# CHAPTER 244

## DISPERSION OF HEAVY SUSPENSIONS IN WAVE BOUNDARY LAYERS

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### 1. Introduction

The transport of suspended particles is of interest to the coastal engineer because it affects the distribution of sediments and the evolution of shorelines. It is also important to pollutant disposal in the sea, and to the transport of nutrients needed for sustaining marine life. Planktonic larvae depend on waves and currents to carry them from offshore to the rock stratum for reproduction and growth. External fertilization of sperms and eggs of marine organisms are further affected by the convection and diffusion processes in the sea.

Taylor's pioneer work in a steady flow through a tube has shown that diffusion, whether molecular or turbulent, is greatly enhanced by transverse shear in the flow. Since in the sea the bottom boundary layer is the zone where shear is the strongest, dispersion must be the most prominent there. For wave-induced boundary layers only the flow field has been studied extensively. For example Stokes laminar boundary layer theory has been extended to turbulent boundary layers by Kajiura(1968), Jonnson & Carlsen(1976), Grant & Madsen(1979) and others. The mean circulation induced in the oscillating boundary layer by Reynolds stresses has been studied under pseudo-laminar model by Longuet-Higgins(1958), Hunt & Johns(1963), Carter, Liu & Mei(1973) etc. Trowbridge & Madsen(1984a,b) further extended it to turbulent flows. So far there has been little theoretical work for predicting dispersion in wave boundary layers.

What is needed is a theory for diffusion of particles in such a transient and nonuniform field of shear. In this paper we outline our recent work obtained by employing the theory of homogenization, which is a systematic technique of taking averages over the period of microscale to obtain the slow variations over the microscale. The central tool is the method of multiple scales which has been recently applied to disperison through spatially periodic porous media. The same technique is shown here to be effective for wave problems where there is periodicity in time. Specifically we consider here the dispersion of a cloud of suspension with a prescribed initial distribution in the boundary layer. Attention will be focussed on small amplitude waves without ambient current so that the flow field including the first order oscillations and second order mean can be worked out by a perturbation method, and use will be made of the fact the a the boundary layer is much thinner than the horizontal length scale so that the vertical variation can be decoupled from the horizontal problem. Starting with a convective diffusion equation for a heavy suspension for one particle size, we expand the concentration as a perturbation series in powers of the wave steepness. At the leading order the concentration is expected to vary only slowly in time and does not oscillate with the wave because of its weak velocity field. The vertical distribution is due to the balance between vertical diffusion and gravity, but the horizontal distribution is undetermined. The corresponding mathematical problem is a homogeneous boundary value problem in the vertical direction. At the next two orders the fluctuation from the mean is caused by the oscillatory velocity field and consists of both oscillating and slowly varying parts. Their time averages over a wave period are governed by two inhomogeneous boundary value problems in the vertical direction. The solvability condition which imposes constraints on the horizontal and temporal variation of the leading order concentration, leads to the the effective diffusion equation. The effective horizontal velocity of convection is shown to be a weighted average of the induced streaming velocity in the boundary layer. For the special model of constant eddy visocity, the convective velocity and dispersivity tensors are found explicitly in terms of the oscillatory velocity field outside the boundary layer. Specific intitial value problems for a prescribed initial concentration cloud will be solved for the following cases, with emphasis on the effects of the initial position of the cloud on its final destiny.

## 2. Perturbation analysis of diffusion in the boundary layer

We assume that the particles are so small that its velocity is nearly equal to the mean velocity of the local fluid, and that the volume concentration is so small that the suspension cloud does not alter the dynamics of the fluid motion.

Let $C$ denote the volume concentration, $-w_o$ the fall velocity of the suspended particles, and $D_H, D_V$ the eddy mass diffusivities. The diffusion equation for the concentration $C$ of a very dilute sediment cloud can be approximated by

$$\frac{\partial C}{\partial t} + \frac{\partial u_j C}{\partial x_j} + \frac{\partial}{\partial z}\left[(-w_o + w)\, C\right] = \frac{\partial}{\partial x_j}\left(D_H \frac{\partial C}{\partial x_j}\right) + \frac{\partial}{\partial z}\left(D_V \frac{\partial C}{\partial z}\right), \quad (2.4)$$

