CHAPTER 69

STATISTICAL CHARACTERISTICS OF OFFSHORE CURRENTS

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ABSTRACT

This paper presents the analytical derivation of probability density functions applicable for tidal (high frequency components), residual (low frequency components), and total (the vector sum of tidal and residual) current velocities. The theoretical probability density functions are compared with histograms constructed from measured data. A method to estimate the extreme current velocities in a specified time period is presented by applying extreme value statistics.

INTRODUCTION

The magnitude of current velocity measured in offshore areas fluctuates randomly with frequencies covering a wide range from 0.0008 to 0.08 cycles per hour (period range from 1200 to 12 hours). Measured current data may be decomposed into high and low-frequency components. Here, high-frequency components are considered to be currents associated with tides, while low-frequency components (called residual currents) are those attributed to wind and all other environmental conditions.

It may be of considerable interest to obtain the statistical distributions applicable for the tidal (high frequency components), residual (low-frequency components), and total (the vector sum of tidal and residual) current velocities, and therefrom to estimate the extreme velocities expected to occur in a specified time period.

This paper presents the analytical derivation of probability density functions applicable for the tidal and residual as well as total current velocities. The theoretical probability density functions are compared with histograms constructed from measured data. Based on these probability density functions, extreme current velocities are estimated from knowledge of variances of high and low-frequency components in two rectangular directions.

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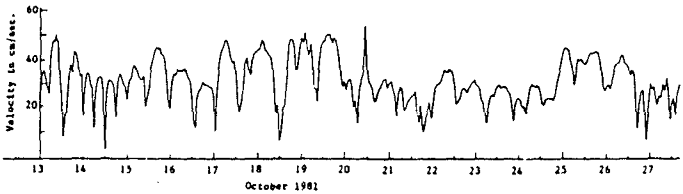
High Frequency (Tidal) and Low Frequency (Residual) Currents

Offshore current velocity records, in general, represent the average velocities consisting of various frequencies measured during a certain time period. For example, Figure 1 taken from Reference 1 shows a portion of measured current velocities recorded at a 26 meter depth off Newfoundland, Canada. The data represent the hourly mean current velocities measured at 10 minute intervals in a dominant hourly current direction. Figure 2 shows a polar diagram indicating the magnitude and direction of the measured current velocities. Unfortunately, statistical information on high frequency (tidal) and low frequency (residual) current velocities cannot be obtained from Figure 2. Nor is it possible to obtain this information by simply separating the time history shown in Figure 1 into high and low frequency components, since the components thusly obtained merely represent the high and low frequency current velocities in the dominant hourly current direction. The tidal and residual currents flow independently in constantly changing direction.



Current velocity

Figure 1 Time history of current velocity

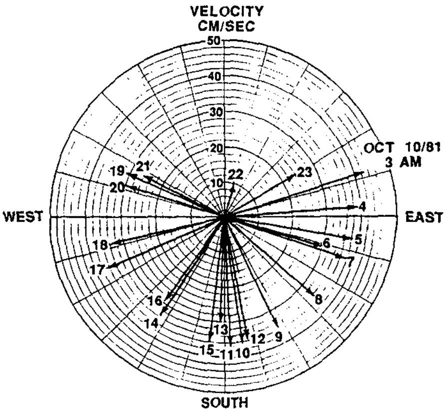


Figure 2 Polar diagram indicating the magnitude and direction of current velocity as a function of time

In order to obtain the statistical properties of tidal components and residual components of current velocity, it is first necessary to decompose the measured current velocity into two rectangular components as shown in Figure 3. Here, for convenience, we consider the East-West and North-South directions. Each of these two velocity components, denoted by U and V respectively, is then further decomposed into a high frequency (tidal) velocity component and a low frequency (residual) velocity component. In the present study, a cutoff frequency of 0.035 cycles/hour (28.5 hour period) is used to separate the two components [1]. This is because the results of spectral analysis of the records indicate that a small amount of energy exists beyond the 24.0 hour period (0.042 cycles/hour). Thus, the time history of current velocity constructed from measurements at one hour intervals is decomposed into the following four components:

