CHAPTER SIXTY SIX

EXPLICIT SOLUTIONS TO PRACTICAL WAVE PROBLEMS

by

Peter Nielsen*

ABSTRACT

Explicit formulae are provided for wave problems covered by linear wave theory. Two goals are pursued. The first is to provide a faster and more flexible tool than usual wave tables. The second is to provide explicit, analytical solutions to problems that have so far demanded time consuming numerical integrations. As an example we solve the problem of wave height variation due to refraction, shoaling and energy dissipation over a soft mud bottom. The obtained explicit solutions are accurate enough for practical purposes and require very little computational effort, in fact they will enable the engineer to solve many wave problems with a handheld calculator. Another advantage of analytical solutions is that they are always much more instructive than numerical results.

INTRODUCTION

Determination of local parameters like wave length, L, celerity, c, and height, H, for linear waves involves solution of the dispersion relation

$$k_o h = kh \tanh kh \qquad (1)$$

for finding the local wave number k at the depth h; k_o is the deep water wave number given by

$$k_o = 4\pi^2/gT^2 \qquad (2)$$

Since (1) is a transcendental equation, k or kh has to be found either by an iterative numerical method or by using a wave table. This is a troublesome process considering the limited accuracy with which linear wave theory represents the physical reality. See Figure 1.

On the other hand linear wave theory is often the only practical option so we will still want to use it, but preferably in a

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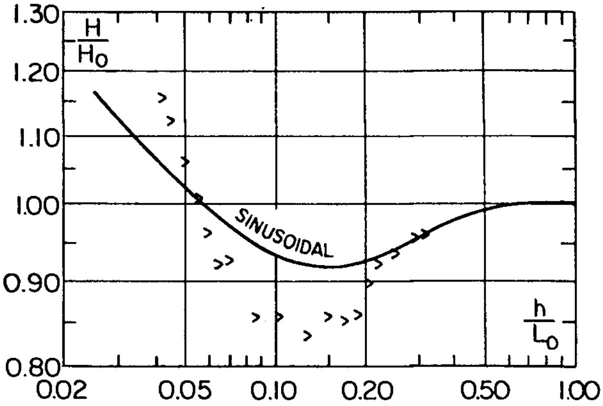

Figure 1: Measured values of the shoaling coefficient H/H_o compared to linear theory (After Brink Kjaer and Jonsson 1973). The error is of the order 10 percent.

mathematical form which is no more complicated than the accuracy of the theory justifies. In other words, there is no reason to solve equation (1) to seven significant digits every time we need a value of kh.

In the following we shall see how kh and many other functions related to linear waves can be expressed very simply in terms of k_oh, and thus become explicit functions of the water depth.

Not only is it possible to write linear wave functions in terms of k_oh, but it turns out that these formulae are very simple and adequate for practical use.

Take of example the function

$$F(k_oh) = \frac{1}{\frac{c_g}{c_o}\cosh^2 kh} \tag{3}$$

which occurs in relation to the problem of wave energy loss over a soft mud bottom. This apparently complicated function becomes

$$F(k_oh) = (k_oh)^{-0.5}[1 - 0.5\, k_oh + \cdots] \tag{4}$$

the accuracy of which is better than 1 percent for $k_oh < 0.92$ that is in depths up to 14 meters for an 8 second wave.

The advantage of using the explicit form (4) is not just to make the evaluation of F easier, in fact the main advantage is that the explicit expression makes it possible to evaluate integrals like

$$I = \int_{h_1}^{h_2} F dh \tag{5}$$

analytically. With this ability we are able to give analytical solutions to many wave problems in intermediate depths, for which explicit solutions have so far only been possible in shallow water.

The improvement of accuracy compared to using shallow water formulae all the way is by no means trivial.

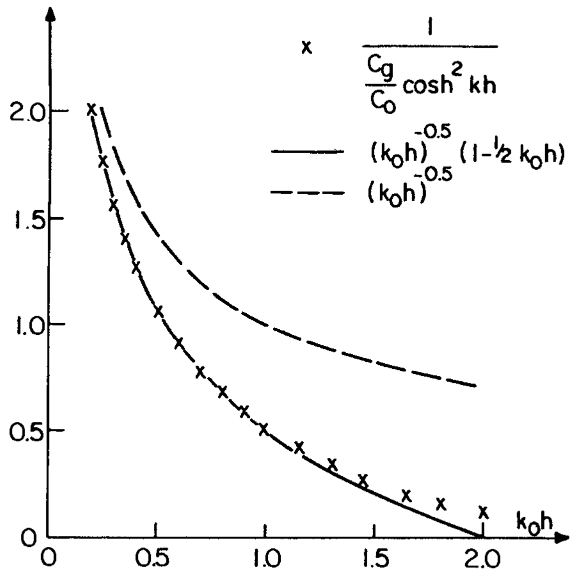

Figure 2: Comparison of the two approximations (4) and (6) to correct values $F(k_oh)$ given by (3).

