INTEGRAL PROPERTIES FOR VOCOIDAL THEORY AND APPLICATIONS

G.P. Bleach

ABSTRACT

A comparison is made between two reference frames that can each be used to define "still water" for finite amplitude waves on water of finite depth. The reference frame characterized by zero mass flux due to the waves is used to find some exact relations between the wave integral properties. The averaged Lagranian (wave action) approach and the energy/momentum approach to the interaction of finite amplitude waves with slowly-varying currents are also derived in this reference frame. Results in many cases are simpler than those in the more commonly chosen reference frame characterized by zero mean horizontal velocity under the waves.

An application of the integral properties is made to Vocoidal wave theory, which is defined in the zero mass flux frame. It is shown that the rotation present in the orbital velocity field of Vocoidal waves is not always negligible.

INTRODUCTION

In the study of periodic surface gravity waves of finite amplitude on water of finite depth, Stokes (1847) considered two reference frames that can be used to define "still water". These are:

(i) reference frame R1, characterized by zero mean horizontal velocity beneath the wave trough;

(ii) reference frame R2, which has zero mass flux associated with the waves.

Stokes showed that these two reference frames are not equivalent (see section 2, equation (2.7); also Peregrine (1976)) and a choice must be made between them. Most surface gravity wave theories have used R1 to define still water; Stokes (1847) and Cokelet (1977) are examples. An exception in this respect is Vocoidal theory, defined in R2. (Swart and Loubser 1978, 1979a, b). Despite the predominance of R1, there has been recent interest in the use of R2. Reinecker and Fenton (1981) note that wave tank measurements often have zero mass flux, which makes comparison with wave theories defined in R1 more difficult. Whitham (1974) and Jonsson (1978) consider R2 to be valuable in the

Department of Physical Oceanography, University of Cape Town, S.A.

analysis of wavetrains on slowly-varying currents in water of slowly-varying depth. Stiassnie and Peregrine (1979) also speculate on the possible benefits of R2 for such wave/current work.

In view of this interest in R2 and prompted by the availability of the Vocoidal wave theory defined in R2, it was decided to investigate in R2 both the wave integral properties of Longuet Higgins (1975) and the equations governing wave/current interactions as given by Crapper (1979) and Stiassnie and Peregrine (1979). Results are presented for the transformations of wave integral properties between R1 and R2, for exact relations between the integral properties in R2 and for the governing equations for finite amplitude waves interacting with currents varying slowly along the stream. In numerous cases the relative simplicity of the results in R2 compared to those in R1 is evident, particularly in the wave/current interaction equations. This is true for both the momentum/energy interaction equations (Phillips (1966)) and for the averaged Lagranian approach to wave/current interaction (Whitham (1974)).

An application of the integral properties is made in R2 to Vocoidal theory in order to ascertain the amount of rotation present in the only orbital velocity field. It is found that the rotation is negligible only for deep water waves of small amplitude (approximating Airy waves) and for shallow water waves of large amplitude (nearly solitary waves).

REFERENCE FRAMES FOR PERIODIC GRAVITY WAVES ON STILL WATER

In this section the reference frames R1 and R2 are defined and compared. Two dimensional inviscid irrotational motion is assumed and periodic surface gravity waves are considered. The waves propagate in the positive x direction with the z axis vertical, positive upwards, and y is reserved for (later) use as the remaining horizontal coordinate. The bottom is flat and located at $z = -h$.

The most general form of the velocity potential for a periodic surface gravity wave is (Peregrine 1976)

$$\phi = \Phi(\chi,z) + \overline{u}x - \gamma t, \qquad (2.1)$$

where the phase χ is related to the wavenumber k and frequency ω (relative to a fixed observer) by

$$\chi \equiv kx - \omega t. \qquad (2.2)$$

Φ is the periodic part of ϕ and the constants $\overline{u}$ and γ are determined by the choice of horizontal and vertical reference frames respectively. $\overline{u}$ represents a depth independent current and γ is related to the mean water level. Here, t is taken as zero for convenience and γ is dispensed with (although it can always be found from the Bernoulli equation) until later.