where $j = 1,2$ and $u_j$ represent the horizontal components of the fluid velocity and $w$ the vertical component. The crucial boundary condition on the seabed is a matter of considerable uncertainty. For example in steady unifom flows there should not be any net exchange of particles, hence

$$D_V \frac{\partial C}{\partial z} + w_o C = 0, \tag{2.5}$$

This leads to the profile of $C(z)/C(0)$ but leaves the value of $C(0)$ undetermined. For spatially nonuniform flows Sayre (1969) proposed the following empirical relation

$$D_v \frac{\partial C}{\partial z} + (1-\alpha) w_o C + W = 0,$$

where $\alpha$ is the bed absorbency coefficient representing the probability that a particle settling to the bed is deposited there, $W$ the average rate of local entrainment. Direct measurements of $\alpha$ and $W$ are obviously difficult and none is known to have been made. To simulate resuspension in wave boundary layers it has been proposed that the rate of concentration flux is an empirical function of time. Others have suggested that the bed concentration is proportional to the excess shear stress over the critical stress to initate the bed load movement. These empirical consideration are necessary practical measures when local scour is an integral part of the sediment process. However, there are situations where the sediments or contaminants are dumped from ships, or released from a sewage outfall, and not supplied locally from the seabed. Then as long as they can be kept in suspension, the no-flux condition is still appropriate. We shall confine our attention to such cases where the bed is non-erodible bed and particles so small that they are kept in suspension for almost any bed shear. It is reasonable to take $\alpha = W = 0$ so that

$$D_V \frac{\partial C}{\partial z} + w_o C = 0, \quad z = 0 \tag{2.6}$$

suffices.

At the upper edge of the boundary layer we assume

$$C = 0, \qquad\qquad z \to \infty. \tag{2.8}$$

In addition, the initial horizontal distribution of the depth averaged concentration is prescribed in some source area. Thus the physical problem is to seek the long-time diffusion of a sediment cloud from a localized source.

In oceanic flows, momentum eddy diffusivities in horizontal and vertical directions may be quite different (see *e.g.*, Pedlosky, 1979), but the value of the horizontal diffusivity is difficult to estimate reliably. In past theories on dispersion, two assumptions are common. One is the equality of momentum and mass diffusivities (Reynolds analogy, Taylor, 1953). The second is the equality of the longitudinal and transverse diffusivities (Taylor, 1953; Sayre, 1969 and Sumer, 1974). In wave boundary layers, there is the added complexity that the eddy viscosity may depend on time. However, for coastal applications, most models of eddy viscosity are time-independent (see Sleath 1992 for survey); this simplifying assumption is also adopted here.

In the present problem there are several characteristic length scales in the vertical direction. The first is the thickness of a steady concentration layer due to the balance of downward sedimentation by gravity and vertical diffusion, $d \sim D_V/w_o$. Associated with fluid oscillations there are two additional vertical length scales, i.e., the oscillatory boundary layer thicknesses $\delta \sim \sqrt{2\nu_e/\omega}$ and $\delta_D \sim \sqrt{2D_V/\omega}$, For generality, all three scales are assumed to be comparable, *i.e.*, implying that

$$Sc = \frac{\nu_e}{D} \sim \left(\frac{\delta}{\delta_D}\right)^2 = O(1), \tag{2.9}$$

where $Sc$ is the Schmidt number.

We now consider small amplitude oscillations of high enough frequency so that both the wave steepness, $kA$, and the ratio of the oscillatory boundary layer thickness to the wave length, $k\delta$, are small, *i.e.*

$$\epsilon = kA \ll 1, \tag{2.10a}$$

$$\beta = k\delta\sqrt{\frac{D_H}{D_V}} \ll 1. \tag{2.10b}$$

Let us expand the velocity field $(u_i, w)$ in powers of $\epsilon$,

$$u_i = u_i^{(1)} + u_i^{(2)} + \ldots, \quad i = 1,2; \qquad w = w^{(1)} + w^{(2)} + \ldots, \tag{2.11}$$