Figure 2 shows equation (4) as well as the shallow water expression

$$F_s(k_oh) = (k_oh)^{-0.5} \tag{6}$$

compared to correct values. We see that the improvement by adding just one more term is remarkable. And the accuracy of equation (4) is probably a lot better than the accuracy by which physical parameters of a natural mud bottom can be described.

The efficiency of formulae like (4) as simple and accurate approximations was first pointed out by Nielsen (1982).

AN ALTERNATIVE WAVE TABLE

Most coastal engineers are familiar with the use of wave tables for linear waves.

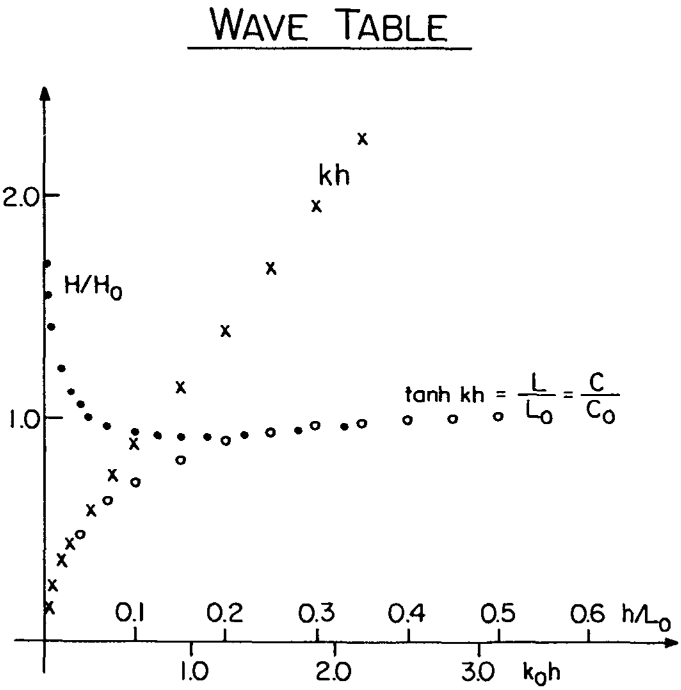

Figure 3: A wave table provides discrete values of commonly used wave parameters in terms of $k_o h$ or h/L_o.

The general form of wave tables is illustrated in Figure 3 and is based on the fact that the dependence of linear wave parameters on the depth h can be expressed in the form

$$P = P_o F(k_o h) \tag{7}$$

where subscript "o" denotes deep water properties. For example, the local wave speed is given by

$$c = c_o \tanh kh = c_o G(k_o h) \tag{8}$$

The function $G(k_oh)$ does not have an exact explicit form in terms of usual functions but it is of course possible to construct explicit approximations to $G(k_oh)$ with any degree of accuracy. The aim of the following is to suggest a standard method for providing such approximations and give a few examples.

The most commonly used form of approximations to transcendental or implicitly given functions are MacLaurin series or power expansions around zero. For example

$$\tanh x = x - \frac{1}{3}x^3 + \frac{2}{15}x^5 - \cdots \tag{9}$$

However, such expansions do not exist for functions like $G(k_oh)$ in equation (8), because $G(k_oh)$ is not analytical at $k_oh=0$.

It is therefore necessary to use a different form of expansions. We choose the form

$$G(k_oh) = G_s(k_oh)\left[1 + a_1k_oh + a_2(k_oh)^2 + \cdots\right] \tag{10}$$

where the subscript s stands for shallow water. The coefficients a_i are constants.

The shallow water expressions are always power functions of k_oh, but the power is often not an integer. Still, functions of the form (10) are convenient in the sense that they are easy to integrate.