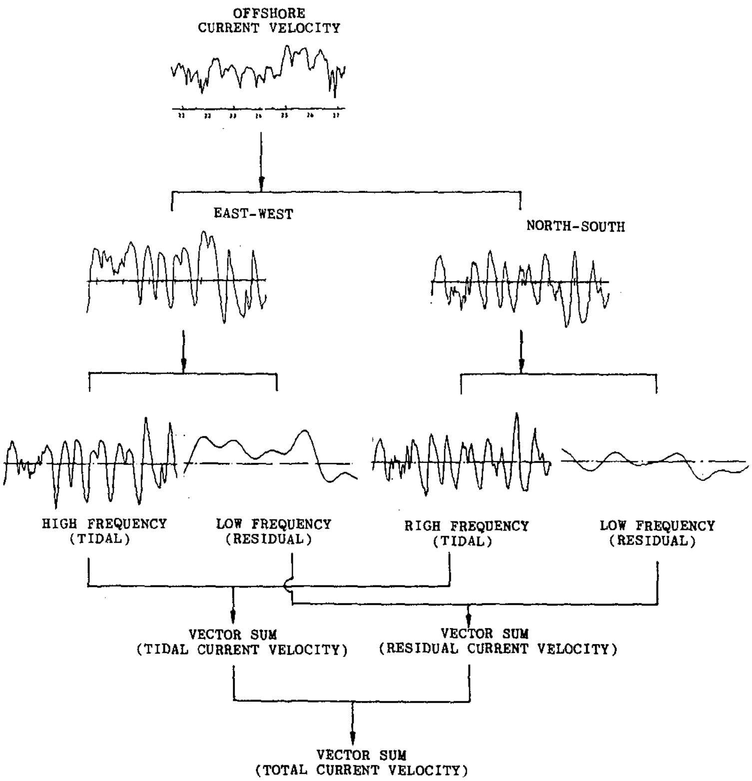


Figure 3 Diagram illustrating analysis of offshore current velocity data

In the East-West direction
High frequency (tidal) component, H_u
Low frequency (residual) component, L_u

In the North-South direction
High frequency (tidal) component, H_v
Low frequency (residual) component, L_v.

From these four time histories, the magnitude of high frequency (tidal) velocity can be obtained by combining H_u and H_v in vector form, while that for the low frequency (residual) velocity is obtained by combining L_u and L_v. The current velocities thusly combined at one hour intervals, denoted by H and L, respectively, have individual directions and provide information on tidal and residual current velocities. The high and low frequency currents are hereafter referred to as tidal and residual currents, respectively.

Figures 4 and 5 show examples of polar diagrams indicating the magnitude and direction of the tidal and residual current velocities, respectively, as a function of time. As can be seen in Figure 4, the tidal current velocities show approximately the same pattern of rotation as the measured current velocities shown in Figure 2. However, the residual velocity rotates very slowly clockwise through 180 degrees in approximately 20 hours in this example, and then rotates counter-clockwise through 90 degrees in approximately 12 hours. The residual current velocity does not have a consistently rotating directionality in contrast to the tidal velocity.

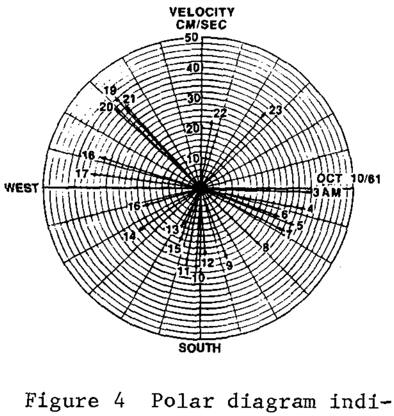


Figure 4 Polar diagram indicating the magnitude and direction of tidal (high frequency) current velocity as a function of time

