CHAPTER EIGHTY ONE

CONFORMAL MAPPING SOLUTION OF A WAVE FIELD ON THE ARBITRARILY SHAPED SEA BOTTOM

by

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ABSTRACT

A new wave equation has been derived for the full nonlinear dispersive waves propagating over an arbitrarily shaped sea bed. The method of the derivation of the equation uses a conformal mapping technique by which the original domain can be transformed onto a domain with a uniform depth to make the basic equation easily integrable vertically. By taking an inverse Fourier transform, the velocity potential obtained by the integration can be expressed in the form which can construct the exact wave equation from the water surface boundary conditions. An algorithm for the numerical integration of the equation is presented with some examples of the solution.

1. INTRODUCTION

Nearshore wave deformation is characterized mainly by the effects of nonlinearity and dispersivity of waves, and non-uniformity of bottom topography. So far there have been many studies on the wave deformation by using the well known KdV equation or Boussinesq equation (e.g., Peregrine, (1967), Madsen and Mei(1969)). Although these equations have been modified to include the bottom effect by Kakutani(1972), Shuto(1974), Peregrine (1967) and others, their applicability is limitted to slowly varying bottom topography. Furthermore, these wave equations can describe only weak nonlinearity and dispersivity of waves.

The present study derives a new wave equation which may be applied to full nonlinear-dispersive waves propagating over a sea bottom with an arbitrary topography. The method of the derivation of the equation uses a conformal mapping technique by which the original domain can be transformed onto a uniform depth region. The mapping does not change the form of the basic equation, i.e., Laplace's equation on a velocity potential, and symplifies bottom boundary condition so that the basic equation becomes easily integrable. The velocity potential function obtained by the integration includes all components of a time-varying wave number spectrum. Further, the velocity potential function so obtained may be expressed as a function with a convolutional integral by an inverse Fourier transform technique. Hence, the water surface boundary conditions can yield the exact wave equation by using the representation of the velocity potential at the water surface. An algorithm of the numerical integration of the equation is presented with some examples of the soluion. In the appendixes, some descriptions on an analytical method to obtain an approximate solution by using a WKB perturbation technique are presented with a method to estimate wave reflection coefficients.

2. FORMULATION OF EQUATIONS

2.1 Basic Equation and Boundary Conditions in a Mapped Space

Physical plane denoted by $z = x+iy$ may be transformed conformally onto a ζ-plane by an analytical complex function,

$$\zeta = f(z). \tag{1}$$

The function f can be chosen as that which transforms the domain D with an arbitrarily shaped bottom topography in the z-plane onto the domain D' with a uniform depth β_0 in the ζ-plane as shown in Fig.1. If we denote ζ-plane as $\zeta = \alpha+i\beta$, Eq.(1) can be written as

$$\alpha = \phi_1(x,y), \quad \beta = \phi_2(x,y), \tag{2}$$

or inversely,

$$x = \psi_1(\alpha,\beta), \quad y = \psi_2(\alpha,\beta). \tag{3}$$

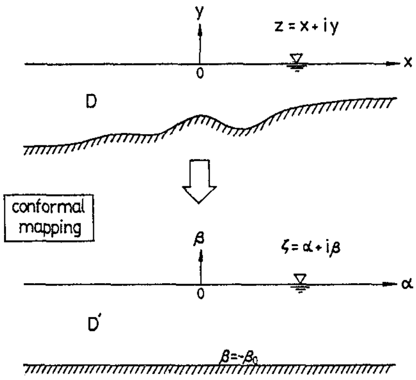

Fig.1 Conformal mapping of wave field

The function f defines the curvilinear coordinates system (α,β) in the original domain. From another point of view, it can be said that the function f is just a complex velocity potential function so that the constant α and β lines correspond to equi-potential lines and stream lines, respectively. Therefore, we can utilize the usual potential flow theory to construct the function f in which one of the stream lines forms the bottom boundary and another stream line coincides with the still water surface. For example, the region over a uniformly sloping beach can be transformed into a constant depth region by the complex function

$$\zeta = \ln z. \tag{4}$$

This is a velocity potential function for the flow due to a point source with a unit strength located at the intersection of bottom line and the still water surface (Fig.2). For the more general case of the bottom topography, we can use a numerical method poroposed by, e.g., Chenin and Schwartz (1982).

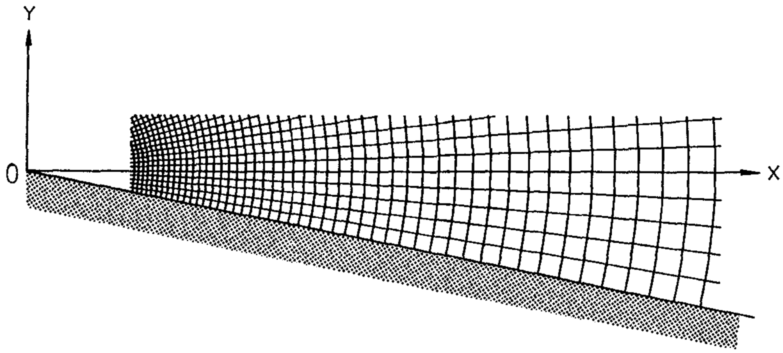

Fig. 2 Orthogonal curvilinear grid system for uniformly sloping beach

The basic equation and the boundary conditions in the z-plane may be written as follows.