Let us now consider the most fundamental example:

$$F(k_oh) = kh \tag{11}$$

i.e., we want to write kh in the form

$$kh = F_s(k_oh)\left[1 + a_1k_oh + a_2(k_oh)^2 + \cdots\right] \tag{12}$$

The fundamental relation between kh and k_oh is the dispersion relation

$$k_oh = kh \tanh kh \tag{1}$$

First we find $F_s(k_oh)$ by letting both kh and k_oh approach zero. We find

$$F_s(k_oh) = \sqrt{k_oh} \tag{13}$$

Next, the coefficients a_i are found by inserting

$$kh = \sqrt{k_o h}\,[1 + a_1 k_o h + a_2 (k_o h)^2 + \cdots] \qquad (14)$$

into the dispersion relation, using the expansion (9) for the hyperbolic tangent. The result is

$$kh = \sqrt{k_o h}\,[1 + \frac{1}{6} k_o h + \frac{11}{360} (k_o h)^2 + \cdots] \qquad (15)$$

This formula is accurate enough for most practical purposes. For $k_o h < 0.31$ which corresponds to 5 meters of water for an 8 second wave, the relative error is less than 0.01 percent. For $k_o h < 0.63$ the error is less than 0.07 percent, and even for $k_o h=2.5$, corresponding to 40 meters of water for an 8 second wave the relative error is only 0.44 percent.

ALTERNATIVE WAVE TABLE

		1% Limit $k_o h$
kh	$=\sqrt{k_o h}\left[1+\frac{1}{6} k_o h + \frac{11}{360}(k_o h)^2\right]$	2.72
$\tanh kh$	$=\sqrt{k_o h}\left[1-\frac{1}{6} k_o h\right]$	1.62
C_g/C_o	$=\sqrt{k_o h}\left[1-\frac{1}{2} k_o h + \frac{7}{72}(k_o h)^2\right]$	2.09
K_s	$=(k_o h)^{-0.25}\left[1+\frac{1}{4} k_o h + \frac{13}{288}(k_o h)^2\right]$	1.34
$\frac{1}{\sinh kh}$	$=(k_o h)^{-0.5}\left[1-\frac{1}{3} k_o h\right]$	1.54

Figure 4: An alternative wave table, the column to the right shows the limiting values of $k_o h$ below which the accuracy is better than one percent. $k_o h=1.5$ corresponds to a depth of 24 meters for an 8 second wave.

Once we have an expression for kh it is straightforward to obtain similar expressions for different functions of kh like

$$\frac{c_g}{c_o} = \tanh kh\left[0.5 + \frac{kh}{\sinh 2kh}\right] \tag{16}$$

and

$$K_s = H/H_o = \sqrt{0.5\ c_o/c_g} \tag{17}$$

c_g is the local group velocity.

Figure 4 shows five commonly used functions and the limits below which the relative error is less than one percent.

All the formulae shown above are tuned to the needs of nearshore coastal engineering work in that they are exact in the shallow water limit. In deep water it is necessary to use another type of expansion. The small parameter is no longer k_oh but $\exp(-k_oh)$ or $\exp(-kh)$.

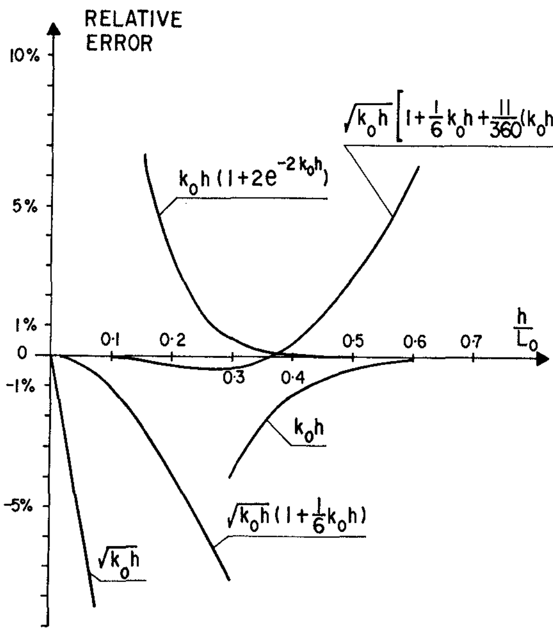

Figure 5: Relative error of (15) and (18), and of some truncated versions.

