CHAPTER 41

LIMIT WAVES ON HORIZONTAL SEA FLOOR

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ABSTRACT

A numerical solution to periodic nonlinear irrotational surface gravity waves on a horizontal sea floor is developed using an iterative Boundary Integral Equation Method (BIEM). This solution technique is subsequently applied to determine the characteristics of limit waves for which the wave crest theoretically ceases to be rounded and become angled with an included angle of 120 degrees.

INTRODUCTION

Analytic wave theories (1,2,3) have shown that for large waves, nonlinear effects become important and wave theories that account for some of these effects give more accurate results. Most of these wave theories are based on the solutions to the irrotational monochromatic wave problem on a horizontal bottom. Application of the numerical results provided by Dean's stream function solution (4), which is of this type, is presently being recommended in the Shore Protection Manual (5) to determine forces on piles, although high order Stokes solutions have also been used. The Dean's stream function and similar solutions (4,6,7,8), are based on truncated Fourier expansions for which the accuracy of the solution decreases as the wave height approaches limit wave conditions. A limit wave has a stagnation point at the crest and, as shown by Stokes (9), the wave crest is sharp and includes an angle of 120 degrees.

A solution to limit waves based on the transformed (complex potential) plane and including terms to specifically account for the crest flow has been given by Williams (10). His solution consists of combinations of periodic motion modes and the mentioned crest terms. Since the solution is performed in the transform plane all but the dynamic surface condition are satisfied. A collocation technique satisfying the dynamic condition at discrete nodal points is used to determine the mode amplitudes. The number of modes required for accurate results and convergence depends on wave steepness and depth and was determined through empirical trial and error. Although the mode amplitudes have been calculated for a number of different wave conditions the application of the results is somewhat cumbersome and interpolation does not seem straightforward.

The numerical solution described here offers an attractive alternative method for determining limit wave parameters. The method is convergent and explicitly accounts for wave set-down.

The Boundary Integral Equation Method (BIEM), (11) solves a govern-

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ing differential equation (e.g. Laplace equation) by evaluating the two principal dependent variables on a given physical boundary. The basic concept employed stems from the application of Green's theorem and the resulting boundary method is now firmly established as an important alternative technique to the prevailing numerical methods of analysis in continuum mechanics. The efficiency of the BIEM is attributed primarily to the fact that in non-linear problems with an unknown free surface which must be determined through interaction, it is much easier to move a few nodes on the surface for each iteration than it is to update an entire spatial grid.

In this study, a numerical solution for limit waves by means of the BIEM technique is developed. It is obtained in a coordinate system translating at wave phase speed and in which the wave motion consequently is steady. The limit waves considered here are monochromatic and periodic. They are assumed to be irrotational and symmetric around a wave crest plane. The limit wave under these conditions is taken as the wave with crest particle velocity equal to the celerity, according to Stokes criterion (9). Although Benjamin and Feir (12) have questioned the stability of a periodic solution, this study does not intend to elaborate on this aspect of the problem, but accepts it as a valuable working assumption.

THEORETICAL FORMULATIONS

Two-dimensional periodic surface gravity waves at the non-breaking limit are considered as shown in Figure 1. With horizontal coordinate x' and vertical coordinate z', the origin is on the bed and moves with the same speed as the waves so that in this moving frame all motion is steady. When the assumption of an incompressible fluid and irrotation-

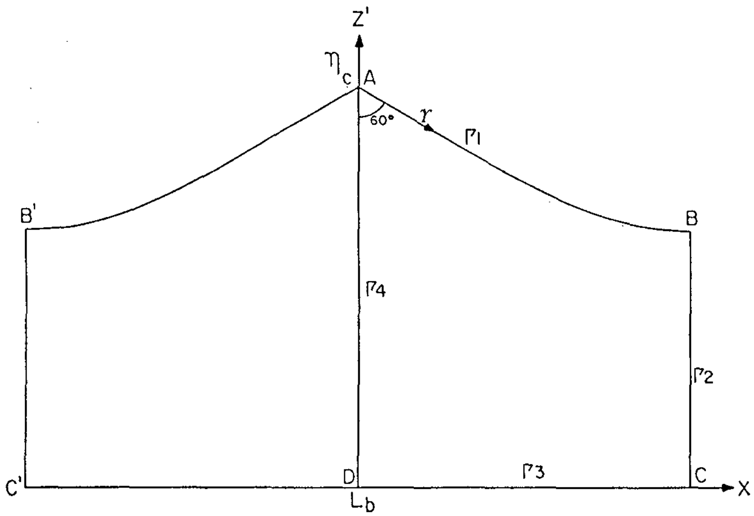

Fig. 1 Definition sketch.