Vertical reference frame

The choice made here is that z = o at the mean water level, i.e.

$$\bar{\eta} \equiv o \qquad (2.3)$$

where

$$\bar{\eta} \equiv \frac{1}{\lambda} \int_o^\lambda \eta dx \qquad (2.4)$$

and $\eta(\chi)$ is the wave elevation.

Horizontal reference frame

The reference frame is to be chosen so that there is no background current. There are two obvious choices, firstly to choose the mean horizontal velocity beneath the wave trough to be zero (reference frame R1) and secondly to choose the mass flux associated with the wave (reference frame R2) to be zero.

zero mean horizontal velocity (R1): $\bar{u} \equiv \frac{1}{\lambda} \int_o^\lambda u dx = o$ (2.5)

zero mass flux (R2): $\bar{u}_m \equiv \frac{1}{\lambda h} \int_o^\lambda \int_{-h}^\eta = o$ (2.6)

$\bar{u}$ is the mean horizontal velocity from the velocity potential ϕ and $\bar{u}_m$ is the (Eulerian) mass flux velocity.

Stokes (1847) showed that $\bar{u}$ and $\bar{u}_m$ are not equivalent, since $\bar{u}_m$ can be rewritten as follows, using ϕ from (2.1).

$$\bar{u}_m \equiv \frac{1}{\lambda h} \int_o^\lambda \int_{-h}^\eta \frac{\partial \phi}{\partial x} dzdx$$

$$= \frac{1}{\lambda h} \int_o^\lambda \int_{-h}^\eta \left(\frac{\partial \Phi}{\partial x} + \bar{u}\right) dzdx$$

Since $\bar{\eta} \equiv o$,

$$\bar{u}_m = \frac{1}{\lambda h} \int_o^\lambda \int_{-h}^\eta \frac{\partial \Phi}{\partial x} dzdx + \bar{u} \qquad (2.7)$$

The remaining integral for Φ is the average mass flux velocity due to the periodic wave motion and turns out to be positive, so $\bar{u}_m > \bar{u}$.

A choice must therefore be made between R1 and R2. The latter is not often used, despite the advantages mentioned earlier. These advantages are now explored below with the definition in R2 of wave integral properties and the relations they satisfy (following Longuet-Higgins' (1975) work in R1), followed by the analysis of wave/current interaction in the following section.

INTEGRAL PROPERTIES AND THE EXACT RELATIONS BETWEEN THEM

Integral properties for periodic surface gravity waves involve wave properties that have been averaged over the phase χ. If the averaging is done at a particular time t, this is equivalent to an average over the wavelength λ. Longuet-Higgins (1975) defined gravity wave integral properties and derived relations between them, working in R1 in almost all cases. (See also Crapper 1979).

Here the transformations between R1 and R2 are given, followed by definition of the integral properties in R2 and the derivation of the relations they satisfy.

Notation: (i) printed quantities are those defined in R2

(ii) an overbar $\overline{(\ )}$ denotes $\frac{1}{\lambda}\int_0^\lambda (\ )dx$ throughout

Let U_r be defined as the velocity of translation of R2 with respect to R1. The velocities u and $\overline{u}$ and the velocity potential ϕ transform as follows:

$$u' = u - U_r \tag{3.1}$$

$$\overline{u}' \equiv \frac{1}{\lambda}\int_0^\lambda u'dx = \frac{1}{\lambda}\int_0^\lambda udx - U_r$$

$$\therefore \quad \overline{u}' = -U_r \quad \text{using (2.5)} \tag{3.2}$$

$$\phi' = \phi - U_r x \tag{3.3}$$

Integral properties for periodic surface gravity waves

The following definitions of the wave integral properties are essentially of Longuet-Higgins with the density ρ inserted. They are not yet specific to any reference frame. All definitions are per unit horizontal surface area.