where the superscripts indicate the order in powers of $\epsilon$, *i.e.*, $u_i^{(n)}, w^{(n)} = O(\epsilon^n)$. Clearly there are two distinct time scales in the diffusion process, one for the vertical diffusion across the boundary layer, $O(\delta^2/D_V)$, and the other for the horizontal diffusion across a wave length, $O(1/k^2 D_H)$. The ratio between the two is $O(k^2\delta^2 D_H/D_V) = O(\beta^2)$. It is therefore natural to introduce multiple scale coordinates for time, $t$ and $t' = \beta^2 t$. Assume for generality $\epsilon = O(\beta)$ so that we may use

$$T = \epsilon^2 t \tag{2.12}$$

instead of $t'$, and expand $C$ as a perturbation series

$$C = C^{(0)} + C^{(1)} + C^{(2)} + \ldots, \tag{2.13}$$

where $C^{(0)}$ is independent of the fast time scale and $C^{(n)}, n = 1, 2, ...$ dependent on $t$ and $T$. At the leading order in $\epsilon$, $C^{(0)}$ satisfies the homogeneous ordinary differential equation

$$-w_o \frac{\partial C^{(0)}}{\partial z} = \frac{\partial}{\partial z}\left(D_V \frac{\partial C^{(0)}}{\partial z}\right), \qquad 0 < z < \infty, \tag{2.14}$$

with the homogeneous boundary conditions

$$w_o C^{(0)} + D_V \frac{\partial C^{(0)}}{\partial z} = 0, \qquad z = 0; \tag{2.15}$$

$$C^{(0)} = 0, \qquad z \to \infty. \tag{2.16}$$

The nontrivial solution is

$$C^{(0)} = \widehat{C} F, \tag{2.17a}$$

where

$$F = \exp\left(-\int \frac{w_o}{D_V} dz\right) \tag{2.17b}$$

gives the vertical structure and

$$\widehat{C} = \widehat{C}(x_i, T), \qquad i = 1, 2 \tag{2.17c}$$

gives the horizontal variation of $C^{(0)}$. As in the steady uniform flow $C^{(0)}$ is so far unknown. At $O(\epsilon)$, the period-average of $C^{(1)}(x, y, t, T)$ can be shown to be zero. Assuming the first order fluid velocity in the boundary layer to be simple harmonic, the time-harmonic part of $C^{(0)}$ is forced by various products of $u^{(1)}$ and $C^{(0)}$ and their derivatives. Therefore $C^{(1)}$ can be solved in terms of $C^{(0)}$ and its horizontal gradient. At $O(\epsilon^2)$, the period-average of $C^{(2)}$ satisfies an inhomogeous differential equation similar to (2.6). Its solution implies that the depth average of th forcing terms must vanish. This leads to

$$\frac{\partial \widehat{C}}{\partial T}\langle F\rangle + \frac{\partial}{\partial x_i}\left[\langle \bar{u}_i^{(2)} F\rangle \widehat{C}\right] = -\frac{\partial}{\partial x_i}\langle\overline{u_i^{(1)} C^{(1)}}\rangle + \frac{\partial}{\partial x_j}\langle D_H \frac{\partial}{\partial x_j}\left(\widehat{C} F\right)\rangle. \tag{2.18}$$

Once a specific choice of $D_V$ and $D_H$ is made, the right hand side of (2.18) can be found in terms of $C^{(0)}$, hence $\widehat{C}$. This then gives the effective convection-diffusion equation for $\widehat{C}$. As will be identified later, $\bar{u}_i^{(2)}$ on the left is Eulerian streaming in the boundary layer. The first term on the right represents the dispersion due to oscillatory shear in the boundary layer and the last term the horizontal turbulent diffusion.

Though the form is entirely expected, every term in (2.18) has been deduced here without *additional* closure hypotheses.

In the rest of the paper the simplifying assumption $D_H = D_V = D =$ constant will be made to enable explicit analytical results.

