CHAPTER 132

DETERMINATION OF COEFFICIENTS IN MORISON FORMULA BY A KALMAN FILTER

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1 INTRODUCTION

In engineering practice semi-empirical formulae are used for the determination of hydrodynamical loads on vertical cylinders due to water waves. The most popular is the Morison formula

$$\frac{dP}{dz} = C_M \rho \frac{\pi d^2}{4} \frac{\partial u}{\partial t} + C_D \frac{\rho d}{2} u|u| \tag{1}$$

where dP is the resultant horizontal force, ρ – the density of the fluid, d – diameter of the cylinder, u – the horizontal component of the orbital velocity in wave motion calculeted at the axis of the cylinder. The drag coefficient C_D and inertia coefficient C_M for uniform oscillating flow were determined experimentaly by Sarpkaya (1975). The value of these coefficients depend upon the Reynolds and Keulegan – Carpentier numbers. The experiments for the determination of the hydrodynamical forces due to water waves were performed by Chakrabarti (1983), Cotter (1984), Bearman (1985), Sparboom (1987). There are not many data available for this case of loading. A few procedures are established for the determination of the drag and inertia coefficients from the measurements. Some authors claim that the coefficients should be variable in the period of the wave to obtain a good aproximation of the real behavior.

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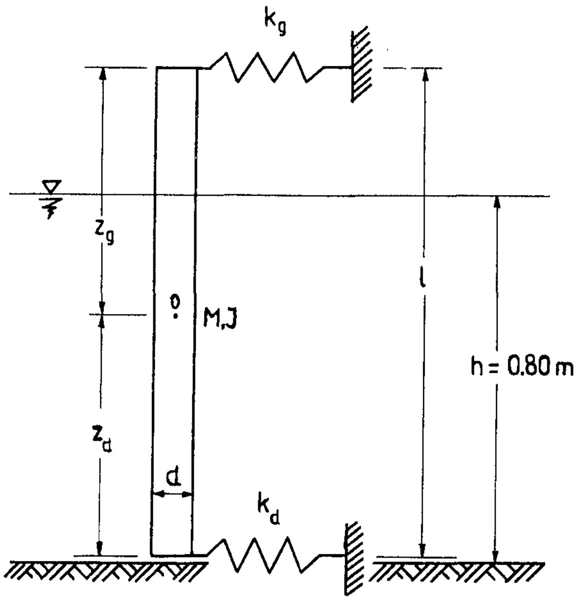

Figure 1: Scheme of dynamical system in one plane.

In the wave flume of the Institute of Hydroengineering in Gdańsk experiments were performed with a vertical rigid cylinder of diameters $d = 76.6$ $[mm]$ and $d = 46.7$ $[mm]$ supported by elastic springs at the bottom and top in the inline and transverse directions (four degrees of freedom). The investigated dynamical system is show in Figure 1. The water depth was 0.80 $[m]$, and the wave heights $H = 12, 20, 25$ $[cm]$ were investigated. The displacements at the positions of the springs were measured. Details are given by Wilde and Sobierajski (in print).

A Kalman filter of the type as presented by Wilde and Kozakiewicz (1984) and by Wilde and Romańczyk (1989) were used to decompose the measurements in components with frequencies equal to the multiple of the wave dominant frequency.

2 THE OUTLINE OF THE PROCEDURE

Let us point to some details of the proposed methods of estimation of the C_M and C_D coefficients in the Morison formula. It is necessary to construct a mathematical model for the decomposition of the elevation of the free surface. Let us define two independent stationary random functions $A_1(t)$, $D_1(t)$ defined by the following ITÔ stochastic differential equation

$$d\begin{bmatrix} A_0(t) \\ A_1(t) \end{bmatrix} + \eta \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} A_0(t) \\ A_1(t) \end{bmatrix} = 2\sqrt{\eta P} \begin{bmatrix} 1 \\ 0 \end{bmatrix} dB(u) \tag{2}$$

where η is a parameter with dimension $[s^{-1}]$, P is the asymptotic value of the variance and $dB(u)$ is a Brownian motion process with the variance equal to one. Two random funtions $X(t)$, $Y(t)$ are defined by the matrix equation

$$\begin{bmatrix} X(t) \\ Y(t) \end{bmatrix} = \begin{bmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{bmatrix} \begin{bmatrix} A_1(t) \\ D_1(t) \end{bmatrix} \tag{3}$$

where ω is the dominant angular frequency. The mean values of the processes $X(t)$ and $Y(t)$ is equal to zero and the covariance functions are for the stationary case are

$$c_{xx}(\tau) = c_{yy}(\tau) = Pe^{-\eta\tau}[1 + \eta\tau]\cos\omega\tau \tag{4}$$

where $\tau = \mid t_2 - t_1 \mid$.