al motion are incorporated, the steady-state solution is represented by a velocity potential function satisfying the Laplace equation, i.e.

$$\nabla^2\phi' = 0 \tag{1}$$

The velocity potential (ϕ') is defined in terms of velocity components u' and w' in the moving frame as

$$-\frac{\partial\phi'}{\partial x'} = u' = u - C_b \tag{2}$$

$$-\frac{\partial\phi'}{\partial z'} = w' = w \tag{3}$$

in which C_b is the speed of the limit wave and u, w are the velocity components in the corresponding fixed frame (here and in the following we use ' to designate variables measured in the translating coordinate system). The kinematic boundary condition applies to both the free surface and the bottom, i.e.

$$w' = 0 \qquad z' = 0 \tag{4}$$

$$\frac{\partial\eta}{\partial x'} = \frac{w'}{u'} \qquad z' = \eta(x') \tag{5}$$

where η is the free surface displacement.

The dynamic boundary condition on the free surface is given as

$$\eta + \frac{1}{2g}[u'^2 + w'^2] = B \qquad z' = \eta(x') \tag{6}$$

in which B is the Bernoulli constant and g is the gravitational constant.

In this study, we are specifically searching for limit wave solutions and therefore impose a further condition that the particle velocity at the crest equals the wave phase speed. This yields the well-known result that the crest has an included angle of 120 degrees and the fluid in the vicinity of the crest is characterized by a 120 degree corner flow (9), i.e.

$$\left.\frac{\partial\eta}{\partial x'}\right|_{x'=0} = -\tan^{-1}\frac{\pi}{6} \tag{7}$$

It is noted that solution of regular non-limit waves can be obtained in a very similar manner to the one described here by eliminating eq. (7) and specifying a value for B, see Lu (13).

The celerity C_b is obtained from the second definition of wave celerity by Stokes (1), i.e.

$$C_b = -\frac{1}{d}\int_0^{\eta} u'\,dz' \,. \tag{8}$$

where d denotes the water depth.

There exists a close relationship between the celerity and the overall volume transport. If Q' is defined as the total volume rate of flow underneath the steady wave per unit width, it and the wave celerity C_b are related by

$$Q' = -\,d(C_b - C_s) \tag{9}$$

in which C_s is the average mass transport velocity. If pure wave motion is assumed to cause no overall volume transport of fluid (i.e., C_s=0), then $C_b = -Q'/d$.

The water depth, d*, is determined so that the crest area equals the trough area, or

$$\int_0^{L_b} \eta(x')\,dx' = d^* \tag{10}$$

in which L_b denotes the limit wave length.

Based upon the assumption of irrotationality and periodicity, the mean water level is always depressed due to existence of waves (14). The mean water level obtained by eq. (10) is lower than the undisturbed water level. The difference Δd is defined as set-down. The set-down can be determined by (see Lu (13))

$$\Delta d = \frac{1}{2g}\left[-C_b^2 + \overline{\left(\frac{\partial\phi'}{\partial x'}\right)^2 + \left(\frac{\partial\phi'}{\partial z'}\right)^2} \right]_{z=\eta(x')} \tag{11}$$

where overbar denotes the spatial average over a wave length along the free surface.

BOUNDARY INTEGRAL EQUATION METHOD (BIEM)

The boundary value problem defined through eq. (1) to eq. (7) can be transformed into an integral equation on the boundary only as delineated by Brebbia and Wrobel (15). Thus, the potential at a point p is given by

$$c\phi(p) = \int_\Gamma \frac{\partial\phi}{\partial n}\,\phi^* d\Gamma - \int_\Gamma \phi\,\frac{\partial\phi^*}{\partial n}\,d\Gamma \tag{12}$$

where $\partial\phi/\partial n$ and ϕ are the unknown independent variables, (normal derivative and velocity potential, respectively) and principal value of integrals is implied. The value of c is: c=1 for an internal point, c=0 for an external point, and c=0.5 for a boundary point on a smooth boundary. When the surface is not smooth at the point p, c can be computed from eq. (12) by setting ϕ = constant over the whole domain Γ is the boundary domain, ϕ^* is the fundamental solution (Green's function) of a concentrated potential at a source point q as determined from the equation $\nabla^2\phi^* + \delta(q) = 0$ ($\delta(q)$ is the dirac delta function). For a two dimensional plane problem, the fundamental solution is $\phi = \ln(1/r)/(2\pi)$, where r is the distance from the source point to the integration point on the boundary. It is noted here that ϕ and $\partial\phi/\partial n$ are two independent variables on the boundary domain since $\partial\phi/\partial n$ cannot be determined from ϕ on the boundary only and vice versa.