## 3. The Stokes effective coefficients

For constant eddy diffusivities the first order velocityu is the well known solution of Stokes the second order mean is the Eulerian streaming inferable from Hunt and Jonhns (see Mei 1983). Details of $C^{(1)}$ is described elsewhere. After considerable algebra we get the effective diffusion equation for $\widehat{C}$:

$$\frac{\partial \widehat{C}}{\partial T} + \frac{\partial}{\partial x_i}\left(\mathcal{U}_i \widehat{C}\right) = D\frac{\partial^2 \widehat{C}}{\partial x_j \partial x_j} + \frac{\partial}{\partial x_i}\left(\mathcal{D}_{ij}\frac{\partial \widehat{C}}{\partial x_j}\right), \tag{3.1}$$

where

$$\begin{aligned}\mathcal{U} &= \frac{1}{\omega}\Re\left(H_1 U_o \frac{\partial U_o^*}{\partial x} + H_2 V_o \frac{\partial U_o^*}{\partial y} + H_3 U_o \frac{\partial V_o^*}{\partial y}\right),\\ \mathcal{V} &= \frac{1}{\omega}\Re\left(H_1 V_o \frac{\partial V_o^*}{\partial y} + H_2 U_o \frac{\partial V_o^*}{\partial x} + H_3 V_o \frac{\partial U_o^*}{\partial x}\right),\end{aligned} \tag{3.2}$$

$$\begin{aligned}\mathcal{D}_{xx} &= \Re\left[\frac{H_4}{\omega}|U_o|^2\right], \qquad \mathcal{D}_{xy} = \Re\left[\frac{H_4}{\omega}(U_o^* V_o)\right],\\ \mathcal{D}_{yx} &= \Re\left[\frac{H_4}{\omega}(U_o V_o^*)\right], \qquad \mathcal{D}_{yy} = \Re\left[\frac{H_4}{\omega}|V_o|^2\right].\end{aligned} \tag{3.3}$$

The coefficients $H_1, H_2, H_3$ and $H_4$ are functions of $M$ and $Sc$.

Equation (3.1) may be normalized by defining $\bar{U}_o$ to be the maximum of the first order velocity $(U_o^2 + V_o^2)^{1/2}$ and letting

$$U'_{o_i} = \frac{U_{o_i}}{\bar{U}_o}, \qquad x'_i = kx_i, \qquad t' = \frac{k^2 \bar{U}_o^2}{\omega}T, \tag{3.4}$$

where primes denote dimensionless quantities. Then

$$\frac{\partial \widehat{C}}{\partial t'} + \frac{\partial}{\partial x'_i}\left(\mathcal{U}'_i \widehat{C}\right) = \frac{\partial}{\partial x'_i}\left(K'_{ij}\frac{\partial \widehat{C}}{\partial x'_j}\right), \tag{3.5}$$

where

$$\begin{aligned}\mathcal{U}' &= \Re\left(H_1 U'_o \frac{\partial U_o'^*}{\partial x'} + H_2 V'_o \frac{\partial U_o'^*}{\partial y'} + H_3 U'_o \frac{\partial V_o'^*}{\partial y'}\right),\\ \mathcal{V}' &= \Re\left(H_1 V'_o \frac{\partial V_o'^*}{\partial y'} + H_2 U'_o \frac{\partial V_o'^*}{\partial x'} + H_3 V'_o \frac{\partial U_o'^*}{\partial x}\right),\\ K'_{xx} &= \mathcal{D}'_{xx} + D' = \Re(H_4)|U'_o|^2 + D',\\ K'_{xy} &= \mathcal{D}'_{xy} = \Re\left(H_4 U_o'^* V'_o\right),\\ K'_{yx} &= \mathcal{D}'_{yx} = \Re\left(H_4 U'_o V_o'^*\right),\\ K'_{yy} &= \mathcal{D}'_{yy} + D' = \Re(H_4)|V'_o|^2 + D',\\ D' &= D/\left(\bar{U}_o^2/\omega\right),\end{aligned}$$

in which $K'_{ij}$ are components of the total dispersivity tensor.

In steady flows in a channel or a tube the longitudinal dispersion coefficient is usually much greater than diffusivity (molecular or eddy). It can be shown that the shear-enhanced dispersion can be much greater than the eddy diffusivity if the ambient oscillation is sufficiently intense or the frequency sufficiently low. Otherwise the two can be comparable.

We now examine the spatial variations of the dispersion phenomenon by specific examples. For brevity, primes on dimensionless quantities will be dropped for convenience in later discussions. The values of $H_1, H_2, H_3, H_4$ are for $M = Sc = 1$.