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = 0, \tag{5}$$

$$\frac{\partial\Phi}{\partial y} = -\frac{\partial\Phi}{\partial x}\cdot\frac{\partial h}{\partial x}, \qquad y = -h(x) \tag{6}$$

$$\frac{\partial\eta}{\partial t} + \frac{\partial\Phi}{\partial x}\cdot\frac{\partial\eta}{\partial x} - \frac{\partial\Phi}{\partial y} = 0, \qquad y = \eta(x,t) \tag{7}$$

$$\frac{\partial\Phi}{\partial t} + \frac{1}{2}\cdot[(\frac{\partial\Phi}{\partial x})^2 + (\frac{\partial\Phi}{\partial y})^2] + g\eta = 0, \qquad y = \eta(x,t) \tag{8}$$

where Φ is a velocity potential, h is a water depth, η is a water surface elevation, and g is the gravitational acceleration.

In the ζ-plane, the boundary conditions become

$$\frac{\partial\Phi}{\partial\beta} = 0, \qquad \beta = -\beta_0 \tag{9}$$

$$\frac{\partial\tilde{\eta}}{\partial t} + s^2\cdot[\frac{\partial\Phi}{\partial\alpha}\cdot\frac{\partial\tilde{\eta}}{\partial\alpha} - \frac{\partial\Phi}{\partial\beta}] = 0, \qquad \beta = \tilde{\eta}(\alpha,t) \tag{10}$$

$$\frac{\partial\Phi}{\partial t} + \frac{s^2}{2}\cdot[(\frac{\partial\Phi}{\partial\alpha})^2 + (\frac{\partial\Phi}{\partial\beta})^2] + g\cdot\psi_2(\alpha,\tilde{\eta}) = 0, \qquad \beta = \tilde{\eta}(\alpha,t) \tag{11}$$

where s is a scale factor related with the mapping defined as

$$s = \left|\frac{d\zeta}{dz}\right|. \tag{12}$$

In the water surface boundary conditions (10) and (11), the scale factor s is coupled with the nonlinear terms. Hence it can be said that the scale factor s directly affects the nonlinearity of waves. Further, the scale factor s represents nonuniformity of the gravity as will be shown later.

In contrast with the boundary conditions, the basic equation remains its form unchanged through the mapping by the nature of the conformal transform, i.e.,

$$\frac{\partial^2\Phi}{\partial\alpha^2} + \frac{\partial^2\Phi}{\partial\beta^2} = 0. \tag{13}$$

Therefore, the basic equation can be easily integrable vertically on the ζ-plane with the simplified bottom boundary condition (9).

2.2 General Solution of the Basic Equation

The Fourier transform of Eq.(13) with regard to α becomes

$$\frac{\partial^2\bar{\Phi}}{\partial\beta^2} + k^2\bar{\Phi} = 0, \tag{14}$$

where $\bar{\Phi}$ is the Fourier transform of Φ with respect to α defined as

$$\bar{\Phi}(k,\beta,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi(\alpha,\beta,t)\cdot e^{-ik\alpha}\cdot d\alpha. \tag{15}$$

Eq.(14) can be integrated with the bottom boundary condition

$$\frac{\partial\bar{\Phi}}{\partial\beta} = 0,$$

to yield

$$\bar{\Phi}(k,\beta,t) = A(k,t)\cdot\frac{\cosh k(\beta+\beta_0)}{\cosh k\beta_0}. \tag{16}$$

Therefore, Φ is represented by the inverse Fourier transform of $\bar{\Phi}$

$$\Phi(\alpha,\beta,t) = \int_{-\infty}^{\infty}A(k,t)\cdot\frac{\cosh k(\beta+\beta_0)}{\cosh k\beta_0}\cdot e^{ik\alpha}\cdot dk, \tag{17}$$

where A(k,t) is a time varing wave number spectrum to be determined by the water surface boundary conditions, Eqs.(10) and (11).

From Eq.(17), Φ_α, Φ_β and Φ_t can be represented as

$$\frac{\partial\Phi}{\partial\alpha} = i\int_{-\infty}^{\infty} k\cdot A(k,t)\cdot\frac{\cosh k(\beta+\beta_0)}{\cosh k\beta_0}\cdot e^{ik\alpha}\cdot dk, \tag{18}$$

$$\frac{\partial\Phi}{\partial\beta} = \int_{-\infty}^{\infty} k\cdot A(k,t)\cdot\frac{\sinh k(\beta+\beta_0)}{\cosh k\beta_0}\cdot e^{ik\alpha}\cdot dk, \tag{19}$$

$$\frac{\partial\Phi}{\partial t} = \int_{-\infty}^{\infty} \frac{\partial}{\partial t}A(k,t)\cdot\frac{\cosh k(\beta+\beta_0)}{\cosh k\beta_0}\cdot e^{ik\alpha}\cdot dk. \tag{20}$$

To execute the inverse Fourier transform of the right hand side of the above equations, we may introduce a new variable $\chi(\alpha,t)$ defined as

$$\chi(\alpha,t) = \int_{-\infty}^{\infty} A(k,t)\cdot e^{ik\alpha}\cdot dk, \tag{21}$$

and use the following inverse Fourier transform formula (e.g., Erdélyi, 1954),

$$F^{-1}\left[\frac{\cosh k(\beta+\beta_0)}{\cosh k\beta_0}\right] = -\frac{2\pi}{\beta_0}\left\{\frac{\sin(\frac{\pi\beta}{2\beta_0})\cdot\cosh(\frac{\pi\alpha}{2\beta_0})}{\cosh(\frac{\pi\alpha}{\beta_0}) - \cos(\frac{\pi\beta}{\beta_0})}\right\} \equiv K_1(\alpha,\beta), \quad (-\beta_0\le\beta<0) \tag{22}$$