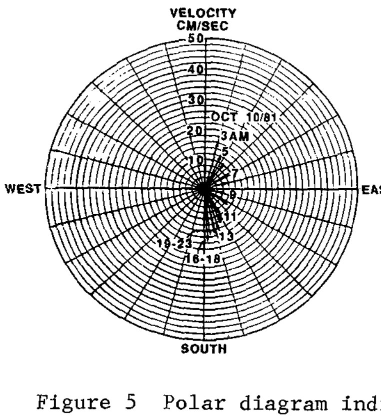


Figure 5 Polar diagram indicating the magnitude and direction of residual (low frequency) current velocity as a function of time

It is noted that the tidal current and the residual current are statistically uncorrelated. The correlation coefficient between them is extremely small, on the order of 0.05. Hence, the possibility of simultaneous occurrence of large tidal current and large residual current is almost nil.

PROBABILITY DISTRIBUTIONS OF TIDAL AND RESIDUAL CURRENT VELOCITIES

The probability density function applicable for the tidal current velocity can be derived as follows:
It is found from statistical analysis of measured data that both the East-West and North-South components of high frequency current velocities approximately follow the Gaussian probability distribution with zero mean and variance which is equal to the area under the spectral density function obtained for each direction [1]. It is also found that the East-West and North-South components, H_u and H_v, are statistically independent. Let us write the variance of H_u and H_v as σ_{HU}^2 and σ_{HV}^2, respectively. Since the two random variables, H_u and H_v are statistically independent, the joint probability density function can be written as

$$f(H_u, H_v) = \frac{1}{2\pi\sigma_{HU}\sigma_{HV}} \exp\left\{-\frac{1}{2}\left(\frac{H_u^2}{\sigma_{HU}^2} + \frac{H_v^2}{\sigma_{HV}^2}\right)\right\}, \tag{1}$$

$$0 \le H_u < \infty , \qquad 0 \le H_v < \infty .$$

Let H be the vector sum of the two components. Then, H_u and H_v can be written as

$$H_u = H \sin\theta \qquad H_v = H \cos\theta , \tag{2}$$

where θ = directional angle of H measured from North.

By applying the technique of changing random variables, the joint probability density function of H and θ can be obtained as

$$f(H,\theta) = \frac{H}{2\pi\sigma_{HU}\sigma_{HV}} \exp\left\{-\frac{H^2}{2}\left[\frac{1}{\sigma_{HU}^2} + \left(\frac{1}{\sigma_{HV}^2} - \frac{1}{\sigma_{HU}^2}\right)\cos^2\theta\right]\right\},$$

$$0 \le H < \infty , \qquad 0 \le \theta \le 2\pi. \tag{3}$$

The probability density function of the vector sum can then be obtained as the marginal probability density function of Eq.(3). That is,

$$f(H) = \frac{H}{2\pi\sigma_{HU}\sigma_{HV}} e^{-\frac{H^2}{2\sigma_{HU}^2}} \int_0^{2\pi} \exp\left\{-\frac{H^2}{2}\left(\frac{1}{\sigma_{HV}^2} - \frac{1}{\sigma_{HU}^2}\right)\cos^2\theta\right\} d\theta . \tag{4}$$

By expanding the exponential function and by integrating with respect to θ, we have

$$f(H) = \frac{H}{\sigma_{HU}\sigma_{HV}} e^{-\frac{H^2}{2\sigma_{HU}^2}} \left[1 - \frac{H^2}{4}\left(\frac{1}{\sigma_{HV}^2} - \frac{1}{\sigma_{HU}^2}\right) + \frac{H^4}{64}\left(\frac{1}{\sigma_{HV}^2} - \frac{1}{\sigma_{HU}^2}\right)^2 + \ldots\right.$$