## 4. Two-dimensional diffusion of a localized cloud in bidirectional waves

We consider a system of short-crested waves whose inviscid velocity amplitudes near the seabed are described in dimensional variables by

$$\begin{aligned} U_o &= 2i\bar{U}_o \cos\theta \sin(kx\cos\theta) e^{iky\sin\theta}, \\ V_o &= 2\bar{U}_o \sin\theta \cos(kx\cos\theta) e^{iky\sin\theta} \end{aligned}. \tag{4.1}$$

This corresponds to a plane wave incident towards and reflected by a sea wall along the $y$ axis. The angle of incidence is $\theta$. In normalized variables (with primes omitted), we have

$$\begin{aligned} U_o &= 2i \cos\theta \sin(x\cos\theta) e^{iy\sin\theta}, \\ V_o &= 2\sin\theta \cos(x\cos\theta) e^{iy\sin\theta} \end{aligned}. \tag{4.2}$$

It is easy to calculate the following dimensionless coefficients,

$$\begin{aligned} \mathcal{U} &= \left[2\Re(H_1)\cos^3\theta + (\Re(H_3) - \Re(H_2))\sin\theta\sin 2\theta\right]\sin(2x\cos\theta), \\ \mathcal{V} &= \left[2\Im(H_1)\sin^3\theta + \Im(H_3)\cos\theta\sin 2\theta\right] 2\cos^2(x\cos\theta), \\ K_{xx} &= 4\Re(H_4)\cos^2\theta\sin^2(x\cos\theta) + D, \\ K_{yy} &= 4\Re(H_4)\sin^2\theta\cos^2(x\cos\theta) + D, \\ K_{xy} &= -K_{yx} = \Im(H_4)\sin 2\theta\sin(2x\cos\theta). \end{aligned} \tag{4.3}$$

Note that the tensor $\{K_{ij}\}$ is not symmetric.

Let us examine the diffusion due to the impulsive release from a localized Gaussian cloud

$$\widehat{C}(x,y,0) = C_o \exp\left\{-\left[(x-x_c)^2 + (y-y_c)^2\right] \Big/ L^2\right\}. \tag{4.4}$$

The assumption of $Sc = M = 1$ is still kept, for which $\Re(H_1) = -0.122$ $\Im(H_1) = \Im(H_3) = 0.659$, $\Re(H_2) = 0.033$, $\Re(H_3) = -0.155$, $\Re(H_4) = 0.024$ and $\Im(H_4) = 0.234$. Choosing $D = 10^{-3}$ and $L^2 = 0.1$, the initial value problem is solved numerically by a Peaceman-Rachford ADI (alternating-direction) finite difference method. Three angles of incidence and various positions of the initial source center have been considered.

As a sample we consider oblique incidence $\theta = \pi/4$, then

$$\begin{aligned} \mathcal{U} &= \frac{1}{\sqrt{2}}\Re(H_1 + H_3 - H_2)\sin\sqrt{2}x = -0.219\sin\sqrt{2}x, \\ \mathcal{V} &= \sqrt{2}\Im(H_1 + H_3)\cos^2\frac{x}{\sqrt{2}} = 1.864\cos^2\frac{x}{\sqrt{2}}, \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} K_{xx} &= 2\Re(H_4)\sin^2\frac{x}{\sqrt{2}} + D = 0.048\sin^2\frac{x}{\sqrt{2}} + D, \\ K_{yy} &= 2\Re(H_4)\cos^2\frac{x}{\sqrt{2}} + D = 0.048\cos^2\frac{x}{\sqrt{2}} + D, \\ K_{xy} &= -K_{yx} = \Im(H_4)\sin(\sqrt{2}x) = 0.234\sin(\sqrt{2}x). \end{aligned} \tag{4.6}$$

Thus for $x_c = 0$, $\mathcal{V}$ is dominant; the cloud is convected along the line $x = 0$ and diffused faster in the $y$ than in $x$ as shown in figure 1. For $x_c\cos\theta = x_c/\sqrt{2} = \pi/2, K_{xx} > K_{yy}$, the nonuniform convection velocity causes the cloud to bifurcate towards the lines $x_c/\sqrt{2} = 0, \pi$ where $K_{yy} > K_{xx}$, as in figure 2. For $x_c\cos\theta = \pi/4$, most of the pulse is convected to the left, with the front leading the rest in the shape of an eel, as shown in figure 3.