$$F^{-1}\left[\frac{\sinh k(\beta+\beta_0)}{\cosh k\beta_0}\right] = i\cdot\frac{2\pi}{\beta_0}\left\{\frac{\cos(\frac{\pi\beta}{2\beta_0})\cdot\sinh(\frac{\pi\alpha}{2\beta_0})}{\cosh(\frac{\pi\alpha}{\beta_0}) - \cos(\frac{\pi\beta}{\beta_0})}\right\} \equiv i\cdot K_2(\alpha,\beta), \quad (-\beta_0\le\beta<0) \tag{23}$$

where $F^{-1}[\cdot]$ is the operator to take inverse Fourier transform. From the above equations and the convolution formula (24) on the inverse Fourier transform,

$$F^{-1}[\bar{f}_1(k)\cdot\bar{f}_2(k)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} f_1(\xi)\cdot f_2(\alpha-\xi)\cdot d\xi. \tag{24}$$

we can express the velocity potential $\Phi(\alpha,\beta,t)$ and its partial derivatives with respects to α, β and t as follows.

$$\Phi(\alpha,\beta,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} K_1(\xi,\beta)\cdot\chi(\alpha-\xi,t)\cdot d\xi, \tag{25}$$

$$\frac{\partial\Phi}{\partial\alpha} = \frac{1}{2\pi}\int_{-\infty}^{\infty} K_1(\xi,\beta)\cdot\frac{\partial}{\partial\alpha}\chi(\alpha-\xi,t)\cdot d\xi, \tag{26}$$

$$\frac{\partial\Phi}{\partial\beta} = \frac{1}{2\pi}\int_{-\infty}^{\infty} K_2(\xi,\beta)\cdot\frac{\partial}{\partial\alpha}\chi(\alpha-\xi,t)\cdot d\xi, \tag{27}$$

$$\frac{\partial\Phi}{\partial t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} K_1(\xi,\beta)\cdot\frac{\partial}{\partial t}\chi(\alpha-\xi,t)\cdot d\xi. \tag{28}$$

Hence the water particle velocity in the ζ-plane can be calculated through these equations by prescribing the value of χ. However it should be noted that the region in which the above equations are valid is $-\beta_0 \leq \beta < 0$.

2.3 Derivation of Linear Full Dispersive Wave Equation

For linear waves, the water surface boundary condition on Φ becomes

$$\frac{\partial^2\Phi}{\partial t^2} + g\cdot s_0(\alpha)\cdot\frac{\partial\Phi}{\partial\beta} = 0, \qquad \beta = 0. \tag{29}$$

where $s_0(\alpha)$ is the scale factor at $\beta = 0$. In the above equation, the gravitational acceleration g is coupled with the scale factor $s_0(\alpha)$. Hence, it can be said that, for the linear waves, the mapping replaces the effect of the water depth variation in the z-plane with that of non-uniformity of the gravity in the ζ-plane.

The limit values of Φ_β and Φ_t represented as Eqs.(26) and (27) when $\beta \to -0$ are

$$\lim_{\beta\to-0}\frac{\partial\Phi}{\partial\beta} = \frac{1}{2\beta_0}\int_{-\infty}^{\infty}K(\xi)\frac{\partial}{\partial\alpha}\chi(\alpha-\xi,t)d\xi,$$

$$\lim_{\beta\to-0}\frac{\partial\Phi}{\partial t} = \frac{\partial}{\partial t}\chi(\alpha,t),$$

where $K(\xi)$ is a kernal function defined as

$$K(\xi) = \mathrm{cosech}(\frac{\pi}{2\beta_0}\xi). \tag{30}$$

Substituting these equations into Eq.(29) and using the relation $\chi_t = g\cdot\eta$, we can obtain the following linear wave equation to describe evolution of water surface elevation η.

$$\frac{\partial^2}{\partial t^2}\eta(\alpha,t) + \frac{g}{2\beta_0}\cdot s_0(\alpha)\cdot\int_{-\infty}^{\infty}K(\xi)\frac{\partial}{\partial\alpha}\eta(\alpha-\xi,t)d\xi = 0. \tag{31}$$

In the second term of the above equation, the non-uniformity of the gravity expressed as $g\cdot s_0(\alpha)$ represents the effect of bottom topography in the original z-plane and the convolutional integral includes the effect of wave dispersivity. For long waves, Eq.(31) becomes

$$\frac{\partial^2}{\partial t^2}\eta(\alpha,t) - g\cdot s_0(\alpha)\cdot\beta_0\cdot\frac{\partial^2}{\partial\alpha^2}\eta(\alpha,t) = 0. \tag{32}$$

2.4 Derivation of Full Nonlinear-Dispersive Wave Equation

Nonlinear wave equation may be derived by the almost same manner as the case of linear waves. In this case, however, Eqs.(26) to (28) on Φ_α, Φ_β and Φ_t are not applicable in the region of $\beta \geq 0$ as already mentioned. This limitation is originated from the formula (22) and (23) on the kernal functions K_1 and K_2. Therefore, alternative methods must be developed to evaluate the value of those kernel functions at the water surface boundary.