For a well posed problem it is sufficient that one of the variables, ϕ or $\partial\phi/\partial n$ be prescribed along every boundary segment. In a numerical solution, the boundary is divided into a number of elements and the independent variables are described by discrete values at element nodes. Consequently a matrix form in terms of ϕ and $\partial\phi/\partial n$ can be written and solved by applying eq. (12) at each node. Once, both nodal velocity potential and normal derivative, ϕ and $(\partial\phi/\partial n)$ are known everywhere on the boundary, the velocity potential and velocity components can be calculated at any interior point as described by Ligget and Liu (16).

NUMERICAL FORMULATION AND SOLUTION

The numerical solution is obtained in the translating coordinate system in which the wave is staionary and symmetric around the wave crest plane. Thus, the model domain need only be half of the wave (i.e. ABCD) as shown in Figure 1.

In the mathematical problem the location of the surface, $\eta(x')$, is not known a priori, but must be determined as part of the solution through the use of conditions (5) and (6). In the present numerical solution an initial location of the free surface is guessed, the dynamic surface condition is imposed and an iterative technique is developed to change the position of the surface until the kinematic condition is also satisfied.

Because only wave length and wave crest elevation can be chosen independently, the problem geometry can be characterized by only a single dimensionless parameter, ξ = ratio between the limit wave crest elevation and limit wave length, making it easier to numerically cover the full range of possible conditions.

Boundary Conditions

Because the wave crest is a stagnation point, the Bernoulli constant $B = \eta_c$, where η_c is the crest elevation. The dynamic free surface condition then becomes

$$V_s = [2g(\eta_c - \eta)]^{1/2} \qquad (13)$$

where V_s is the tangential velocity along the free surface. Eq. (13) can therefore be used to determine the velocity potential along a given free surface by performing a line integration along contour AB with the curvilinear coordinate γ.

$$\phi_s'(\gamma) = \int_A^{\gamma} V_s \, d\Gamma + \phi_s'\Big|_A \qquad (14)$$

Taking ϕ' in the crest plane equal to 0 and noting $\gamma = \gamma(x')$

a Dirichlet condition is obtained,

$$\phi' = \phi_s'(x') \qquad \text{on } \Gamma_1 \tag{15}$$

A Neumann condition at the bottom is defined by the kinematic bottom boundary condition

$$w' = \frac{\partial \phi'}{\partial n} = 0 \qquad \text{on } \Gamma_3 \tag{16}$$

The assumption that waves are propagating without changing form implies that in the crest and trough planes vertical velocity components are zero, hence, ϕ' is constant

$$\phi' = 0 \qquad \text{on } \Gamma_4 \tag{17}$$

$$\phi' = \phi_T \qquad \text{on } \Gamma_2 \tag{18}$$

where ϕ_T obtained by eq. (14) is the velocity potential at the trough.

Initial Estimate of Surface

A cubic polynomial conforming to the known slopes at both ends (-30 degrees at the crest and 0 degrees at the trough) and passing through the crest and an arbitrarily chosen trough is used as the initial surface. Subsequently, the surface is discretized into a number of elements.

Numerical Schematization

For best possible fit a number of quadratic isoparametric elements are used along the free surface, while normal quadratic elements are used on the rest of the boundary. The nodes are located at the ends and the middle of these elements and have values of ϕ_j and $\partial\phi_j/\partial n$ associated with them, where j=1,2,3...n and n denotes the total number of nodal points.

Because the gradients are expected to be greater near the crest a variable element length is used with elements smallest at that location. Extensive testing showed that a total of 96 boundary elements provided a reasonable compromise between cost of computation and accuracy. The same grid was used for all computations regardless of the value of ξ.

Surface Iteration Technique

Based on the initial surface the BIEM can be executed. The results will in general not satisfy the kinematic boundary condition, but can be made to do so by changing the surface.