$$\sim \frac{H}{\sigma_{HU}\sigma_{HV}} e^{-\frac{H^2}{2\sigma_{HU}^2}} \cdot e^{-\frac{H^2}{4}\left(\frac{1}{\sigma_{HV}^2} - \frac{1}{\sigma_{HU}^2}\right)}$$

$$= A \frac{H}{\sigma_{HU}\sigma_{HV}} \exp\left\{-\frac{H^2}{4}\left(\frac{1}{\sigma_{HU}^2} + \frac{1}{\sigma_{HV}^2}\right)\right\} , \tag{5}$$

where A is a normalization factor to be determined such that the area under the probability density function becomes unity. It can be obtained as

$$A = 2\sigma_{HU}\sigma_{HV}\left(\frac{1}{\sigma_{HU}^2} + \frac{1}{\sigma_{HV}^2}\right) . \tag{6}$$

Hence, the probability density function of tidal current velocity becomes

$$f(H) = \frac{H}{2}\left(\frac{1}{\sigma_{HU}^2} + \frac{1}{\sigma_{HV}^2}\right)\exp\left\{-\frac{H^2}{4}\left(\frac{1}{\sigma_{HU}^2} + \frac{1}{\sigma_{HV}^2}\right)\right\} . \tag{7}$$

This is the Rayleigh probability density function whose parameter is given as a function of the variances of the East-West and North-South components. Figure 6 shows a comparison of the probability density function given in Eq.(7) and the histogram of tidal current velocity constructed from measured data. Good agreement between them can be seen.

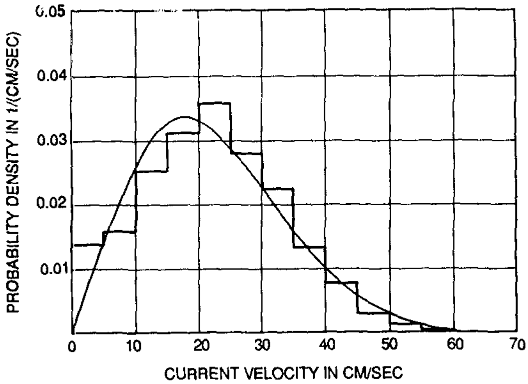


Figure 6 Comparison between histogram and probability density function of tidal (high frequency) current velocity

The probability density function of the residual current velocity can be derived through the same procedure as that used for the tidal current velocity. That is, by writing the variances of the East-West and North-South components of residual currents as σ_{LU}^2 and σ_{LV}^2, respectively, the probability density function of the residual current velocity, L, can be given by

$$f(L) = \frac{L}{2}\left(\frac{1}{\sigma_{LU}^2} + \frac{1}{\sigma_{LV}^2}\right) \exp\left\{-\frac{L^2}{4}\left(\frac{1}{\sigma_{LU}^2} + \frac{1}{\sigma_{LV}^2}\right)\right\} \quad . \tag{8}$$

A comparison between the probability density function and the histogram constructed from measured data is shown in Figure 7.

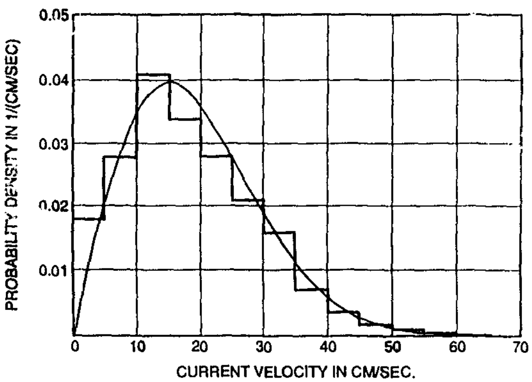


Figure 7 Comparison between histogram and probabiility density function of residual (low frequency) current velocity

PROBABILITY DISTRIBUTION OF TOTAL CURRENT VELOCITY

In this section, the probability density function of the vector sum of tidal and residual current velocities, hereafter called the total current velocity, will be derived. Here, the total current velocity is equal to the measured current velocity illustrated in Figure 1.