The one method is to reevaluate K_1 and K_2 by replacing the denominator of Eq.(16) with $\cosh k(a_0+\beta_0)$ where a_0 is a parameter chosen as $a_0 > \tilde{\eta}_{max}$. The other method is to take Taylor expansion of the functions to be transformed in Eqs.(22) and (23) around $\beta = 0$. The results by the latter method become as follows.

$$K_1(\alpha,\beta) = 2\pi \sum_{n=0}^{\infty} (-1)^n \frac{\beta^{2n}}{(2n)!} \cdot \delta^{(2n)}(\alpha) + \frac{\pi}{\beta_0} \int_{-\infty}^{\infty} K(\xi) \cdot \sum_{n=0}^{\infty} (-1)^n \frac{\beta^{2n+1}}{(2n+1)!} \cdot \delta^{(2n+1)}(\alpha-\xi)\, d\xi, \tag{33}$$

$$K_2(\alpha,\beta) = \frac{\pi}{\beta_0} \int_{-\infty}^{\infty} K(\xi) \cdot \sum_{n=0}^{\infty} (-1)^n \frac{\beta^{2n}}{(2n)!} \cdot \delta^{(2n)}(\alpha-\xi)\, d\xi - 2\pi \sum_{n=0}^{\infty} (-1)^n \frac{\beta^{2n+1}}{(2n+1)!} \cdot \delta^{(2n+1)}(\alpha), \tag{34}$$

where $\delta^{(n)}(\alpha)$ is the n-th derivative of Dirac's delta function. From Eqs. (33) and (34), Φ_α, Φ_β and Φ_t can be expressed as

$$\frac{\partial \Phi}{\partial \alpha} = \chi^{(1)}(\alpha,t) + \frac{\beta}{2\beta_0} \int_{-\infty}^{\infty} K(\xi) \cdot \chi^{(2)}(\alpha-\xi,t)\, d\xi + \cdots\cdots\cdots, \tag{35}$$

$$\frac{\partial \Phi}{\partial \beta} = \frac{1}{2\beta_0} \cdot \int_{-\infty}^{\infty} K(\xi) \cdot \chi^{(1)}(\alpha-\xi,t)\, d\xi - \beta \cdot \chi^{(2)}(\alpha,t) + \cdots\cdot, \tag{36}$$

$$\frac{\partial \Phi}{\partial t} = \frac{\partial}{\partial t}\chi(\alpha,t) + \frac{\beta}{2\beta_0} \cdot \int_{-\infty}^{\infty} K(\xi) \cdot \frac{\partial}{\partial t}\chi^{(1)}(\alpha-\xi,t)\, d\xi + \cdots\cdots, \tag{37}$$

where the superscripts (n) for χ indicate the n-th order partial derivative with respect to α.

Substitution of Eqs.(35) to (37) into the surface boudary conditions (10) and (11) leads to the following integro-differential equations on $\tilde{\eta}$ and χ.

$$\frac{\partial\tilde{\eta}}{\partial t} + \frac{1}{2\pi}\cdot s^2(\alpha,\tilde{\eta})\cdot\left[\int_{-\infty}^{\infty}K_1(\xi,\tilde{\eta})\frac{\partial}{\partial\alpha}\chi(\alpha-\xi,t)d\xi\cdot\frac{\partial\tilde{\eta}}{\partial\alpha}\right.$$
$$\left. - \int_{-\infty}^{\infty}K_2(\xi,\tilde{\eta})\frac{\partial}{\partial\alpha}\chi(\alpha-\xi,t)d\xi\right] = 0, \qquad (38)$$

$$\int_{-\infty}^{\infty}K_1(\xi,\tilde{\eta})\frac{\partial}{\partial t}\chi(\alpha-\xi,t)d\xi$$
$$+ \frac{1}{4\pi}\cdot s^2(\alpha,\tilde{\eta})\cdot\left[\left\{\int_{-\infty}^{\infty}K_1(\xi,\tilde{\eta})\frac{\partial}{\partial\alpha}\chi(\alpha-\xi,t)d\xi\right\}^2\right.$$
$$\left. + \left\{\int_{-\infty}^{\infty}K_2(\xi,\tilde{\eta})\frac{\partial}{\partial\alpha}\chi(\alpha-\xi,t)d\xi\right\}^2\right] + 2\pi g\cdot\psi_2(\alpha,\tilde{\eta}) = 0. \qquad (39)$$

Since the equations are derived without any assumptions and constraints, they constitute a set of exact wave equations on $\tilde{\eta}$ and χ. Therefore they can essentially express arbitrary degree of nonlinearity, dispersivity and the effect of non-uniformity of the water depth.

Fenton & Rienecker (1980) proposed a numerical method for the accurate solution of nonlinear equations for water waves over a horizontal bed. The method uses the finite Fourier series approximation on the velocity potential Φ and predicts the evolution of water surface elevation η with that of the Fourier coefficients. On the contrary, the wave equations (38) and (39) include all components of the continuous wave number spectrum $A(k,t)$, and they can describe directly the evolution of $\tilde{\eta}$ and χ in the physical space.

3. NUMERICAL SOLUTION OF WAVE EQUATIONS

3.1 Algorithm of the Numerical Integration of the Equations

The kernel function $K(\xi)$ included in the linear wave equation (31) and the nonlinear wave equations (38) and (39) has a singularity at $\xi = 0$ such as

$$\lim_{\xi\to\pm 0} K(\xi) = \pm\,\infty.$$

To avoid this singularity the following procedure has been developed.

If the interval of integaration of a convolutional integral $I(\alpha,t)$ for $K(\alpha)$ and a continuous function $f(\alpha,t)$ is divided into the three parts as

$$I(\alpha,t) = \int_{-\infty}^{\infty}K(\xi)\cdot f(\alpha-\xi,t)d\xi$$
$$= \left(\int_{-\infty}^{-\Delta\xi} + \int_{-\Delta\xi}^{\Delta\xi} + \int_{\Delta\xi}^{\infty}\right)\cdot K(\xi)\cdot f(\alpha-\xi,t)d\xi$$
$$= I_1 + I_2 + I_3, \qquad (40)$$

then the singularity is included in the second integral I_2. Substitution of the Taylor expansion of $f(\alpha-\xi,t)$ around $\xi = 0$ into the second integral I_2 leads to

$$I_2(\alpha,t) = \sum_{n=0}^{\infty} \int_{-\Delta\xi}^{\Delta\xi} (-1)^n \cdot \frac{\xi^n}{n!} \mathrm{cosech}(\frac{\pi}{2\beta_0}\xi) d\xi \cdot \frac{\partial^n}{\partial\alpha^n} f(\alpha,t).$$