Mean mass flux $$I \equiv \frac{\rho}{\lambda}\int_0^\lambda\int_{-h}^\eta u\ dzdx \tag{3.4}$$

$$= \rho h\,\overline{u}_m \qquad \text{by (2.6)}$$

Mean kinetic energy $$T \equiv \frac{\rho}{2\lambda}\int_0^\lambda\int_{-h}^\eta (u^2+w^2)dzdx \tag{3.5}$$

Mean potential energy $$V \equiv \frac{\rho}{\lambda}\int_0^\lambda\int_0^\eta gz\ dzdx \tag{3.6}$$

Radiation stress component $$S_{xx} \equiv \frac{\rho}{\lambda}\int_0^\lambda\int_{-h}^\eta [\frac{p}{\rho} + u^2]dzdx - \tfrac{1}{2}\rho gh^2 \tag{3.7}$$

Mean energy flux $$F \equiv \frac{\rho}{\lambda}\int_0^\lambda\int_{-h}^\eta [\frac{p}{\rho} + \tfrac{1}{2}(u^2+w^2)+gz]u\ dzdx \tag{3.8}$$

Additional quantities used by Longuet Higgins are as follows:

Mass flux in the steady flow relative to an observer moving with the phase velocity c

$$-Q \equiv \rho \int_{-h}^{\eta} (u-c)dz \quad (3.9)$$

Bernoulli constant $B \equiv p + \frac{\rho}{2}[u^2 - 2uc+w^2] + \rho gz$ (3.10)

Total head $R \equiv p + \frac{\rho}{2}[(u-c)^2 + w^2] + \rho g(z+h)$ (3.11)

$$= B + \rho gh + \frac{\rho c^2}{2}$$

<u>Transformations of the integral properties between R1 and R2</u>

The quantities defined in (3.4-11) are transformed from R1 to R2 by using (3.1-3). The relations (2.5, 2.6) that define R1, R2 are then used to simplify the results.

eg. mass flux $I' \equiv \frac{\rho}{\lambda}\int_{o}^{\lambda}\int_{-h}^{\eta} u'dzdx$ by (3.4)

$$= \frac{\rho}{\lambda}\int_{o}^{\lambda}\int_{-h}^{\eta} (u-U_r)dzdx$$

$$= I - \rho hU_r$$

Since $I' \equiv o$, $I = \rho hU_r$ (3.12)

$$T' = T - \frac{\rho hU_r^2}{2} \quad (3.13)$$

$$S' = S_{xx} - \rho hU_r^2 \quad (3.14)$$

$$F' = F - U_r[3T-2V + \rho h\overline{(u_b)^2}] \quad (3.15)$$

where u_b is the u velocity component at the bottom ($z = -h$), the pressure $p' = p$ and the vertical velocity $w' = w$.

The constants Q', B', R' are found to be:

$$Q' = Q \quad (3.16)$$

$$B' = B + \rho U_r[c-U_r] \quad (3.17)$$

$$R' = R \quad (3.18)$$

Exact relations between the integral properties in R2

A number of exact relations between the integral properties are given by Longuet-Higgins (1975). All except his relation (2.4) are in R1 and here his approach is followed to derive the equivalent relations in R2, making use of the fact that $I' \equiv o$ and using some of the above transformations for confirmation of the results. The results of L.H. are given for comparison, with ρ inserted and referenced by his equation numbers (or letters).

$$Q' = \rho c'h \quad (3.19)$$

$$Q = \rho ch - I \quad \text{L.H.(A)}$$

$$2T' = \rho c'hU_r \quad (3.20)$$

$$= Q'U_r \quad \text{essentially L.H.(2.4)}$$

$$2T = c\ I \quad \text{L.H.(B)}$$

$$2T' = \left(\frac{c'}{\lambda}\right) \int_o^\lambda \eta\ \{c' - [1+(\frac{\partial\eta}{\partial x})]^{\frac{1}{2}}[2R' - 2\rho g(h+\eta)]^{\frac{1}{2}}\}dx \quad (3.21)$$