A convergent iteration technique based on "influence" solutions and a Newton-Raphson method is developed to perform the surface updating automatically. The technique first computes the change in the kinematic condition residual for unit changes in the elevation of each surface node and organizes the influence solutions in a Jacobian matrix. Based on the current residuals the Jacobian is then used in conjunction with a Newton-Raphson method to determine the optimal change to the elevation of each surface node. Because of the nonlinear nature of the problem, the Jacobian should in theory be computed every time the surface is changed, however, we have found that in practice the Jacobian does not change much after the first few updates and need not be recomputed.

The iteration technique is continued until an error measure in the kinematic boundary condition defined as

$$\varepsilon_2(\theta) = \frac{\partial \eta}{\partial x'} - \frac{w'}{u'} \qquad (19)$$

is below a threshold value. We have chosen this threshold value to be on the order of the round off errors of the method on the 36 bit UNIVAC machine employed for the computations. A flow chart of the solution method is shown in Fig. 2.

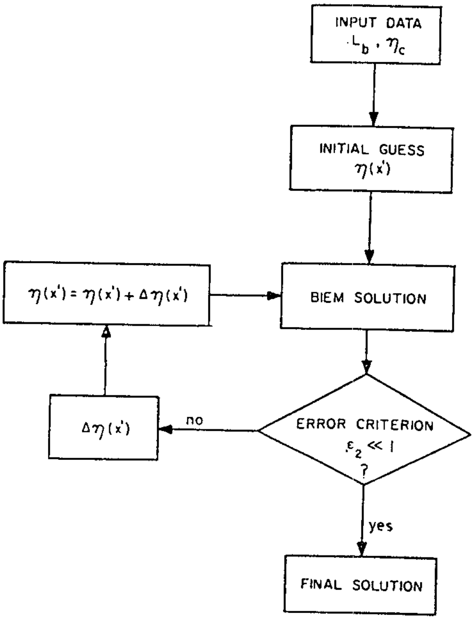

Fig. 2 Flow chart of the BIEM technique.

RESULTS

A unique limit wave is obtained with the BIEM for each discrete value of ξ. (The uniqueness is confirmed by choosing two different initial guesses of surface and obtaining an identical final solution.) However, in practical applications the crest height and wave length necessary to determine ξ are usually not known. The known or more easily obtained parameters are undisturbed water depth (d) and wave period (T). Our results can be made more amenable to practical applications by determining a relationship between ξ and the single dimensionless parameter, d/L_o, with $L_o = gT^2/2\pi$. In addition, it will be very useful to establish relationships for L_b/L_o, H_b/L_o and η_c/L_o as functions of d/L_o to make possible direct calculation of limit wave length, wave crest elevation and wave height. Due to the assumption of limit waves unique relations can be written as

$$L_b/L_o = f_1(d/L_o) \quad (20)$$

$$\eta_c/L_o = f_2(d/L_o) \quad (21)$$

$$H_b/L_o = f_3(d/L_o) \quad (22)$$

and

$$\xi/L_o = f_4(d/L_o) \quad (23)$$

The functions f_1, f_2, f_3 and f_4 are determined numerically from our results as described in the following.

Based on the BIEM solutions and for each doublet (η_c, L_b) the limit wave height H_b can be calculated by $H_b = \eta_c - \eta_t$, where η_t is the surface elevation at the trough. Furthermore, the undisturbed water depth can be calculated from

$$d = \frac{2}{L_b} \int_o^{L_b/2} \eta(x')\, dx' + \frac{1}{2g} \left[-C_b^2 + \overline{\left(\frac{\partial\phi'}{\partial x'} \right)^2 + \left(\frac{\partial\phi'}{\partial z'} \right)^2} \right] \quad (24)$$

in which the first term is the mean water level with waves and the second term is the correction for wave set-down.

Invoking the second definition of phase velocity (zero transport), the phase velocity is determined by

$$C_b = - Q'/d \quad (25)$$

in which Q' is the volume flux in the moving frame. Then, the wave period is determined by $T = L_b/C_b$. Consequently, four dimensionless doublets (L_b/L_o, d/L_o), (η_c/L_o, d/L_o), (H_b/L_o, d/L_o), and (ξ, d/L_o) are obtained for each BIEM solution. The relationship among these dimensionless parameters is plotted as shown in Figs. 3, 4, 5, and 6. Thus, for each given wave period and undisturbed water depth the input data, ξ, for the BIEM and the resulting limit wave length, wave height and crest elevation can be determined immediately.