The magnitude of the total current velocity, denoted by C, can be written by the following vector sum of tidal and residual velocities:

$$C = \left\{ H^2 + L^2 + 2HL\cos(\theta_H - \theta_L) \right\}^{1/2} , \tag{9}$$

where H = magnitude of tidal current velocity, L = magnitude of residual current velocity, θ_H = directional angle of tidal current velocity, θ_L = directional angle of residual current velocity.

It was obtained in the previous section that the random variables H and L both follow the Rayleigh probability law. Hence, the probability distribution of the sum of the first two terms of Eq. (9), $(H^2 + L^2)$, can be obtained as follows:

It can easily be derived that the square of the Rayleigh probability distribution becomes the exponential probability distribution. That is, by writing H^2 and L^2 as H_* and L_*, respectively, we can derive the following probability density functions of H_* and L_*.

$$f(H_*) = \frac{1}{R_H} \exp\left\{-H_*/R_H\right\} , \qquad f(L_*) = \frac{1}{R_L} \exp\left\{-L_*/R_L\right\} , \tag{10}$$

where

$$R_H = 4/\left(\frac{1}{\sigma_{HU}^2} + \frac{1}{\sigma_{HV}^2}\right) \quad \text{and} \quad R_L = 4/\left(\frac{1}{\sigma_{LU}^2} + \frac{1}{\sigma_{LV}^2}\right) . \tag{11}$$

Since the random variables H and L are statistically independent [1], the probability density function of the sum of the random variables H_* and L_*, denoted by X, can be obtained as

$$f(x) = \int_0^x \frac{1}{R_H} \exp\left\{-H_*/R_H\right\} \cdot \frac{1}{R_L} \exp\left\{-(x-H_*)/R_L\right\} dH_*$$

$$= \frac{1}{R_H - R_L}\left(e^{-\frac{x}{R_H}} - e^{-\frac{x}{R_L}}\right) , \qquad 0 \le x < \infty . \tag{12}$$

Next, let us derive the probability density function of the third term of Eq.(9). For this, first the probability density function of $L\cos(\theta_H - \theta_L)$ is considered. Results of analysis show that the difference of the two angles $\theta_H - \theta_L$ scatters in random fashion over the range from $-\pi$ to $+\pi$; therefore, we may assume that $\cos(\theta_H - \theta_L)$ is a random variable and it has a uniform distribution over the range -1 to +1 with a density of 1/2. It can also be assumed that the random variables L and $\cos(\theta_H - \theta_L)$ are statistically independent. Then the probability distribution of the product of $L\cos(\theta_H - \theta_L)$, denoted by Y, can be written as

$$f(y) = \begin{cases} \displaystyle\int_{y}^{\infty} \frac{1}{2}\frac{1}{L} f(L)\, dL & \text{for } y > 0 \\ \displaystyle\int_{-|y|}^{\infty} \frac{1}{2}\frac{1}{L} f(L)\, dL & \text{for } y < 0 , \end{cases} \tag{13}$$

where 1/L is the Jacobian associated with the change of random variables, and f(H) is given in Eq.(8). The probability density function f(L) is symmetric with respect to y=0. Hence, we may write the function as

$$f(y) = \frac{1}{R_L}\int_{|y|}^{\infty} \exp\left\{-L^2/R_L\right\} dL = \sqrt{\pi/R_L}\ \Phi\left(-\sqrt{2/R_L}\ |y|\right) , \tag{14}$$

where $Y = L\cos(\theta_H - \theta_L)$ and R_L is given in Eq.(11).

The probability density function of $HL\cos(\theta_H - \theta_L)$ can be derived as the density function of the produce of two random variables H and $Y = L\cos(\theta_H - \theta_L)$ which are statistically independent. However, the derivation is extremely complicated in practice because of the density function f(y). Hence, the probability density function f(y) is approximated by the following normal probability density function:

$$f(y) = \sqrt{2/\pi R_L}\ \exp\left\{-2|y|^2/R_L\right\} . \tag{15}$$

A comparison of Eq.(14) with its approximation given by Eq.(15) is shown in Figure 8 in which the random variable Y is non-dimensionalized by letting $Y = \sqrt{R_L/2}\ X$.