The corresponding expression for 2T is identical in form (L.H.(E)). (A typographical error in L.H.(E) is the omission of the first η symbol). Note that the choice $\overline{\eta} \equiv o$ (2.3) removes the $\eta c'$ term after integration.

$$S'_{xx} = 4\ T' - 3V' + \rho h\overline{(u'_b)^2} \quad (3.22)$$

where $$\overline{(u'_b)^2} = \overline{(u_b{}^2)} + U_r{}^2 \quad (3.23)$$

The corresponding Sxx expression (L.H.(c)) is again identical in form to the R2 expression.

$$F' = [3\ T' - 2\ V' + \tfrac{1}{2}\ \rho h\overline{(u'_b)^2}]c' \quad (3.24)$$

$$F = [3\ T - 2\ V]c + \tfrac{1}{2}\ [\rho hc + I]\overline{u_b{}^2} \quad \text{L.H.(3.10)}$$

Note that the energy flux relation is simpler in R2 since the I' term is zero. This type of simplification occurs frequently in the conservation equations to be derived below and is a primary advantage of working in R2.

CONSERVATION EQUATIONS FOR WAVES ON SLOWLY-VARYING CURRENTS OVER SLOWLY-VARYING TOPOGRAPHY

Phillips (1966) derived a set of conservation equations for waves interacting with depth-independent currents that vary slowly along or across the stream. These equations are mass, momentum and energy equations that have been vertically integrated and then horizontally averaged. Whitham (1974) derived an alternative set via a variational approach and the two sets were made fully compatible by Stiassnie and Peregrine (1979). A general review of wave/current interactions is given by Peregrine (1976).

All the above work is effectively in R1; the mean horizontal velocity at a fixed point below the wave trough is the current velocity. Stiassnie and Peregrine speculate that it may be better to work in a reference frame in which the total mass flux divided by ρd (where d is the water depth) is the current; i.e. the wave mass flux is zero, which is the defining property of R2.

The present work follows Phillips' approach in R2 and confirmed that the governing equations take a simpler form than in R1. The alternative approach of Whitham is also investigated in R2 and expressions are found for Luke's Lagrangian (Luke 1967), the Bernoulli term γ' (2.1) and Whitham's averaged Lagrangian. Once a suitable wave theory has been chosen, the wave action equation can be derived from the averaged Lagrangian given here.

Notation and definitions

Motion is now in three dimensions and Greek subscripts refer to components in the horizontal ($x_1 = x$, $x_2 = y$) plane. Since slowly varying waves and currents may change the water level from its undisturbed value at $z = o$, the water depth is denoted by $d(x_\alpha t)$. The slowly varying bottom is at $z = -h(x_\alpha)$, and $\partial_{\alpha\beta} \equiv 1$ if $\alpha = \beta$, else $\partial_{\alpha\beta} \equiv o$.

$$d = \frac{1}{\lambda}\int_o^\lambda (\eta+h)dx \tag{4.1}$$

The integral properties (3.4-3.8) are easily generalised for motion in the (x,y) plane with mean water level at $z = d-h$; the overbar represents (as before) the average over the wavelength.

$$I'_\alpha \equiv \overline{\rho\int_{-h}^{\eta} u'_\alpha dz} \equiv o \tag{4.2}$$

$$T' \equiv \overline{\frac{\rho}{2}\int_{-h}^{\eta} (u'_\alpha u'_\alpha + w^2)dz} \tag{4.3}$$

$$V' \equiv \frac{\rho g}{2}[\overline{\eta^2} - (d-h)^2] \tag{4.4}$$

$$S'_{\alpha\beta} \equiv \overline{\rho\int_{-h}^{\eta} (u'_\alpha u'_\beta + \frac{p}{\rho}\partial_{\alpha\beta})dz} - \tfrac{1}{2}\rho g d^2 \partial_{\alpha\beta} \tag{4.5}$$

$$F'_\alpha \equiv \rho\int_{-h}^{\eta} u'_\alpha[\tfrac{1}{2}(u'_\beta u'_\beta + w^2) + \frac{p}{\rho} + g(z+h-d)]dz \tag{4.6}$$

Averaged mass, momentum and energy conservation equations.