Let us define a complex signal

$$Z(t) = X(t) + iY(t) = W(t)e^{-i[\omega t - \psi(t)]} \tag{5}$$

where

$$W(t) = \sqrt{X^2(t) + Y^2(t)} = \sqrt{A_1^2(t) + D_1^2(t)} \tag{6}$$

is the random amplitude and $\psi(t)$ is the random phase shift. Let us assume that the process $X(t)$ has a narrow - band spectral density, such that the amplitude $W(t)$ and phase shift change very little in the wave period.

Two finite sequences of stochastic processes $X_n(t)$, $Y_n(t)$ for $n = 1, 2, \ldots$ are defined by the formula

$$X_n(t) + iY_n(t) = Z^n(t) = W^n(t)e^{in[\omega t - \psi(t)]} \tag{7}$$

which have dominant frequencies equal to $n\omega$. It follows from eguations (6) and (7) that the function $W^n(t)$ are envelops for both functions $X^n(t)$ and $Y^n(t)$. If the expression in the square brackets is equal to $r\pi$, where r is an integer then $Y_n(t)$ and $X_n(t)$ touches the envelope for all n. If in the mathematical model (2) the parameter η is equal to zero then the solutions correspond to constant in time functions and the amplitudes and phase shifts are random variables and the real parts correspond to cos and imaginary parts to sin functions. In the stochastic case when η is not equal to zero but is small copared to ω there is a similar behaviour for the sample functions.

From eguations (5) and (7 it follows that $X_n(t)$ is the real and $Y_n(t)$ the imaginary part of $Z(t)$ to the power n and thus it is possible to express them as polynomials in $X(t)$ and $Y(t)$ of order n. For example

$$\begin{aligned} X_2(t) &= X^2(t) - Y^2(t) \\ Y_2(t) &= 2X(t)Y(t). \end{aligned}$$

The inverse problem is to express $X^r(t)$ and $Y^r(t)$ as a series in terms of $X_p(t)$ and $Y_s(t)$ where p and s are smaller or equal to r. It may be easily seen that such a representation is possible with coefficients which depend upon $W^q(t)$ where $q+p=r$ and $q+s=r$. For example

$$\begin{aligned} X^2(t) &= 0.5W^2(t) + 0.5X_2(t) \\ X^4(t) &= 0.375W^4(t) + 0.5W^2(t)X_2(t) + 0.125X_4(t). \end{aligned}$$

In general any polynomial function in $X_r(t)Y_q(t)$ may be expressed as a linear combination of terms of the type $X_p(t)$ and $Y_s(t)$ with coefficients which depend upon $W(t)$.

The surface elevation for the second order Stokes' wave $\zeta(t)$ is described by the formula

$$\zeta(t) = \frac{H}{2}\{cos(kx - \omega t) + \frac{2\gamma}{h}\cos[2(kx - \omega t)]\} \tag{8}$$

where

$$\gamma = \frac{kh}{2}\coth(kh)\left[1 + \frac{3}{2sinh^2(kh)}\right],$$

H is the wave height, h the water depth, ω the angular frequency and k is the wave number. When the wave height and phase shift change very little in the wave period for a fixed position in space $x = 0$, the relation (8) may be generalized for random waves by assuming

$$\zeta = X(t) + \frac{\gamma}{h}X_2(t) \tag{9}$$

where the wave number k is assumed constant and calculated from the dispersion relation for the peak frequency. When the process has very narrow- band spectral density the relations for the regular waves may be used as a first approximation to find the horizontal components $u(t)$ of the velocity vectors and the accelerations. For example the accelerations are

$$\frac{\partial u}{\partial t} = \omega^2\frac{\cosh[k(h+z)]}{\sinh(kh)}Y(t) + \frac{3}{2}\omega^2 k\frac{\cosh[2k(h+z)]}{\sinh^4(kh)}Y_2(t) \tag{10}$$

where z is the vertical coordinate measured from the still water level.