A measure of the error committed in the kinematic boundary condition used by Dean (4) is defined by

$$E_2 = \left[\sum_{j=1}^{N} \varepsilon_2(\theta)^2 \right]^{1/2} \quad (26)$$

in which N is the number of sampling points along the wave profile and $\varepsilon_2(\theta)$ is the error at each sampling point. Comparison of E_2 between the BIEM solution with 96 discretized boundary elements and those computed by Dean (4) for other existing wave theories is shown in Fig. 7.

It should be noted that the present solution satisfies the dynamic surface condition "exactly" while the other solutions have errors in the dynamic condition. Except for very shallow water the BIEM using 96 boundary element provides results with an order of magnitude smaller error. This error could of course be made arbitrarily small by using more elements and a computer with higher precision. Direct comparison with Dean's stream function solution is not possible because it satisfies the kinematic condition exactly while containing errors in the dynamic condition, however, due to the previously mentioned convergence problems significantly better results should be expected from the BIEM solution.

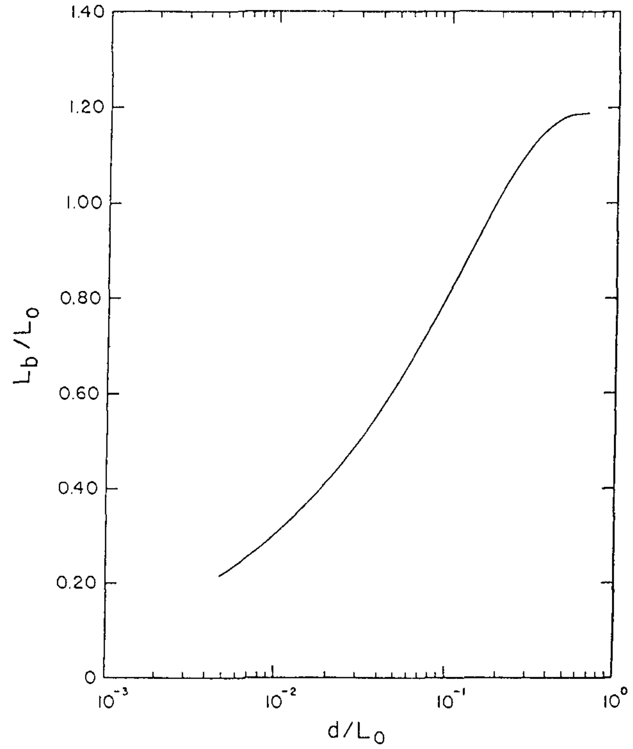

Fig. 3 L_b/L_o versus d/L_o.

The results obtained for H_b/L_o as function of d/L_o (figure 6) may be compared with the theoretical value derived by Michell (17) and Miche (18) who found $H_b/L = 0.142$ in deep water and $H_b/L = .140 \tanh(d/L)$ in intermediate depth water respectively. Due to numerical limitations that the theoretical limit of infinitely deep water cannot be calculated with BIEM approach. The deepest case calculated in this study corresponds to $d/L_o = 0.7$ and the corresponding $H_b/L_b = 0.140$. Note that these expressions use the nonlinear wave-length which is greater than L_o. An earlier study has indicated a ratio between deep water limit wave length and linear wave length of 1.2, (4), while our results yield a value of 1.186 for $d/L_o = 0.7$. As a further verification of our BIEM solutions comparison is also made with the results by

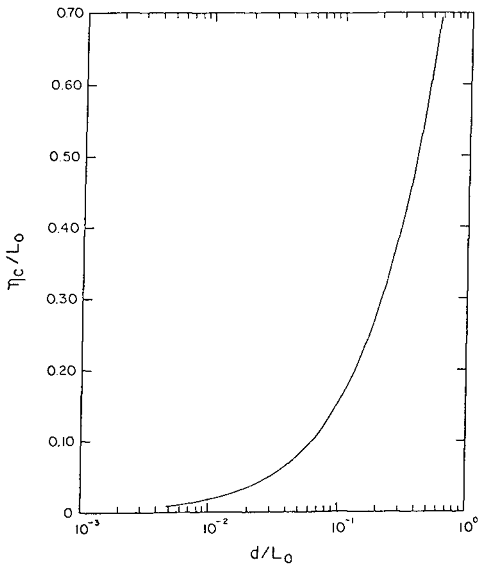

Fig. 4 η_c/L_o versus d/L_o.