The kinematic conservation equation is common to both approaches and is (Peregrine (1976)):

$$\omega' = \sigma' + k'_\alpha U'_\alpha \qquad (4.7)$$

where ω is (as before) the wave frequency relative to a fixed observer, σ' is the frequency relative to the current and U'_α are the components of the depth independent current.

The mass, momentum and energy conservation equations in R2 are derived following Crapper (1979) - essentially Phillips' approach. In each case, the flow velocity is split into $U'_\alpha + u'_\alpha$ in the local conservation equation and the equation is then integrated over depth and averaged over a wavelength, which introduces the integral properties (4.2-4.6). Boundary conditions and the zero mass flux condition (4.2) are used finally to simplify the results. The method is sketched below for the mass conservation equation and the results for mass, momentum and energy conservation are compared with the corresponding results in R1 obtained by Stiassnie and Peregrine.

Mass conservation.

Local continuity equation: $\frac{\partial}{\partial x_\alpha}(U'_\alpha + u'_\alpha) + \frac{\partial w}{\partial z} = o \qquad (4.8)$

$$\text{Now } \frac{\partial}{\partial x_\alpha} \int_{-h}^{\eta} (U'_\alpha + u'_\alpha)dz - (U'_\alpha + u'_\alpha)_{z=\eta} \cdot \frac{\partial \eta}{\partial x_\alpha} - (U'_\alpha + u'_\alpha)_{z=-h} \frac{\partial h}{\partial x_\alpha}$$

$$+ w|_{z=\eta} - w|_{z=-h} = o \qquad (4.9),$$

where Leibnitz' rule has been used to reverse the order of differentiation and integration.

The kinematic surface boundary condition is

$$\frac{\partial \eta}{\partial t} + (U'_\alpha + u'_\alpha) \frac{\partial \eta}{\partial x_\alpha} = w \qquad \text{at } z = \eta \qquad (4.10)$$

and the corresponding bottom condition is

$$-(U'_\alpha + u'_\alpha) \frac{\partial h}{\partial x_\alpha} = w \qquad \text{at } z = -h \qquad (4.11)$$

The mass conservation equation is obtained after inserting (4.10) and (4.11) into (4.9), averaging over a wavelength and using $I'_\alpha \equiv o$.

$$\rho\frac{\partial d}{\partial t} + \frac{\partial}{\partial x_\alpha}[\rho d U'_\alpha] = 0 \qquad (4.12)$$

Compare the result (in R1) of Stiassnie and Peregrine:

$$\rho\frac{\partial d}{\partial t} + \frac{\partial}{\partial x_\alpha}[\rho d U_\alpha + I_\alpha] = 0 \qquad (4.13)$$

(The mass conservation equation also appears in the averaged Lagrangian approach described later.)

Momentum conservation.

The momentum conservation equation is found similarly, and is:

$$\frac{\partial}{\partial t}(\rho d U'_\alpha) + \frac{\partial}{\partial x_\beta}[\rho d U'_\alpha U'_\beta + S'_{\alpha\beta} + \tfrac{1}{2}\rho g d^2 \partial_{\alpha\beta}] - \rho g d\frac{\partial h}{\partial x_\alpha} = 0 \qquad (4.14)$$

Compare the R1 expression:

$$\frac{\partial}{\partial t}(\rho d U_\alpha + I_\alpha) + \frac{\partial}{\partial x_\beta}[(\rho d U_\alpha + I_\alpha)(\frac{I_\beta}{\rho d} + U_\beta) + S_{\alpha\beta} + \tfrac{1}{2}\rho g d^2 \partial_{\alpha\beta} - \frac{I_\alpha I_\beta}{\rho d}]$$
$$- \rho g d\frac{\partial h}{\partial x_\alpha} = 0 \qquad (4.15)$$

Energy conservation.