Again based on the dispersion relation (1) we find the following deep water approximation for kh:

$$kh = k_o h\left[1 + 2e^{-2k_o h}\right] \qquad (18)$$

Figure 5 shows the relative erros of eq. (18) and (15) and of some truncated versions.

DISSIPATION PROBLEMS

Explicit formulae are necessary for analytical evaluation of integrals. For example the integral

$$I = \int_{h_1}^{h_2} \sinh kh \, dh \qquad (19)$$

cannot be evaluated analytically because kh is an implicit function of h. Such integrals occur frequently in coastal engineering and the only way of evaluating them has so far been by numerical integration.

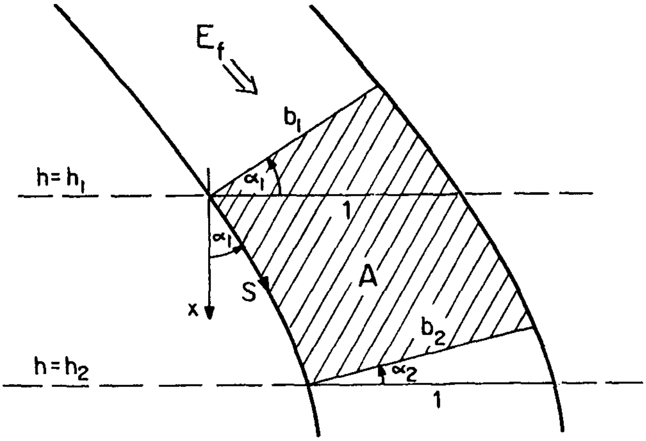

Figure 6: Definition diagram for wave height predictions over straight parallel bottom contours.

One of the most common problems for coastal engineers is to predict wave height variation due to shoaling, refraction and different sorts of energy dissipation. Such calculations are based on energy flux (E_f) considerations and if the bed contours can be assumed straight and parallel as in Figure 6, the fundamental differential equation is

$$\frac{d}{ds}(E_f \cos\alpha) = D_E \cos\alpha \qquad (20)$$

where D_E is the energy dissipation per unit area, and s is the distance measured along the wave orthogonals.

If the sea bed consists of sand and the effects of winds are neglected, most energy dissipation will be due to bed friction and the solution can be based on

$$D_E = \frac{2}{3\pi} \rho f_e u_{b,max}^3 \tag{21}$$

where f_e is Jonsson's energy dissipation factor (Jonsson, 1966), and $u_{b,max}$ is the wave induced velocity amplitude near the bed.

The solution, as given by Nielsen (1984), is

$$H_2 = \frac{H_1 \sqrt{\dfrac{c_{g1}\cos\alpha_1}{c_{g2}\cos\alpha_2}}}{1 + H_1 \dfrac{k_o f_e}{3\pi} \sqrt{\dfrac{c_{g1}\cos\alpha_1}{c_o}}\, J} \tag{22}$$

where J is given by

$$J = \int_{k_o h_1}^{k_o h_2} \frac{c_o^{1.5} \cos^{-1.5}\alpha}{c_g^{1.5} \sinh^3 kh} \frac{dx}{dh}\, dk_o h \tag{23}$$

It turns out that the fairly complicated integrand in (23) has a reasonably simple approximation of the form (10) so that the solution can be evaluated on a handheld calculator with sufficient accuracy for practical purposes. In fact the accuracy of the approximate solution which takes about 100 program steps on an HP-15c is probably far better than the accuracy of any available procedure for prediction of the energy dissipation factor f_e.

Figure 7 shows wave height variation due to refraction, shoaling and friction, calculated with numerical evaluation of J on a major computer and by the explicit solution

$$J = \frac{k_o^{1-1/p}}{pA^{1/p}} \left\{ \frac{(k_o h_1)^{1/p-2.25}}{\frac{1}{p} - 2.25} \left[1 - \left(\frac{h_1}{h_2}\right)^{2.25-1/p}\right] \right.$$

$$+ \frac{(\delta - \frac{1}{4})(k_o h_1)^{1/p-1.25}}{\frac{1}{p} - 1.25} [1 - (\frac{h_1}{h_2})^{1.25-1/p}]$$

$$+ \frac{(\mu - \frac{\delta}{4} - \frac{61}{480})(k_o h_1)^{1/p-0.25}}{\frac{1}{p} - 0.25} [1 - (\frac{h_1}{h_2})^{0.25-1/p}] \Bigg\} \quad (24)$$