When n is an even number, the value of the integral becomes zero in the sense of Cauchy principal value because the kernel function $K(\xi)$ is an odd function. Thus, if $\Delta\xi$ is taken to be enough small to neglect the higher order terms, $I_2(\alpha,t)$ can be evaluated by

$$I_2(\alpha,t) \cong -\int_{-\Delta\xi}^{\Delta\xi} \xi \cdot \mathrm{cosech}(\frac{\pi}{2\beta_0}\xi) d\xi \cdot \frac{\partial}{\partial\alpha} f(\alpha,t)$$

$$= -2 \cdot (\frac{2\beta_0}{\pi})^2 \cdot \mathrm{Shi}^*(\frac{\pi}{2\beta_0}\Delta\xi) \cdot \frac{\partial}{\partial\alpha} f(\alpha,t), \quad (41)$$

where

$$\mathrm{Shi}^*(x) = \int_0^x \frac{\xi}{\sinh\xi} d\xi = \sum_{k=0}^{\infty} \frac{(2-2^{2k})B_{2k}}{(1+2k)(2k)!} x^{1+2k}$$

$$= x - \frac{x^3}{3 \cdot 3!} + \frac{7x^5}{15 \cdot 5!} - \frac{31x^7}{21 \cdot 7!} + \frac{127x^9}{15 \cdot 9!} - \cdots\cdots, \quad (42)$$

$$(|x| < \pi)$$

and B_n is Bernoulli numbers.

Using the above procedure, we can obtain an alternative representaion of the linear wave equation.

$$\frac{\partial^2}{\partial t^2}\eta(\alpha,t) = -\frac{g}{2\beta_0} s_0(\alpha) \cdot \int_{-\infty}^{-\Delta\xi} K(\xi) \frac{\partial}{\partial\alpha}\eta(\alpha-\xi,t) d\xi$$

$$+ (\frac{2}{\pi})^2 \beta_0 g \cdot s_0(\alpha) \cdot \mathrm{Shi}^*(\frac{\pi}{2\beta_0}\Delta\xi) \cdot \frac{\partial^2}{\partial\alpha^2}\eta(\alpha,t)$$

$$- \frac{g}{2\beta_0} s_0(\alpha) \cdot \int_{\Delta\xi}^{\infty} K(\xi) \frac{\partial}{\partial\alpha}\eta(\alpha-\xi,t) d\xi. \quad (43)$$

Since the above equation has no singularity, it may be easily solved by a usual finite difference scheme.

For the nonlinear wave equation, the almost same procedure as the linear case can be used except that the nonlinear wave equation includes higher order derivatives of the unknown variable χ.

3.2 Examples of Numerical Solutions

(1) Linear waves

As an example of the numerical solution for the linear wave equation (43), a calculation has been made for waves propagating over a stepped sea bed with a slope of arbitrary inclination as shown in Fig.3. The scale factor $s_0(\alpha)$ for this case becomes

$$s_0(\alpha) = \frac{\beta_0}{h} e^{\theta l/(2\beta_0)} \cdot \left[\cosh\left(\frac{\pi l}{2\beta_0}\right) - \tanh\left(\frac{\pi\alpha}{2\beta_0}\right) \cdot \sinh\left(\frac{\pi l}{2\beta_0}\right) \right]^{-\theta/\pi}, \quad (44)$$

where

$$l = \frac{\beta_0}{\theta} \cdot \ln\left(\frac{h}{b}\right). \quad (45)$$

The above equation may be obtained by using the Poisson-Schwartz's integral formula. The solid lines in Fig.3 represent the constant α and β lines. The initial conditions for the integration adopted here are

$$\eta(\alpha,0) = a \cdot \exp\{-(\alpha-\alpha_0)^2/(2\beta_0)^2\}, \quad (46)$$

with $a = 1.5\text{m}, \quad \beta_0 \ (= h \ ) = 10\text{m},$

and

$$\frac{\partial}{\partial t}\eta(\alpha,0) = 0.$$

The inclination of the slope θ and the water depth ratio b/h are chosen as the value of 45° and 1/4, respectively. The upper figure in Fig.3 shows the scale factor s_0 as a function of x. From the meaning of the scale factor as previously mentioned, we can say that, in the ζ-plane, the gravitational accerelation for the shallower region is four times as great as that for the large depth region.

The result of the computation is shown in Fig.4 where the full lines represent the solution for the linear full-dispersive wave equation (31), while the broken lines show the solution for the linear long wave equation (32). The full-dispersive wave solution shows that, in contrast with the non-dispersive wave solution, the longer waves propagate faster than the shoter waves. This is a direct manifestation of the wave dispersivity. The figure also shows that the right going waves divide into reflected waves and transmitted waves on the slope.

(2) Nonlinear waves

Nonlinear wave solution has been obtained for the same stepped bottom topography as the above. For the reason of simplicity, only the first terms in the power series of Eqs.(35) to (37) are taken for the calculation here. The initial condition for $\tilde{\eta}$ is chosen as the same as Eq.(46) but with a = 2.0 m. The initial condition for χ is taken as $\chi(\alpha,0) = 0$ which means that there is no fluid motion at the initial stage.