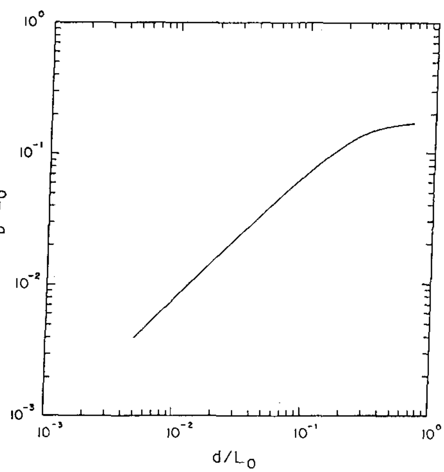

Fig. 5 H_b/L_o versus d/L_o.

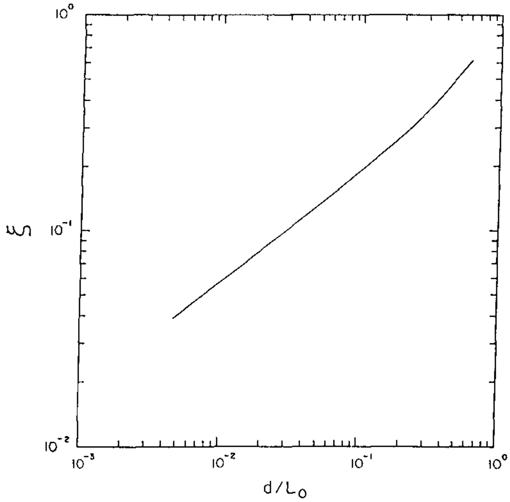

Fig. 6 ξ versus d/L_0.

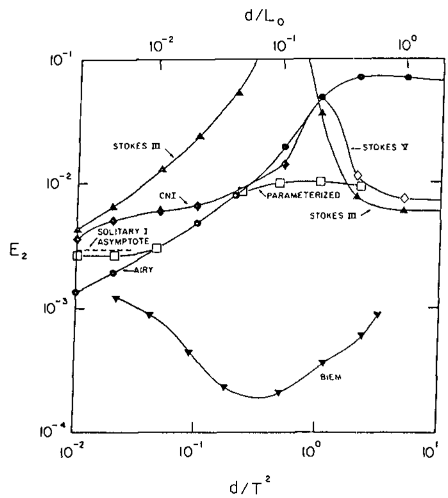

Fig. 7 Comparison of analytic kinematic criterion for limit waves (Dean, 1974)

Schwartz (6), Cokelet (7), and William (10) for three different d/L_b. As is shown in Table 1, the solution obtained by the BIEM technique agrees with the results obtained from these three different limit wave solutions.

$\frac{d}{L_b}$	$\frac{H}{L_b}$ (Schwartz, 1974)	$\frac{H}{L_b}$ (Cokelet, 1977)	$\frac{H}{L_b}$ (Williams, 1981)	$\frac{H}{L_b}$ (BIEM)
0.3665	0.1380	0.1378	0.1378	0.1377
0.0355	-	0.0279	0.0283	0.0279
0.0168	-	0.015	0.0137	0.0139

Table 1: Comparison of limit wave height for various water depths.

CONCLUSION

A numerical solution for periodic nonlinear irrotational surface gravity limit waves is developed by means of the Boundary Integral Equation Method (BIEM). The formulation for limit waves employs Stokes limit criterion.

The BIEM solution theoretically satisfies exactly the Laplace equation, the dynamic surface boundary condition, and Stokes limit criterion, although any numerical solution will contain finite discretization errors. Solution compliance with the kinematic surface condition is forced by iterating the surface until any error is made arbitrarily small. The final surface is taken as the true limit wave. The wave properties including the dimensionless maximum wave height can be determined as functions of a given water depth and wave period. For the cases considered involving 96 boundary elements and iterating to an error $E_2 < 10^{-3}$ one solution requires approximately 45 minutes of CPU time on a UNIVAC 1100/81 system.

ACKNOWLEDGEMENTS

This study was sponsored by the National Science Foundation as part of a study done under Grant No. ENG-11930.

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