According to the Morison formula it is necessary to calculate the values of the function $\mid u \mid u$. It is desireable to express this random function as a sum of random functions with dominant frequencies equal to $n\omega$ and coefficients wich are varying slowly in time. Thus we want to approximate a random function $g(X,Y)$ by the following series

$$g(X,Y) = a_0(W) + \sum_{p=1}^{n} a_p(W)X_p + \sum_{r=1}^{n} b_r(W)Y_r \tag{11}$$

in the sens that the squared difference J between the left and right side in (11) is minimum. From (7) it follows

$$\begin{aligned} X_p(t) &= W^p(t)\cos[p(\omega t + \psi)] \\ Y_p(t) &= W^p(t)\sin[p(\omega t + \psi)]. \end{aligned} \tag{12}$$

When these functions are substituted it follows that the function J is a function of the random amplitude $W(t)$ and random phase shift $\psi(t)$, and for a fixed time t is a random variable. It is easy to see that when $A_1(t)$ and $D_1(t)$ are independent random variables with Gaussian distributions then $W(t)$ has a Rayleigh distribution and $\psi(t)$ has a uniform distribution in any interval of lenght 2π and that they are statistically independent. We may substitute

$$U(t) = -\omega t + \psi(t) \tag{13}$$

and for a fixed t the random variable U has a uniform distribution in the interval $-\pi$, π.Thus as a criterion of approximation for a fixed t, in the random formulation we may take that the expected value with respect to the phase

$$I = \int_{-\pi}^{\pi} J(W,u)\frac{1}{2\pi}du \tag{14}$$

is minimized , where the integral is a Rieman integral and $W(t)$ is a fixed value and u is dummy variable.

Similar as in the case of Fourier series, when the necessary condition and the orthogonality properties of the trigonometric functions are used the following expressions for the coefficients are obtained

$$\begin{aligned} a_0(W) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(W,u)du \\ a_s(W) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(W,u)\cos(su)du \\ b_s(W) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(W,u)\sin(su)du. \end{aligned} \tag{15}$$

When these formulas are used for the polynomial functions the same expressions result as before by a direct calculation. For the case $\mid X \mid X$ it follows

$$\mid X \mid X \simeq \frac{8}{3\pi}WX + \frac{8}{15\pi}W^{-1}X_3 - \frac{8}{105\pi}W^{-3}X_5 + \ldots + \tag{16}$$

In Figure 2 a realization of a random function $\mid X \mid X$ calculated by a computer is shown and an aproximation by two terms. It may be seen that the difference are very small.

With the same approximations as in the expression for the acceleration (10), the horizontal components of the orbital velocities are given by the formula

$$u = \omega\frac{\cosh[k(h+z)]}{\sinh(kh)}X(t) + \frac{3}{4}\omega k\frac{\cosh[2k(h+z)]}{\sinh^4(kh)}X_2(t). \tag{17}$$

Thus the function $\mid u \mid u$ depends upon the values of $X(t)$ and $X_2(t)$ and when the integrals according to the expressions (16) are calculated the interval of integration has to be split into parts (absolute value) which depend upon the values of W and the values of z, where the velocity is determined. The coefficients may be determined by numerical methods and the final result is that the expansion in terms is useful and only a few terms are sufficient to obtain a good approximation.

To obtain the resultant force and moment the loads given by the Morison formula (1) have to be integrated along the pile from the bottom to the free surface which changes in time. This is an approximate procedure because the real velocity field in the neighbourhood of the free surface is not very well described by the Stockes' second order approximation. It should be however mentioned that such a procedure is used in an engineering approach and the differences in resultant forces and moments are not great. Due to the integration to the time–dependent free–surface, terms with the double frequency are generated even when the second term

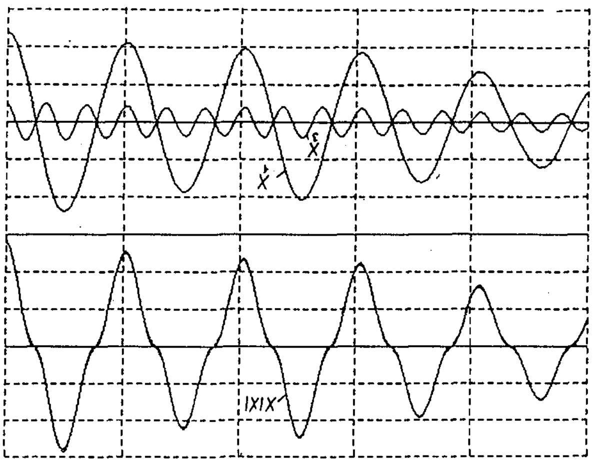

Figure 2: Example of $\mid X \mid X$ process and it's decomposition

in the Stockes'approximation is very small and the position of the free surface is well described by linear theory of waves.