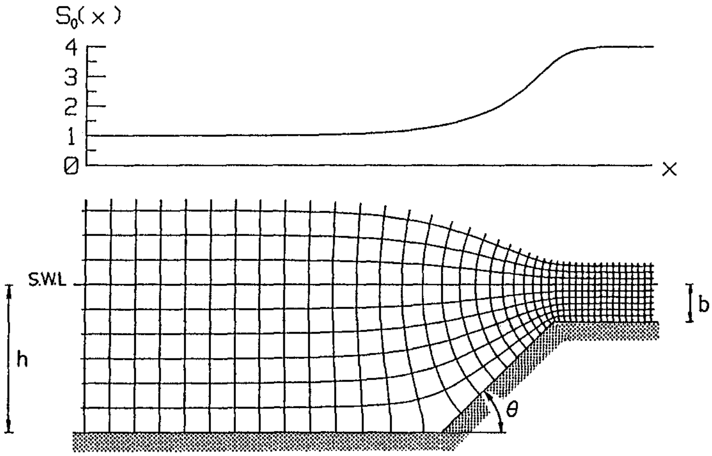

Fig.3 Orthogonal curvilinear grid system and scale factor $s_0(\alpha)$ for a stepped bottom profile.

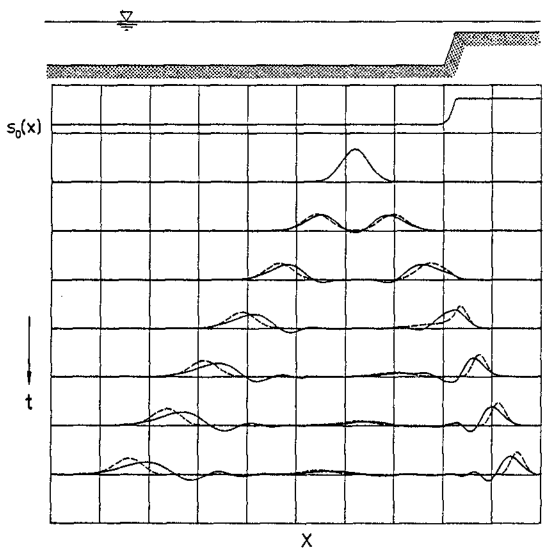

Fig.4 Linear wave evolution over a stepped sea bed.

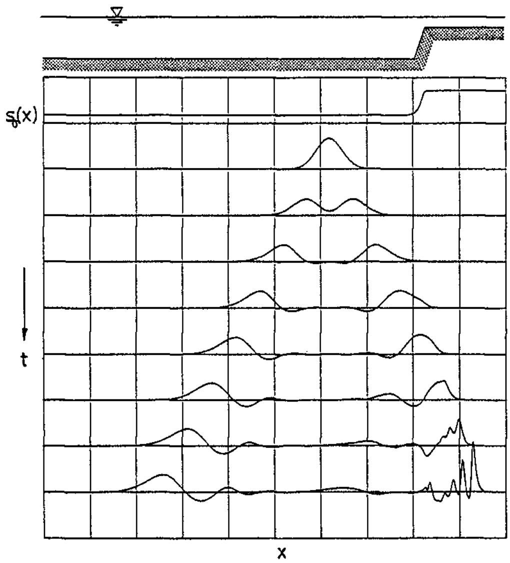

Fig.5 Nonlinear wave evolution over a stepped sea bed.

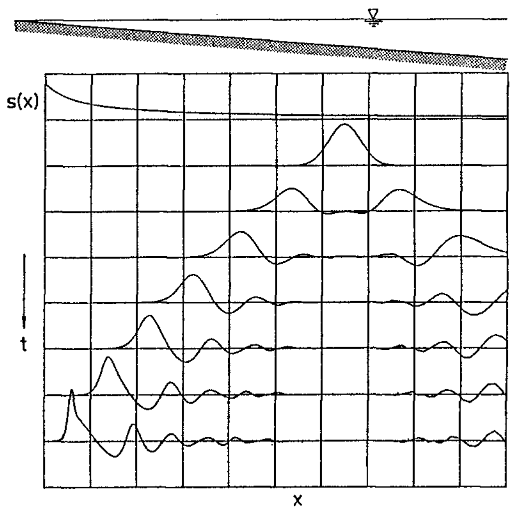

Fig.6 Nonlinear wave evolution over a uniformly sloping beach.
($\tan\beta_0 = 1/30$)

Figure 5 shows the result of the computaion and indicates that the dispersion and reflection of waves take place as in the case of linear waves. The figure also reveals that the right going waves steepen their front face as they propagate over the slope and give rise to fission into the several solitons on the step. This phenomena of solitons has been already reported by Madsen and Mei (1969) and is considered to be due to the combined effects of nonlinearlity and dispersivity of waves and bottom topography. Figure 6 shows another example for waves on a uniformly sloping beach for which the mapping function f is difined as Eq.(4).

From these examples, it may be concluded that the wave equation derived in the present study is effective to describe the wave evolution under the combined effects of nonlinearity and dispersivity of waves and nonuniformity of the water depth.

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APPENDIX A. WKB Solution for Progressive Waves over a Gradually Varing Bottom

If the bottom variation is gradual, the use of the conformal mapping technique becomes not so essential and an approximate solution can be obtained by a usual WKB method (e.g., Chu and Mei, 1970). However, for the problems such as sediment transport, conformal mapping solutions gives more usefull estimation of bottom velocity field compared with other methods because the conformal mapping solution satisfies exactly the bottom boundary condition.

This appendix describes an approximate method to obtain an analytical conformal mapping solution. The method uses a WKB perturbation technique developed by Hamanaka and Kato (1982).

