)\left[\frac{e^{k(d+\eta_1)}}{2e^{kH}}-\frac{e^{kH}\cdot e^{-k(d+\eta_1)}}{2}\right]$$
$$(38)$$

Substitute (33) *to* (38)

$$H-\eta_1=\frac{\sqrt{(1+\omega)^2-k^2\eta_1{}^2}}{2k}\left[\frac{\sqrt{(1+\omega)+k\eta_1}}{e^{kH}\sqrt{(1+\omega)-k\eta_1}}-\frac{e^{kH}\sqrt{(1+\omega)-k\eta_1}}{\sqrt{(1+\omega)+k\eta_1}}\right]$$
$$=\frac{1}{2k}\left[\frac{(1+\omega)(1-e^{2kH})+k\eta_1(1+e^{2kH})}{e^{kH}}\right]$$
$$=\frac{1}{k}\left[k\eta_1\cosh kH-(1+\omega)\sinh kH\right]$$

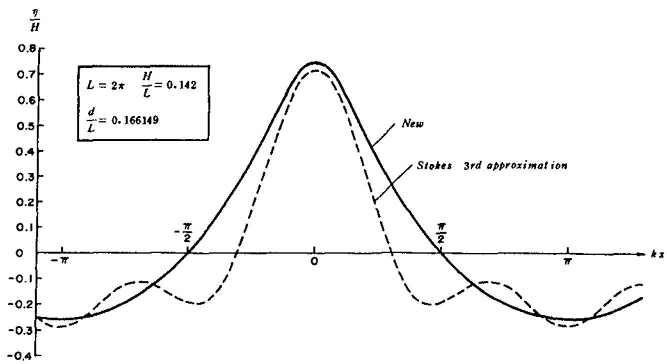
From equation (32)

$$kH=k\eta_1\left[(1+\cosh kH)-\coth k(d+\eta_1)\sinh kH\right] \qquad (39)$$

or

$$\delta = \frac{H}{L} = \frac{\eta_1}{L} \left[(1 + \cosh 2\pi\delta) - \coth k(d+\eta_1) \sinh 2\pi\delta \right] \tag{40}$$

η_1 can also be computed by Newton – Raphson's method, then the stream function and wave profile are to be worked out, The curve of the new theory is shown in following figure, it is much reasonable in comparison with Stokes 3rd approximate curve.



Wave profile in finite water depth

The wave celerity can be calculated by adopting the same method in deep water. Substitute equation (37) to the dynamic boundary condition of free surface,

$$\frac{\eta_1 \sinh k(d+\eta)}{\sinh k(d+\eta_1)} \cos kx + \frac{1}{2g} \left\{ c^2 - 2c^2 k\eta_1 \frac{\cosh k(d+\eta)}{\sinh k(d+\eta_1)} \cos kx + c^2k^2\eta_1{}^2 \frac{\sinh^2 k(d+\eta) + \cos^2 kx}{\sinh^2 k(d+\eta_1)} \right\} = \frac{Q}{g} = h \tag{41}$$

Put the coefficient of variable term to be zero, the wave celerity is worked out to be

$$c^2 = \frac{g}{k} \tanh kd / \{ 1 - k^2\eta_1^2 \frac{\sinh^2 kd}{\sinh^2 k(d+\eta_1)} \} \tag{42}$$

For theoretical consistancy, wave celerity formulas in deep and shallow water must be continuous. Set $d \rightarrow \infty$ in equation (42)

$$(c^2)_{d\rightarrow\infty} = \frac{g}{k} [1 - (k^2\eta_1^2 / e^{2k\eta_1})]^{-1}$$

Substitute equations (13) *and* (19) *to this equation*

$$(c^2)_{d\rightarrow\infty} = \frac{g}{k} [1 - \frac{\left(\frac{2\pi\delta}{1+e^{-2\pi\delta}}\right)^2}{\exp\left(\frac{4\pi\delta}{1+e^{-2\pi\delta}}\right)}]^{-1} \tag{43}$$

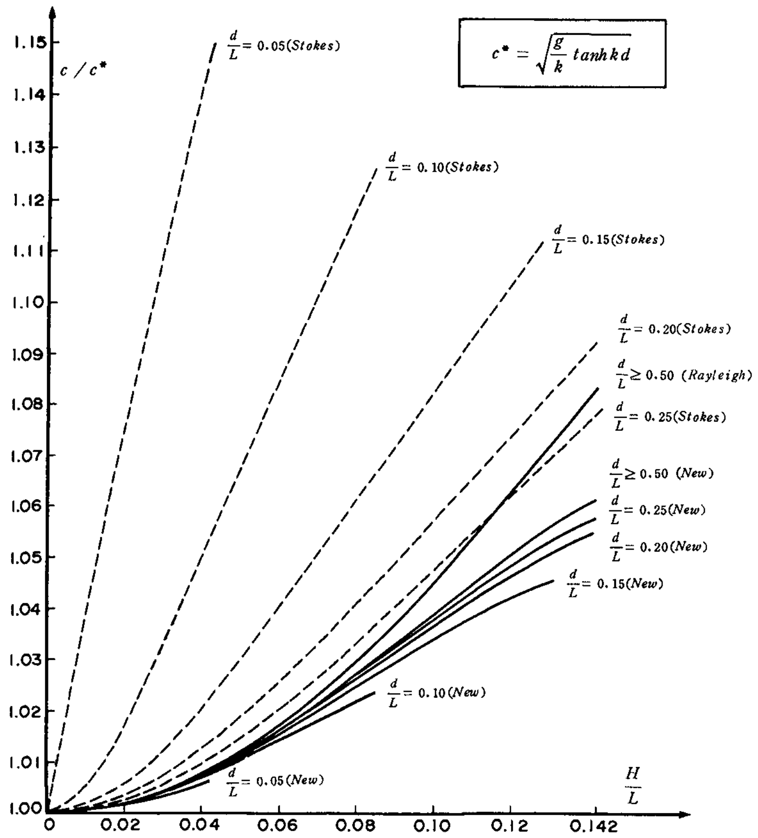
The reality shows that the new theory is superior to the Rayleigh's and Stokes.