To avoid complicated algebraic formulae the integrals corresponding to the resultant forces and moments were not expanded in series with the help of the relations (16) but a numerical procedure was applied. From the measured surface elevations by a Kalman filter the functions $X(t)$ and $Y(t)$ with the dominant frequency equal to the peak frequency of the wave were determined. By the relations (10) and (17) the accelerations and velocities as function of z for the times of measurement were calculated and then according to the Morison formula seperately for the inertia forces and drag forces. These functions of z were integrated from the bottom of the pile to the free surface to obtain the following relations

$$\begin{bmatrix} R(k) \\ M(k) \end{bmatrix} = \begin{bmatrix} R^M(k) & R^D(k) \\ M^M(k) & M^D(k) \end{bmatrix} \begin{bmatrix} C_M \\ C_D \end{bmatrix} \tag{18}$$

where $R^M(k)$ is the resultant force due to the term wiht $C_M = 1$, and $R^D(k)$ due to the drag term with $C_D = 1$, $M^M(k)$, $M^D(k)$ denotes moments due to the inertia and drag terms with $C_M = 1$, $C_D = 1$ and k denotes the time step t_k.

When the resultant forces and moments are know the displacements as functions of time may be calculated. In the plane of propagation of water waves the dynamical system has two degrees of freedom. The position of the rigid cylinder is determined by the measured horizontal displacements $x_g(t)$ and $x_d(t)$ at the levels of upper and lower spring respectively. The damping is very small and the natural frequencies are high compared to the peak wave frequency. In the determination of displacements it is possible to assume the equation for undamped vibrations

$$\begin{bmatrix} M & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} \ddot{x}(t) \\ \ddot{\varphi}(t) \end{bmatrix} + \begin{bmatrix} k_g + k_d & k_g z_g - k_d z_d \\ k_g z_g - k_g z_d & k_g z_g^2 + k_d z_d^2 \end{bmatrix} = \begin{bmatrix} R(t) \\ M(t) \end{bmatrix} \tag{19}$$

where $x(t)$ is the horizontal displacements of the center of mass, $\varphi(t)$ is the angle of rotation around the center of mass, k_g, k_d are the spring constants of the upper and lower springs, z_g, z_d are the distances from the center of mass to the upper and lower springs, $R(t)$, $M(t)$ are the resultant force and moment of the hydrodynamic forces with respect to the center of mass. The position of center of mass and the mass matrix were calculated with added mass of water and the spring constants were obtained from static loads. The dynamic equation was verified by mesuring free vibrations in still water.

The measured displacements $x_g(t)$, $x_d(t)$ are related to the displacements $x(t)$, $\varphi(t)$ by the equation

$$\begin{bmatrix} x_g(t) \\ x_d(t) \end{bmatrix} = \begin{bmatrix} 1 & z_g \\ 1 & -z_d \end{bmatrix} \begin{bmatrix} x(t) \\ \varphi(t) \end{bmatrix}. \tag{20}$$

When the equation of undamped vibrations (19) is used the measured vibrations can not be determined by the solution of this equation because the initial conditions will never be forgotten. The steady - state solution for harmonic vibrations will be used. The hydrodynamic loads (18) are decomposed into components with dominant frequencies corresponding to the multiples of the wave peak frequency and for each component the steady - state harmonic response is calculated. Thus, after simple manipulations the observations model for the time t_k assume the following form

$$\begin{bmatrix} x_g(k) \\ x_d(k) \end{bmatrix} = \begin{bmatrix} H_g^M(k) & H_g^D(k) \\ H_d^M(k) & H_d^D(k) \end{bmatrix} \begin{bmatrix} C_M \\ C_D \end{bmatrix} + \sigma_0 \begin{bmatrix} v_g(k) \\ v_d(k) \end{bmatrix} \tag{21}$$

where $H_g^M(k)$ and $H_d^M(k)$ are the calculated displacements due to the inertia term in the Morison formula for $C_M = 1$, and $H_g^D(k)$, $H_d^D(k)$ due to the drag term for $C_D = 1$, $v_g(k)$, $v_d(k)$ are independent sequences of gaussian white noise with distribution $N(0,1)$ and σ_0 is the standard deviation of the observation noise.