By applying this approximation, the probability density function of $HL\cos(\theta_H - \theta_L)$, denoted by Z, can be derived from Eqs.(7) and (15). That is,

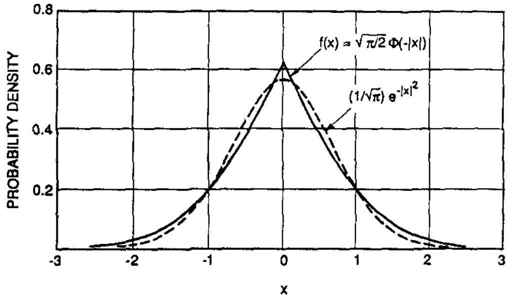


Figure 8 Comparison between f(x) and its approximation by $(1/\sqrt{\pi})\exp\{-|x|^2\}$, where $y = \sqrt{R_1/2}\ x$ in Eq.(15)

$$f(z) = \int_0^u \frac{2}{R_H}\sqrt{\frac{2}{\pi R_L}}\ \exp\left\{-H^2/R_H\right\}\ \exp\left\{-(2/R_L)(z/H)^2\right\} dH$$

$$= \sqrt{\frac{2}{R_H R_L}}\ e^{-2\sqrt{\frac{2}{R_H R_L}}\,|z|}\ , \qquad -\infty < z < \infty\ . \tag{16}$$

From the above equation, the probability density function of $2Z = 2HL\cos(\theta_H - \theta_L)$ which is the third term of Eq.(9) can be obtained. That is, by writing $2Z = W$, we have

$$f(w) = \frac{1}{\sqrt{2R_H R_L}}\ e^{-\frac{2|w|}{\sqrt{2R_H R_L}}}\ , \qquad -\infty < w < \infty\ . \tag{17}$$

The probability density function of C^2 can now be evaluated as the density function of the sum of the two random variables X and W which are considered to be statistically independent and whose probability density functions are given in Eqs.(12) and (17), respectively. Since the random variable C^2 is positive only, we consider the following two cases for both of which the sum of X and W is positive.

(i) For $X > 0$ and $W > 0$.

By writing the random variable $C^2 = U = X + W$, we can derive the probability density function f(u) from Eqs.(12) and (17) as

$$f(u) = \int_u^\infty \frac{1}{R_H - R_L}\left(e^{-\frac{x}{R_H}} - e^{-\frac{x}{R_L}}\right)\frac{1}{\sqrt{2R_H R_L}}\ e^{-\frac{2(u-x)}{\sqrt{2R_H R_L}}}\ dx$$

$$= \frac{1}{\sqrt{2}} \left[\frac{1}{(\sqrt{2R_H} - \sqrt{R_L})(\sqrt{2R_L} - \sqrt{R_H})} e^{-\sqrt{\frac{2}{R_H R_L}}\, u} \right.$$

$$\left. + \frac{\sqrt{R_H}}{(R_H - R_L)(\sqrt{2R_H} - \sqrt{R_L})} e^{-\frac{u}{R_H}} - \frac{\sqrt{R_L}}{(R_H - R_L)(\sqrt{2R_L} - \sqrt{R_H})} e^{-\frac{u}{R_L}} \right] \tag{18}$$

(ii) For $x > 0$ and $W < 0$ but $U = X + W > 0$.