Energy conservation in R2:

$$\frac{\partial}{\partial t}[\tfrac{1}{2}\rho d(U')^2 + \tfrac{1}{2}\rho g(d-h)^2 + T' + V']$$
$$+ \frac{\partial}{\partial x_\alpha}[U'_\alpha\{\frac{\rho d}{2}(U')^2 + \rho g d(d-h) + T' + V'\} + F'_\alpha + S'_{\alpha\beta} U_\beta] = 0 \quad (4.16)$$

Corresponding equation in R1:

$$\frac{\partial}{\partial t}[\tfrac{1}{2}\rho d U^2 + \tfrac{1}{2}\rho g(d-h)^2 + T + V + U_\alpha I_\alpha]$$
$$+\frac{\partial}{\partial x_\alpha}[U_\alpha\{\tfrac{1}{2}\rho d U^2 + \rho g d(d-h) + T + V + U_\beta I_\beta\} + F_\alpha + I_\alpha[g(d-h)+\tfrac{1}{2}U^2]$$
$$+ S_{\alpha\beta}U_\beta = 0 \qquad (4.17)$$

The relative simplicity of the equations in R2 is clear, since all terms involving I'_α are eliminated.

Averaged Lagrangian approach to wave/current interactions

Whitham's method (Whitham (1974), Peregrine (1976), Crapper (1979)) requires a Lagrangian to be chosen and an averaged Lagrangian to be obtained for the system. Here, Luke's (1967) lagrangian is used in R2.

Luke's Lagrangian: $$L' \equiv -\rho\int_{-h}^{\eta}[\frac{\partial\phi'}{\partial t} + \tfrac{1}{2}(\frac{\partial\phi'}{\partial x_\alpha}\cdot\frac{\partial\phi'}{\partial x_\alpha} + (\frac{\partial\phi}{\partial z})^2 + gz]dz \quad (4.18)$$

where $$\phi' \equiv U'_\alpha x_\alpha - \alpha' t + \Phi(\chi', z), \quad (4.19)$$

and $$\chi' \equiv k'_\alpha x_\alpha - \omega t \quad (4.20)$$

The averaged Lagrangian $\mathcal{L}'$ is defined as

$$\mathcal{L}' \equiv \frac{1}{2\pi}\int_0^{2\pi} L' d\chi' \quad (4.21)$$

Use of L' from (4.18) and choosing a particular value of t to make the phase integral equivalent to an integral over the wavelength, one obtains:

$$\mathcal{L}' = \rho d(\gamma' - \tfrac{1}{2}U'_\alpha U'_\alpha) - T' - V' - \tfrac{1}{2}\rho g[(d-h)^2 - h^2] \quad (4.22)$$

Compare $$\mathcal{L} = \rho d(\gamma - \tfrac{1}{2}U_\alpha U_\alpha) + T - V - \tfrac{1}{2}\rho g[(d-h)^2 - h^2], \quad (4.23)$$

obtained by Crapper (1979) in R1. The discrepancy in sign of the kinetic energy term is removed if expressions for γ, γ' are found. The Bernoulli equation for the whole flow is evaluated at $z = -h$, averaged and then (3.20) is used in the R2 expression to obtain:

$$\gamma' = g(d-h) + \frac{U'_\alpha U'_\alpha}{2} + \frac{u'_\alpha u'_\alpha}{2}\Big|_{z=-h} + \frac{2T'}{\rho d} \quad (4.24)$$

$$\gamma = g(d-h) + \frac{U_\alpha U_\alpha}{2} + \frac{u_\alpha u_\alpha}{2}\Big|_{z=-h} \quad (4.25)$$

On inserting these expressions into (4.22) and (4.23) respectively, the form of the averaged Lagrangians $\mathcal{L}'$ and $\mathcal{L}$ is identical:

$$\mathcal{L}' = \rho d\left(\frac{u'_\alpha u'_\alpha}{2}\Big|_{z=-h} + \frac{gd}{2} + \frac{T'}{\rho} - \frac{V'}{\rho}\right) \qquad (4.26)$$

The averaged Lagrangian can be used to obtain a set of conservation equations for the wave/current interaction. These are the mass conservation equation (identical to (4.12)), consistency conditions replacing the momentum equations and the wave action equation which replaces the energy equation. (See Peregrine (1976)). This completes the analysis of the conservation equations in R2 and shows their relative simplicity compared to the more commonly used equations derived in R1.