In the mathematical models for the coefficients C_M and C_D it is assumed that they are statastically independent have constant mean values and fluctuations which are described by the $IT\hat{O}$ stochastic differential equation (2).

The standard Kalman filter procedure is used to estimate the values $C_M(k)$, $C_D(k)$ on the basis of the measured values $x_g(k)$, $x_d(k)$. The procedure was written in the form of an IBM PC computer program.

3 THE EXPERIMENTAL RESULTS

The water depth was 0.80 $[m]$. Different waves were considered. In Figure 3 the the results for a wave with a small wave height are shown. In this case the wave period was $T = 1.4$ $[s]$ the amplitude of the wave was $\frac{H}{2} = 0.0596$ $[m]$, the horizontal component of the velocity at the still water level calculated on basis of Stockes' theory was $u_{max} = 0.2915$ $[ms^{-1}]$, the corresponding KC number 5.327 and the Reynolds number 1.96×10^4. According to the data shown in Figure 3 the second term in the Stockes'approximation is negligible. The measured displacements are

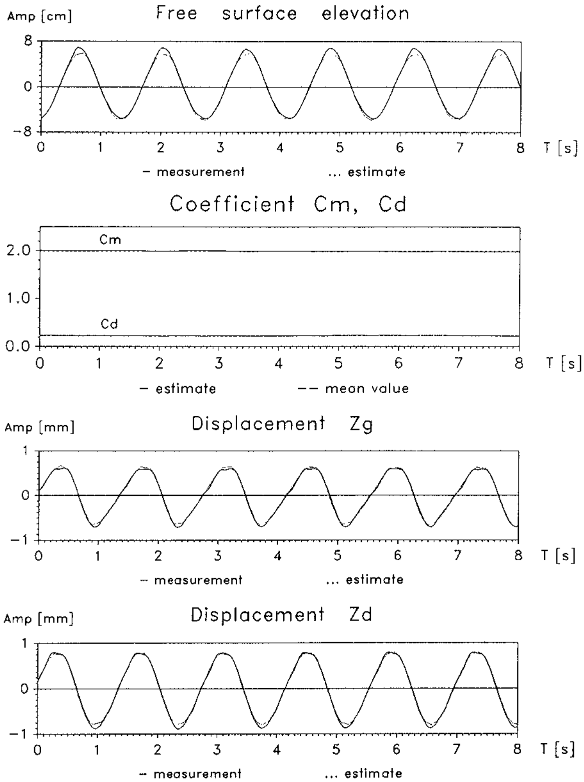

Figure 3: The measured and estimated wave and displacements and the estimated C_M and C_D coefficients for test db1.dat.

very close to trigonometric functions. The estimated coefficients C_M, C_D are very close to constant functions. The average values are $C_M = 1.99$ $C_D = 0.23$. It may be seen that the fluctuations are very small and not regular. Thus the assumption of constant values for the coefficients is justified.

In Figure 4 the data for a steaper wave are shown. In this case the wave period was $T = 1.9$ $[s]$, the amplitude $\frac{H}{2} = 0.113$ $[m]$, the horizontal velocity $u_{max} = 0.527$ $[ms^{-1}]$, the KC number 13.08 the Reynolds number 3.54×10^4. The wave is a second order Stockes' wave. The measured displacements are not symmetric and the influence of the double frequency of the peak wave is large. It may be seen that also in this case the estimted coefficients C_M, C_D are very close to constant functions. The average values are $C_M = 1.60$, $C_D = 0.70$. It should be stressed however that in this case there are differences between the measured values and the values calculated on the basis of the estimated coefficints. It may be seen from the graph that there are higher harmonics which are neglected in the analysis. The Morison formula introduces approximations, the determination of the velocity and acceleration fields from the Stokes' second order approximation does not correspond exactaly to the real behaviour. The description of the dynamical system is simplified because damping is neglected and steady – state harmonic solutions are assumed. It is surprising that with such rude approximation the constant coefficients C_M, C_D may be so adjusted that the calculated solution gives the the basic features of the real behaviour.