By writing $|W| = X - U$ in Eq.(17), we have

$$f(u) = \int_u^{\infty} \frac{1}{R_H - R_L} \left(e^{-\frac{x}{R_H}} - e^{-\frac{x}{R_L}} \right) \frac{1}{\sqrt{2R_H R_L}} e^{-\frac{2(u-x)}{\sqrt{2R_H R_L}}} dx$$

$$= \frac{1}{\sqrt{2}} \frac{1}{R_H - R_L} \left[\frac{\sqrt{R_H}}{\sqrt{2R_H} + \sqrt{R_L}} e^{-\frac{u}{R_H}} - \frac{\sqrt{R_L}}{\sqrt{2R_L} + \sqrt{R_H}} e^{-\frac{u}{R_L}} \right] . \tag{19}$$

The sum of Eqs.(18) and (19) yields the probability density function of U. However, we consider only positive U in the above and discard the sum of the two random variables X and W for which U becomes negative. Hence, it is necessary to normalize the resultant probability density function so that the area under the density function is unity. By taking normalization into consideration, we have

$$f(u) = L \left[\frac{1}{\sqrt{2}} \frac{1}{(\sqrt{2R_H} - \sqrt{R_L})(\sqrt{2R_L} - \sqrt{R_H})} e^{-\sqrt{\frac{2}{R_H R_L}}\, u} \right.$$

$$\left. + \frac{2}{R_H - R_L} \left(\frac{R_H}{2R_H - R_L} e^{-\frac{u}{R_H}} - \frac{R_L}{2R_L - R_H} e^{-\frac{u}{R_L}} \right) \right], \quad 0 \le u < \infty \quad , \tag{20}$$

where

$$L^{-1} = \frac{1}{2} \frac{\sqrt{R_H R_L}}{(\sqrt{2R_H} - \sqrt{R_L})(\sqrt{2R_L} - \sqrt{R_H})} - \frac{2(R_H^2 - R_H R_L + R_L^2)}{(2R_H - R_L)(2R_L - R_H)} . \tag{21}$$

Finally, by changing the random variable U to C where C is the square-root of U, we can derive the probability density function applicable for the total current velocity given in Eq.(9) as follows:

$$f(C) = L \left[\frac{\sqrt{2}\, C}{(\sqrt{2R_H} - \sqrt{R_L})(\sqrt{2R_L} - \sqrt{R_H})} e^{-\sqrt{\frac{2}{R_H R_L}}\, C^2} \right.$$

$$\left. + \frac{4\, C}{R_H - R_L} \left(\frac{R_H}{2R_H - R_L} e^{-\frac{C^2}{R_H}} - \frac{R_L}{2R_L - R_H} e^{-\frac{C^2}{R_L}} \right) \right] , \quad 0 \le C < \infty . \qquad (22)$$

A comparison between the theoretical probability density function derived in Eq.(22) and the histogram of the total current velocity constructed from measured data is shown in Figure 9. Good agreement can be seen in the figure. The values of R_H and R_L of Eq.(22) are computed by Eq.(11) in which variances are evaluated through spectral analysis [1]. These are
σ^2_{HU} = 334.3 $(cm/sec)^2$, σ^2_{HV} = 293.7 $(cm/sec)^2$,
σ^2_{LU} = 248.7 $(cm/sec)^2$, and σ^2_{LV} =214.2 $(cm/sec)^2$.

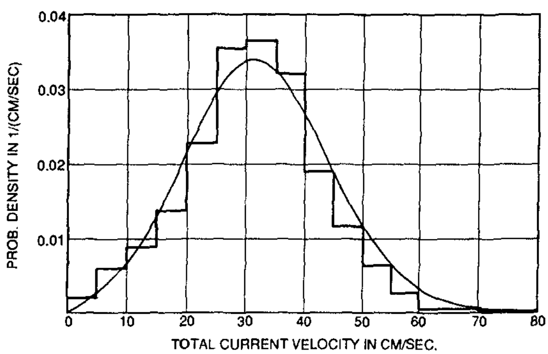


Figure 9 Comparison between histogram and probability density function of total current velocity