The wave celerities in various relative depth are shown in following figure.

In this figure we also see that the new theory is better than Stokes which is unrcasonable that the wave celerities increase more rapidly in shallow water area.

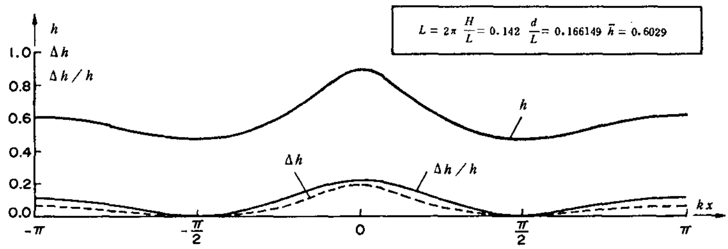
The relative accuracy of water head is estimated as follows.

$$\Delta h = \frac{1}{2g} \{ -2c^2k\eta_1 \frac{\cosh k(d+\eta) - \cosh kd}{\sinh k(d+\eta_1)} \cos kx + c^2k^2\eta_1^2 \cdot \frac{\sinh^2 k(d+\eta) - \sinh^2 kd - k\eta \sinh 2kd + \cos^2 kx}{\sinh^2 k(d+\eta_1)} \} + \eta_1 \frac{\sinh k(d+\eta) - \sinh kd}{\sinh k(d+\eta_1)} \cos kx \tag{44}$$



Comparison of wave celerity in finite water depth

The result of calculation is depicted in the following diagram, for very high waves, the relative errors are up to 20 %.



Relative accuracy in finite water depth

The mean level in the waves in this case is to be calculated as follows.

$$\xi = \frac{1}{L}\int_0^L \eta\, dx = \frac{2}{L}\int_0^{L/2} \eta\, dx$$

Equation (37) *can be expressed approximately by* :

$$\eta = \frac{\eta_1 \sinh k(d+\eta)}{\sinh k(d+\eta_1)} \cos kx$$

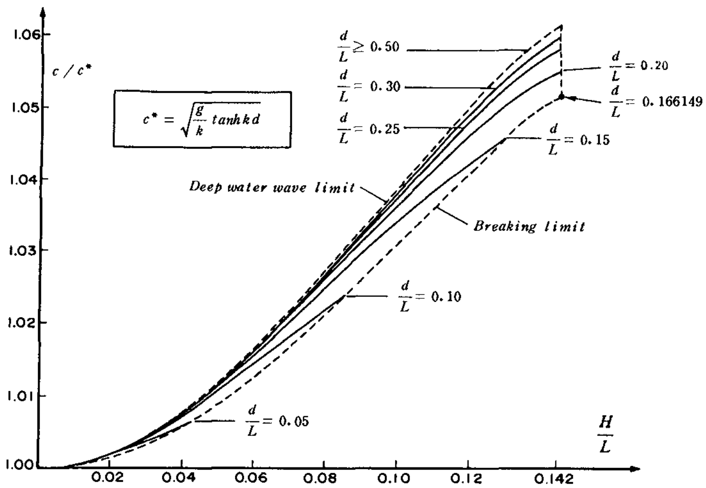
$$\doteqdot \frac{\eta_1 \sinh kd\, e^{k\eta}}{\sinh k(d+\eta_1)} \cos kx \doteqdot B(1+k\eta)\cos kx$$

$$\eta \doteqdot \frac{B\cos kx}{1-Bk\cos kx} \qquad B = \frac{\eta_1 \sinh kd}{\sinh k(d+\eta_1)}$$

$$\xi = \frac{2}{L}\int_0^{L/2} \frac{B\cos kx}{1-Bk\cos kx}\, dx = \frac{1}{k}\left[\frac{1}{\sqrt{1-B^2k^2}} - 1\right] \qquad (45)$$

It can be proved thad while $d \to \infty$, $B = A = \theta / k e^{\theta}$

Finally the significant range of this theory is shown in the next figure.



Significant range of the new wave theory

CONCLUSION

1. *The new wave profile equation describes the wave motion by an implicit function, can be seid to be an exact solution because no approximation approach is adopted.*
2. *In the procedure of calculating wave celerity, some approximate expressions are used, so that some errors will be acknowledged in total water head calculation. However, the wave celerity equation has been proved to be continuous, such a fact shows the new theory is superior to the Rayleigh's and Stoke's.*
3. *The relationship between wave celerity and relative depth d/L in Stokes theory is unreasonable. It is more consistant and*

will coincide with the reality in the new theory.

4. *In the era of electronic computer, the new wave profile equation is suggested to be adopted in practical use after some complement such as the exact position of x — axis is made.*

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