Introducing a small parameter δ which characterizes the horizontal scale of the bottom variation, we can normalize the variables as

$$(\alpha',\beta') = (\delta\alpha,\beta)\cdot\frac{\omega^2}{g}, \qquad t' = \omega t,$$
$$(\tilde{\eta}',\beta_0') = (\tilde{\eta},\beta_0)\cdot\frac{\omega^2}{g}, \qquad \Phi' = \Phi\cdot\frac{\omega^3}{g^2}. \tag{A-1}$$

For convenience the primes will be dropped from here on. From this normalization the linearized governing equations become

$$\delta^2\frac{\partial^2\Phi}{\partial\alpha^2} + \frac{\partial^2\Phi}{\partial\beta^2} = 0, \tag{A-2}$$

$$\frac{\partial\tilde{\eta}}{\partial t} - s_0^2(\alpha)\cdot\frac{\partial\Phi}{\partial\beta} = 0, \qquad \beta = 0 \tag{A-3}$$

$$\tilde{\eta} + s_0(\alpha)\cdot\frac{\partial\Phi}{\partial t} = 0, \qquad \beta = 0 \tag{A-4}$$

$$\frac{\partial\Phi}{\partial\beta} = 0, \qquad \beta = -\beta_0. \tag{A-5}$$

In the next step, let us introduce a local wave number k and transform the independent variables as

$$(\alpha,\beta,t) \rightarrow (\alpha,\beta,\xi), \qquad \xi = \delta^{-1}\int k d\alpha - t. \tag{A-6}$$

Then Eqs.(A-2) to (A-5) become

$$k^2\frac{\partial^2\Phi}{\partial\xi^2} + \frac{\partial^2\Phi}{\partial\beta^2} = - \delta^2\frac{\partial^2\Phi}{\partial\alpha^2} - \delta(\frac{\partial k}{\partial\alpha}\cdot\frac{\partial\Phi}{\partial\xi} + 2k\cdot\frac{\partial^2\Phi}{\partial\alpha\cdot\partial\xi}) \tag{A-7}$$

$$s_0^2(\alpha)\cdot\frac{\partial\Phi}{\partial\beta} + \frac{\partial\tilde{\eta}}{\partial\xi} = 0, \qquad \beta = 0 \tag{A-8}$$

$$\tilde{\eta} - s_0(\alpha)\cdot\frac{\partial\Phi}{\partial\xi} = 0, \qquad \beta = 0 \tag{A-9}$$

$$\frac{\partial\Phi}{\partial\beta} = 0, \qquad \beta = -\beta_0. \tag{A-10}$$

The unknown variables Φ, $\tilde{\eta}$, and k may be expanded by δ as

$$\begin{aligned} \Phi &= \Phi_0 + \delta\Phi_1 + \delta^2\Phi_2 + \cdots\cdots\cdots , \\ \tilde{\eta} &= \tilde{\eta}_0 + \delta\tilde{\eta}_1 + \delta^2\tilde{\eta}_2 + \cdots\cdots\cdots , \\ k &= k_0 + \delta k_1 + \delta^2 k_2 + \cdots\cdots\cdots . \end{aligned} \tag{A-11}$$

Substituting (A-11) into Eqs. (A-7) to (A-10) and separating the orders we obtain a set of systematically solvable equations. The solutions can be summarized as follows.

At $O(\delta^0)$:

$$\begin{aligned} \Phi_0 &= a\cdot \cosh k_0(\beta+\beta_0)\cdot \sin\xi , \\ \tilde{\eta}_0 &= a\cdot s_0(\alpha)\cdot \cosh k_0\beta_0 \cdot \cos\xi , \\ s_0(\alpha)&\cdot k_0\cdot \tanh k_0\beta_0 = 0. \end{aligned} \tag{A-12}$$

At $O(\delta^1)$:

$$\begin{aligned} \Phi_1 &= -\left\{\frac{1}{2}a(\beta+\beta_0)^2\frac{\partial k_0}{\partial\alpha}\cosh k_0(\beta+\beta_0) + \frac{\partial a}{\partial\alpha}(\beta+\beta_0)\sinh k_0(\beta+\beta_0)\right\}\cdot\cos\xi , \\ \tilde{\eta}_1 &= s_0(\alpha)\cdot\left\{\frac{1}{2}a\beta_0^2\frac{\partial k_0}{\partial\alpha}\cosh k_0\beta_0 + \frac{\partial a}{\partial\alpha}\beta_0\sinh k_0\beta_0\right\}\cdot\sin\xi , \\ k_1 &= 0. \end{aligned} \tag{A-13}$$

$$\vdots$$

For nonlinear waves, the solution can be obtained by the almost same manner as the above by introducing another small parameter ε which characterizes nonlinearity of waves and by representing the unknown variables in the form of double perturbation with respect to δ and ε.

APPENDIX B. Reflection Coefficients for Linear Steady Waves

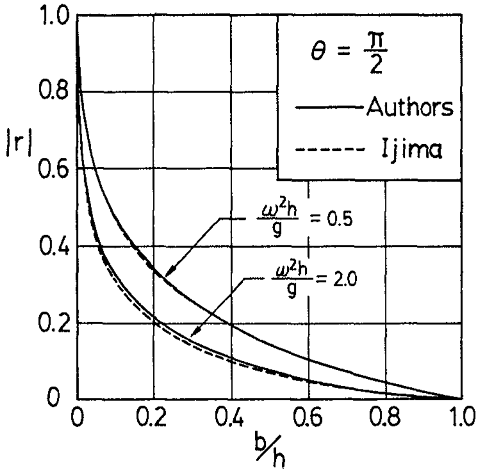
For the case of the steady waves with angular frequency ω, the linear wave equation (31) becomes

$$\hat{\eta}(\alpha) = \frac{g}{2\beta_0\omega^2}\cdot s_0(\alpha)\cdot\int_{-\infty}^{\infty}K(\xi)\cdot\frac{d}{d\alpha}\hat{\eta}(\alpha-\xi)d\xi , \tag{B-1}$$

where $\hat{\eta}(\alpha)$ is an amplitude of the water surface fluctuation and defined as