APPLICATIONS OF THE INTEGRAL PROPERTIES TO VOCOIDAL WAVE THEORY

Swart and Loubser (1978, 1979a,b) developed the Vocoidal gravity wave theory in an attempt to provide a theory that was relatively simple to use yet accurate. Swart et al (1979) found that Vocoidal theory matched 600 experimental data sets better than twelve other commonly used wave theories. (Cokelet's theory was not considered due to its complexity). Two rather unusual features of Vocoidal theory are that it is defined in R2 and that the orbital velocity field contains a small amount of rotation. The integral properties (3.4) - (3.8) have been defined for Vocoidal theory, but not the exact relations between them and the wave/current results above. These relations and results are valid only for irrotational wave motion. In this section, the amount of rotation in Vocoidal theory is checked by investigating the vertical dependence of $\overline{u^T}$ since $\overline{u^T}$ is independent of depth for an irrotational motion. This is shown by considering $\frac{\partial \overline{u'}}{\partial z}$.

$$\frac{\partial \overline{u'}}{\partial z} = \frac{\partial}{\partial z}\left[\frac{1}{\lambda}\int_o^\lambda u'dx\right]$$

$$= \frac{1}{\lambda}\int_o^\lambda \frac{\partial u'}{\partial z}\,dx$$

Irrotationality $\Rightarrow$ $\frac{\partial u'}{\partial z} = \frac{\partial w'}{\partial x}$

$$\therefore \frac{\partial \overline{u'}}{\partial z} = w'(\lambda) - w'(o)$$

$$= o \text{ by periodicity of } w' \qquad (5.1)$$

Numerical integration of u' at various depths below the wave trough for Vocoidal waves of various steepnesses in a variety of water depths indicates that the theory is virtually irrotational both for low waves in deep water, where Vocoidal theory tends to Airy theory, and for high waves in shallow water, where the Vocoidal wave is virtually a solitary wave. For situations in between these extremes, the rotation causes $\overline{u^T}(z)$ to vary considerably with z. Two typical results are shown below in Table 1 for a wave of period 4,0 seconds in 1.0 m water depth. (The low wave/deep water equivalence of Vocoidal and Airy waves is not shown). The parameter Z_r shows the relative depth under the wave trough at which $\overline{u^T}(z)$ has been calculated; Z_r = o at the bottom and Z_r = 1 at the trough. H is the wave height (crest to trough).

Table 1: Mean horizontal velocity $\overline{u^T}(z)$ at various depths

H(m)	Z_r	$\overline{u'(z)}$ (m/s)
0,20	1,0	- 0,006
	0,9	- 0,009
	0,8	- 0,012
	0,7	- 0,014
	0,6	- 0,016
	0,5	- 0,018
	0,4	- 0,020
	0,3	- 0,021
	0.2	- 0,022
	0,1	- 0,022
	0,0	- 0,022
1,40	1,0	- 0,522
	0,9	- 0,530
	0,8	- 0,536
	0,7	- 0,542
	0,6	- 0,547
	0,5	- 0,551
	0,4	- 0,554
	0,3	- 0,557
	0,2	- 0,559
	0,1	- 0,560
	0,0	- 0,560

Because of the rotation present in the theory, the exact relations (3.21), (3,22), (3.24) are unlikely to be obeyed accurately.

Numerical integration of the appropriate Vocoidal expressions on the right hand side of (3.21,2,4) and comparison with the Vocoidal values for $2T'$, S'_{xx}, F' respectively shows that there is good agreement in deep water but divergences of up to 50% elsewhere.