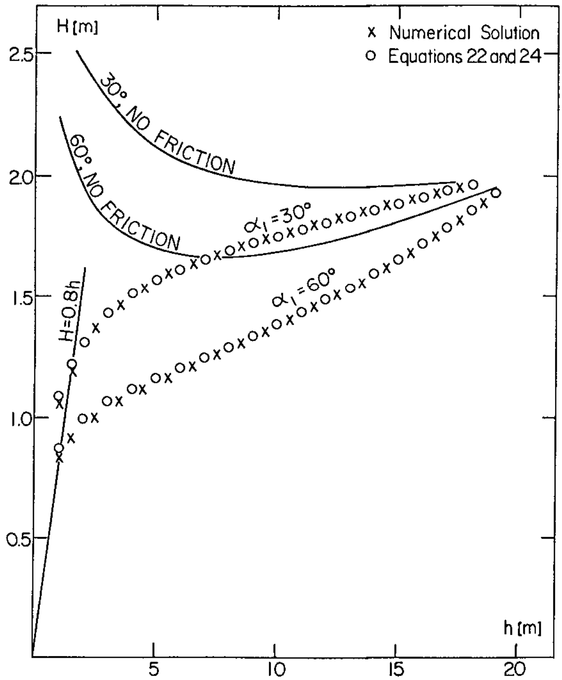

Figure 7: Comparison between the explicit solution (equations 22 and 24) and a numerical solution. The shown example corresponds to T = 8s, f_e = 0.1 and a beach profile given by $h = 0.1(x_o - x)^{2/3}$.

The beach profile is assumed to have the form

$$h = A(x_o - x)^p \quad (25)$$

and the coefficients δ and μ represent the effects of refraction via Snell's law. They are given by

$$\delta = \frac{0.75 \sin^2\alpha_1}{k_o h_1 (1 - \frac{1}{3} k_o h_1)} \tag{26}$$

$$\mu = \frac{1.4 \sin^4\alpha_1}{(k_o h_1)^2} - \frac{1}{3}\delta \tag{27}$$

α_1 is the initial angle between wave crests and bed contours.

ENERGY DISSIPATION OVER A MUD BOTTOM

Waves propagating over a bed of soft mud will tend to induce a wave motion in the mud and thus feed energy into the mud at a rate of

$$D_M(t) = -p(t) \frac{d\eta}{dt} \tag{28}$$

where p is the pressure at the interface and η is the local elevation of the deformed interface. See e.g., Gade (1958).

If the pressure has the form

$$p(t) = p_o + p_1 \cos \omega t \tag{29}$$

and the interface elevation is given by

$$\eta(t) = \eta_o - \eta_1 \cos(\omega t - \phi) \tag{30}$$

Then the time averaged energy flux downwards through the interface is

$$\overline{D_M} = \frac{1}{T} \int_t^{t+T} -p(t) \frac{d\eta}{dt} dt \tag{31}$$

$$= \frac{1}{2} p_1 \omega \eta_1 \sin\phi \tag{32}$$

And introducing the mud response parameter (Tubman and Suheyda, 1976)

$$M = \frac{\rho g \; \eta_1}{p_1} \tag{33}$$

we get

$$D_M = \frac{1}{2} M\sin\phi \, \frac{\omega p_1^{\;2}}{\rho g} \tag{34}$$

For linear waves we have

$$p_1 = \rho g \, \frac{H}{2} \, \frac{1}{\cosh kh} \tag{35}$$

and thus the energy flux into the mud can be written

$$D_M = \frac{\pi \;\; \rho g \;\; M\sin\phi}{4T \;\; \cosh^2 kh} \, H^2 \tag{36}$$

WAVE HEIGHT VARIATION OVER A MUD BOTTOM

For sinusoidal waves propagating over a mud bottom equation (20) becomes

$$\frac{d}{ds} \left(\frac{1}{8} \, \rho g H^2 c_g \cos\alpha\right) = - \frac{\pi \;\; \rho g \;\; M\sin\phi}{4T \;\; \cosh^2 kh} \, H^2 \cos\alpha \tag{37}$$

which has the solution

$$\left(\frac{H_2}{H_1}\right)^2 = \frac{c_{g_1} \cos\alpha_1}{c_{g_2} \cos\alpha_2} \exp(M\sin\phi \;\, I) \tag{38}$$

with

$$I = \int_{s_1}^{s_2} - \frac{2\pi}{T} \, \frac{ds}{c_g \cosh^2 kh} \tag{39}$$

To evaluate this integral we must bring the integrand to an explicit form in terms of the independent variable.