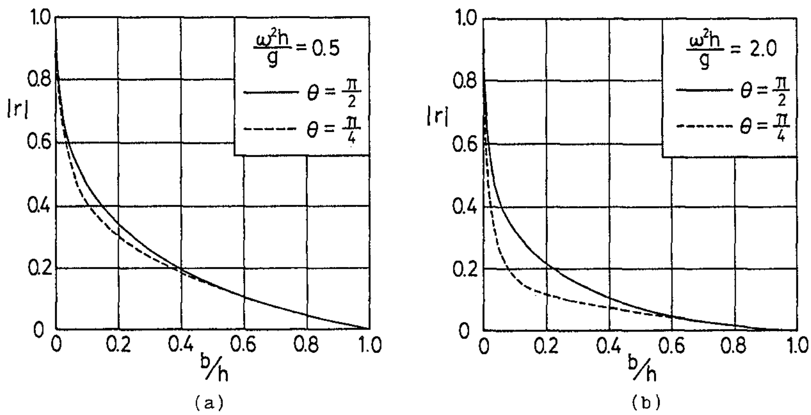
$$\eta(\alpha,t) = \hat{\eta}(\alpha)\cdot e^{i(\omega t-\pi/2)} . \tag{B-2}$$

Although Eq.(B-1) can be used to calculate reflection coefficients for linear steady waves, a simpler and more direct method can be developed with an approximation on Φ. At first, the assumption is made on the velocity function Φ such as



1.0
0.8
0.6
0.4
0.2
0
|r|
0
0.2
0.4
0.6
0.8
1.0
b/h
$\theta = \frac{\pi}{2}$
Authors
Ijima
$\frac{\omega^2 h}{g} = 0.5$
$\frac{\omega^2 h}{g} = 2.0$

Fig. B1



1.0
0.8
0.6
0.4
0.2
0
|r|
0
0.2
0.4
0.6
0.8
1.0
b/h
$\frac{\omega^2 h}{g} = 0.5$
$\theta = \frac{\pi}{2}$
$\theta = \frac{\pi}{4}$
(a)
1.0
0.8
0.6
0.4
0.2
0
|r|
0
0.2
0.4
0.6
0.8
1.0
b/h
$\frac{\omega^2 h}{g} = 2.0$
$\theta = \frac{\pi}{2}$
$\theta = \frac{\pi}{4}$
(b)

Fig. B2

$$\Phi = F(\alpha)\cdot\frac{\cosh\{k(\alpha)(\beta+\beta_0)\}}{\cosh\{k(\alpha)\beta_0)\}}\cdot e^{i\omega t}, \tag{B-3}$$

which satisfies the bottom boundary condition (9). This equation is valid for the case of $\Phi_{\alpha\alpha} << \Phi_{\beta\beta}$. In particular Eq.(B-3) becomes exact for long waves. $k(\alpha)$ in the above equation must hold the following relation from the surface boundary condition (29) on Φ.

$$\omega^2 = g\cdot s_0(\alpha)\cdot k(\alpha)\cdot\tanh\{k(\alpha)\beta_0\}. \tag{B-4}$$

From the linearized boundary condition and Laplace's equation on Φ, Eq. (B-3) yields the following differential equation on $\hat{\eta}(\alpha)$.

$$\frac{d^2}{d\alpha^2}\hat{\eta}(\alpha) + k^2(\alpha)\cdot\hat{\eta}(\alpha) = 0. \tag{B-5}$$

The reflection coefficient $|r(\alpha)|$ for the wave field expressed by the above equation can be calculated from the following Riccati's equation derived by the invariant imbedding method (Bellman and Kalaba,1959).

$$\frac{d}{d\alpha}r(\alpha) = -2ik(\alpha)\cdot r(\alpha) - \frac{k'(\alpha)}{2k(\alpha)}\cdot\{1 - r^2(\alpha)\}. \tag{B-6}$$

The reflection coefficient to be obtained is the value of $|r(\alpha)|$ at $\alpha = -\infty$. Therefore, it is more convenient to change the independent variable from α to ξ by the function $\xi = \tanh\alpha$. From this transform, the domain of the independent variable becomes $[-1< \xi <1]$.

The reflection coefficient can be obtained by a numerical integration using a usual integration scheme such as Runge-Kutta-Gill method. The integration is to be performed backward from $\xi = 1$ to $\xi = -1$ under the initial condition, $r = 0$ at $\xi = 1$.

As an example, the calculation was carried out for the stepped bottom shown in Fig.3. Figure B1 shows the results for the case of $\theta = \pi/2$. In this special case, the reflection coefficient can be also estimated accurately by, e.g., Ijima's method (1971) in which the velocity potential is found by matching conditions at the junction of sea bed. The full lines in the figure indicate the result by the present method, while the broken lines show the result by the Ijima's method. It is shown that, for the case of $\omega^2h/g = 0.5$ corresponding to shallow water waves, the result by the present method shows fairly good agreement with the Ijima's result, and even for the case of $\omega^2h/g = 2.0$ corresponding to almost deep water waves, the present method gives still good results.

The effect of the inclination of the slope has been investigated by the present method. The results are shown in Figs. B2 (a) and (b). From these figures, it is found that the effect of the inclination on the reflection coefficients is more significant for the case of the large value of ω^2h/g. This is considered to be due to the difference of the ratio of the horizontal scale of the slope to the wave length.