CONCLUSIONS

Although the existence of the two reference frames R1 and R2 have long been known, (see Stokes 1847) there has been little use made of R2. In this paper, some advantages of R2 are demonstrated, namely that the exact relations between the integral properties are as simple or simpler in form than those in R1 and that the equations for the interaction of finite amplitude waves with slowly varying currents in water of slowly varying depth are considerably simplified. Transformations from R1 to R2 are also given to facilitate the use in R2 of a wave theory originally defined in R1.

Vocoidal theory is defined in R2 from the outset and could easily be used in the wave/current interaction equations presented here if it were not for the presence of a small amount of rotation. This rotation is investigated via the mass horizontal velocity below the trough and it is found that the rotation is small for deep water waves of small height and for steep shallow water waves.

The presence of rotation is also reflected in the inaccuracies found when the exact relations (3.21, 2, 4) are tested for Vocoidal theory.

A further test for rotation that would be of interest would be that proposed by Truesdell (see Serrin, 1959 - section 27) who defines the following parameter Ω as a measure of rotation:

$$\Omega \equiv \frac{\omega_r}{\sqrt{2\, D_{ij}\, D_{ij}}} \qquad (6.1)$$

where ω_r is the magnitude of the angular velocity. Dij are the components (xz plane) of the deformation tensor

$$D_{ij} \equiv \tfrac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad (6.2)$$

For irrotational motion, $\Omega = o$ and for rigid rotation $\Omega = \infty$. Serrin uses $\Omega < 10\%$ as an indicator of the presence of significant rotation.

REFERENCES

COKELET, E.D. 1977: Steep gravity waves in water of arbitrary uniform depth. Phil. Trans. Roy. Soc. A286, 183-260.

CRAPPER, G.D. 1979: Energy and momentum integrals for progressive capillary-gravity waves. J. Fluid Mech. 94, 13-24.

JONSSON, I.G. 1978: Energy flux and wave action in gravity waves propagating on a current. J. Hydraulic Res. 16, 223-234.

LONGUET-HIGGINS, M.S. 1975: Integral properties of periodic gravity waves of finite amplitude. Proc. Roy. Soc. A342, 157-174.

LUKE, J.C. 1967: A variational principle for a fluid with a free surface. J. Fluid Mech. 27, 395-397.

PEREGRINE, D.H. 1976: Interactions of water waves and currents. Advances in Applied Mechanics 16, 9-117.

PHILLIPS, O.M. 1966: "The Dynamics of the Upper Ocean", Cambridge University Press.

REINECKER, M.M. and FENTON, J.D. 1981: A Fourier approximation method for steady water waves. J. Fluid Mech. 104, 119-137.

SERRIN, J. 1959: "Encyclopaedia of Physics, Volume VIII/I: Fluid Dynamics I". p 125-263.

STIASSNIE, M. and PEREGRINE, D.H. 1979: On averaged equations for finite-amplitude water waves. J. Fluid Mech. 94, 401-407.

STOKES, G.G. 1847: On the theory of oscillatory waves. Trans. Camb. Phil. Soc. 8, 441-455 (or: Math. Phys. Paper 1, 197-229 (1880)).

SWART D.H. et al. 1979: Vocoidal water wave theory. Data report 1. Documentation of computer program for evaluation of experimental and analytical validity of 13 wave theories. NRIO (CSIR) Memorandum 7917.

SWART, D.H. and LOUBSER, C.C. 1978: Vocoidal theory for all non-breaking waves. Proc. 16th Int. Conf. Coastal Eng.

SWART, D.H. and LOUBSER, C.C. 1979a: Vocoidal water wave theory. Volume 1. Derivation. CSIR Research Report 357.

SWART, D.H. and Loubser, C.C. 1979b: Vocoidal water wave theory. Volume 2. Verification. CSIR Research Report 360.

WHITHAM, G.B. 1974: "Linear and Non-Linear Waves". Wiley Interscience.