The results of all experiments are presented in Figure 5. It should be mentioned that in all experiments the Reynold numbers are in the range of 10^4 and thus only the relationship wiht respect ot the Keulegan – Carpenter number KC may be shown. If one compares the results with previos investigations they are within the range of the published values. It should be noted that the coefficients C_D are smaller than the values uselly used for $Re < 10^5$. As a crude recomendation the results shows that for $KC \leq 5$ $C_M = 2$ and it drops to $C_M = 1.5$ for $KC \sim 20$, the values for C_D are $C_D = 0.25$ for $KC = 5$ and they grow up to $C_D = 0.8$ for $KC = 15$ (there are points $C_D = 1.0$). One can not expect to obtain exact values for the considered dynamical system because measurements have a small range in Reynolds numbers.

4 CONCLUSIONS

1. The investigations shown that the Kalman filter procedure is a suitable tool for the analysis of the problem.
2. In the mathematical models it is enough to assume once differentiable random functions described by the $IT\hat{O}$ stochastic differential equation (2).
3. The nonlinear term due to the drag force in Morison formula may be included in the observation model which becomes nonstationary.
4. The simpifying assumptions introduced in the analysis of the dynamical system are justified for practical applications.
5. The results of experiments shown that constant values for the C_M and C_D coefficients may be taken.
6. The procedure may be refined by considering the influence of damping but it looks like it is not possible to expect much better results and the scattering of results must be considerable in view of the complicated physics of the problem.

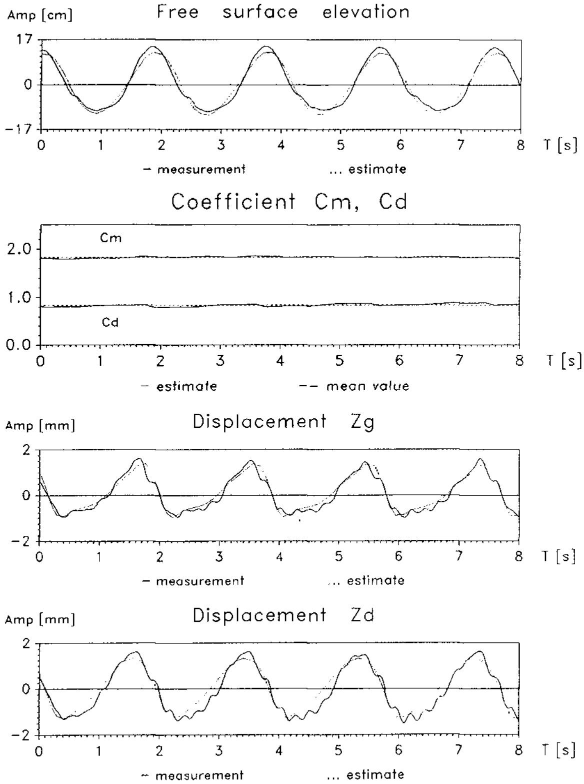

Figure 4: The measured and estimated wave and displacements and the estimated C_M and C_D coefficients for test db5.dat.

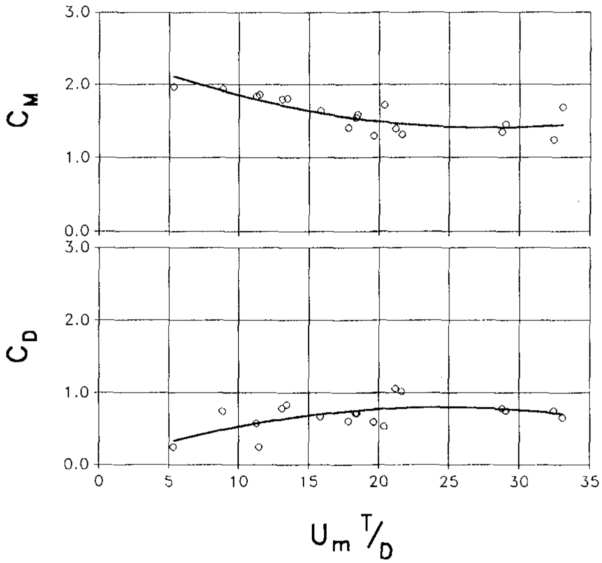

Figure 5: The estimations of the C_M and C_D coefficients

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