Next, let us evaluate the magnitude of extreme current velocity expected to occur in 20 and 50 years. For this, the cumulative distribution function of the total current velocity is obtained by integrating Eq.(22) as follows:

$$F(C) = \int_0^C f(C)\,dC$$

$$= L\left[\frac{\sqrt{R_H R_L}}{2(\sqrt{2R_H} - \sqrt{R_L})(\sqrt{2R_L} - \sqrt{R_H})}\left(1 - \exp\left\{-\sqrt{\frac{2}{R_H R_L}}\,C^2\right\}\right)\right.$$

$$+ \frac{2R_H^2}{(R_H - R_L)(2R_H - R_L)}\left(1 - \exp\left\{-\frac{C^2}{R_H}\right\}\right)$$

$$\left. - \frac{2R_L^2}{(R_H - R_L)(2R_L - R_H)}\left(1 - \exp\left\{-\frac{C^2}{R_L}\right\}\right)\right] \tag{23}$$

Then, by applying extreme value statistics, the probable extreme value most likely to occur in n-observations can be evaluated as a solution of the following equation:

$$\frac{1}{1 - F(C)} = n \tag{24}$$

Here, n is the number of samples in a specified time period. Since n = 1911 in 4 months for this example, the extreme values in 4 months, 20 years and 50 years are estimated as shown in Figure 10. For convenience, the logarithm of Eq.(24) is shown in the figure. The estimated extreme current velocity in 4 months is 76 cm/sec as compared with 74 cm/sec observed in the data. The extreme values expected in 20 and 50 years are 92.0 cm/sec and 94.5 cm/sec, respectively, as indicated in the figure.

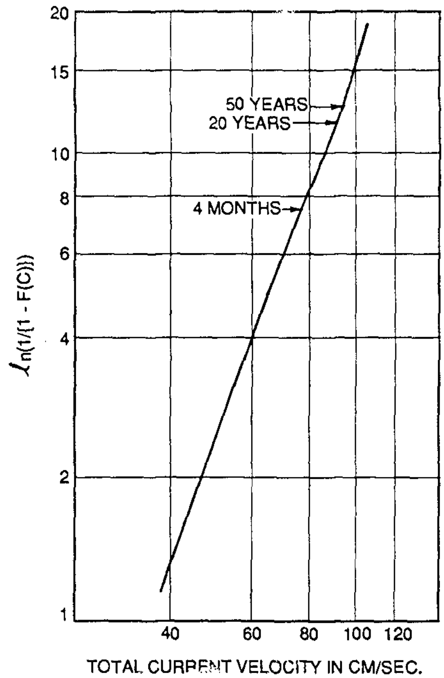


Figure 10 Estimation of extreme current velocity in 4 months, 20 and 50 years

CONCLUSIONS

This paper presents the results of a study on stochastic analysis of offshore currents. The objective of the study is to estimate the extreme magnitude of offshore current velocities based on the probability density functions applicable for tidal (high frequency components), residual (low frequency components), and total (the vector sum of tidal and residual) current velocities. From the results of the analysis, the following conclusions are derived:

(1) It is found theoretically that the tidal (high frequency components) current velocity follows the Rayleigh probability law. The probability density function is given in Eq.(7). The parameter of the probability density function is given as a function of variances of high frequency components in two rectangular directions.

(2) Similarly, the magnitude of residual (low frequency components) current velocity follows the Rayleigh probability law and its density function is given in Eq.(8).

(3) The probability density function of the total current velocity (the vector sum of tidal and residual current velocities) is analytically derived as given in Eq.(22).

(4) The theoretical probability density functions are compared with histograms constructed from measured data and good agreement between them is obtained.

(5) A method for estimating the extreme value of the total current velocity expected in a specified time period is developed from knowledge of the variances of high and low frequency components in two rectangular directions.

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REFERENCES

1. Ochi, M.K. and McMillen, R.I.,"Stochastic Analysis of Offshore Currents", Proc. 21st Int. Conf. on Coastal Eng., Vol.3, 1988, pp.2536-2546.