If h varies monotonically with s, we can change the variable of integration into $k_o h$. From Figure 6 we get

$$ds = \frac{d\,k_o h}{k_o \frac{dh}{dx} \cos\alpha} \tag{40}$$

and since $c_o k_o = 2\pi/T$ we find

$$I = -\int_{k_o h_1}^{k_o h_2} \frac{d\,k_o h}{\frac{dh}{dx} \frac{c_g}{c_o} \cos\alpha \cosh^2 kh} \tag{41}$$

Now using the alternative wave table (Figure 4) and a bit of algebra we find

$$\frac{1}{\frac{c_g}{c_o} \cosh^2 kh} \approx (k_o h)^{-0.5} [1 - 0.5\, k_o h] \tag{42}$$

the accuracy of which is discussed in Figure 2. Since most natural mud bottoms are quite flat it is not very restrictive to assume constant bed slope and straight, parallel contours. Then Snell's law gives

$$\frac{1}{\cos\alpha} = [1 - (\frac{c}{c_1})^2 \sin^2\alpha_1]^{-0.5} \tag{43}$$

and we use

$$\frac{c}{c_o} = \tanh kh \approx \sqrt{k_o h}\, (1 - \frac{1}{6} k_o h) \tag{44}$$

to get

$$\frac{1}{\cos\alpha} = [1 - \beta k_o h (1 - \frac{1}{3} k_o h)]^{-0.5} \tag{45}$$

where

$$\beta = \frac{\sin^2\alpha_1}{k_o h_1 (1 - \frac{1}{3} k_o h_1)} \tag{46}$$

After some algebra we then get

$$I = -\frac{dx}{dh}\int_{k_o h_1}^{k_o h_2} [(k_o h)^{-0.5} + \frac{1}{2}(\beta-1)(k_o h)^{0.5}$$

$$+ (\frac{3}{8}\beta^2 - \frac{5}{12})(k_o h)^{1.5}]\, dk_o h \tag{47}$$

and

$$I = -\frac{dx}{dh}\{2(k_o h_1)^{0.5}[(\frac{h_2}{h_1})^{0.5} - 1] + \frac{1}{3}(\beta-1)(k_o h_1)^{1.5}[(\frac{h_2}{h_1})^{1.5} - 1]$$

$$+ (\frac{3}{20}\beta^2 - \frac{1}{6}\beta)(k_o h_1)^{2.5}[(\frac{h_2}{h_1})^{2.5} - 1]\} \tag{48}$$

This formula together with (38) provides a very simple tool for predicting wave height variation over a soft mud bottom, and because the solution is explicit we can quite easily use it "in reverse", i.e.: If the final wave height, H_2, is known we can solve directly for $M\sin\phi$.

DISCUSSION

A set of simple explicit formulae have been provided (Figure 4) for easy calculation of linear wave properties. These formulae provide a handy alternative to wave tables, thus the basic

$$kh = \sqrt{k_o h}\,[1 + \frac{1}{6}k_o h + \frac{11}{360}(k_o h)^2 + \cdots] \tag{15}$$

can easily be memorized by people who deal with linear waves frequently.

The major advantage of introducing explicit formulae is probably that it makes it possible to give analytical solutions to problems that involve integration. Such a solution to the problem of wave height variation due to shoaling refraction and frictional dissipation was given by Nielsen (1984) and a similar solution has been given above for the case of energy absorption by a soft mud bottom.

The accuracy of the solution (38) and (48) will depend on the depths h_1 and h_2 and on the starting angle α_1. An error estimate can be obtained by comparing discrete values of the integrands in (41) and (47).

At the presentation of this paper, Dr. O. Skovgaard of the Technical University of Denmark pointed out that a reduction by 20% or more in computing time could be obtained by using (15) as the first estimate in iterative procedures for determination of kh. It could be added that extra accuracy gained by iteration is often of no practical consequence because of the limited accuracy with which the physical environment can be described and of the crudeness of other underlying assumptions. - This is a matter of opinion, but it seems reasonable to omit iteration and rely entirely on (15) at least in the "debugging" phase for big wave programs.

ACKNOWLEDGEMENT

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