We have also considered a sustained source and solved the inhomogeneous diffusion equation

$$\frac{\partial\widehat{C}}{\partial t} + \frac{\partial}{\partial x_i}\left(\mathcal{U}_i\widehat{C}\right) = \frac{\partial}{\partial x_j}\left(K_{ij}\frac{\partial\widehat{C}}{\partial x_j}\right) + S\left(x_i\right), \quad t > 0. \tag{4.7}$$

The source function $S$ is chosen to be a cosine-shaped distribution centered around $(x_c, y_c)$ and maintained at a steady rate after $t = 0$. A plume is formed with the front evolving in the same way as the impulsively released cloud, but the plume is always connected back to the steady source. We shall, therefore, omit the results here.

## 5. Concluding remarks

In this paper we have given general formulas for convections and diffusion in a wave boundary layer. As an example we have examined a localized cloud in a bidirectional wave system which may represent an obliquely incident and reflected wave system near a sea wall. The cloud drifts to the nodal line close to its initial position. When the particle cloud is initially midway between two nodal lines along the $y$ axis. The cloud then bifurcates towards the two nodal lines on each side while the peaks diminish.

Future improvments must include better models of turbulence, deposition and resuspension. For fine cohesive sediments possible coagulation is worth studying. Finally it is interesting to examine dispersion in the surf zone where breaking waves induce longshore currents.

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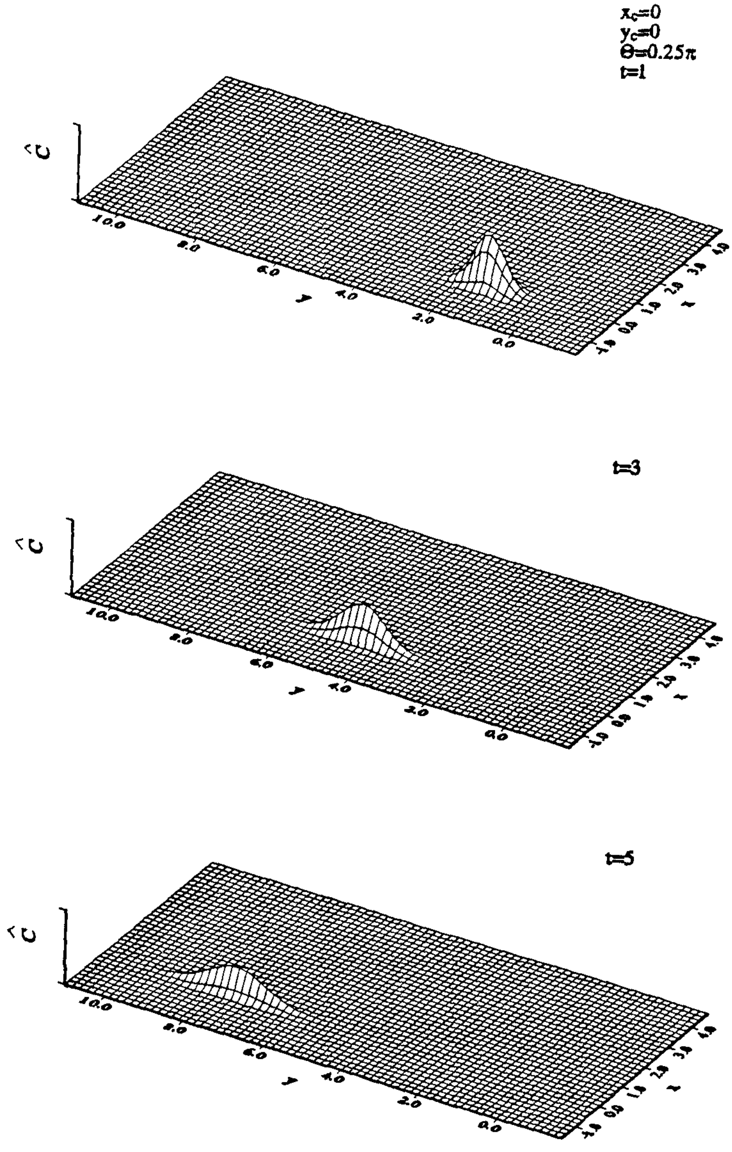


**Figure 1:** Evolution of a concentration released in a bidirectional wave. Initial position at $x_c = 0$.

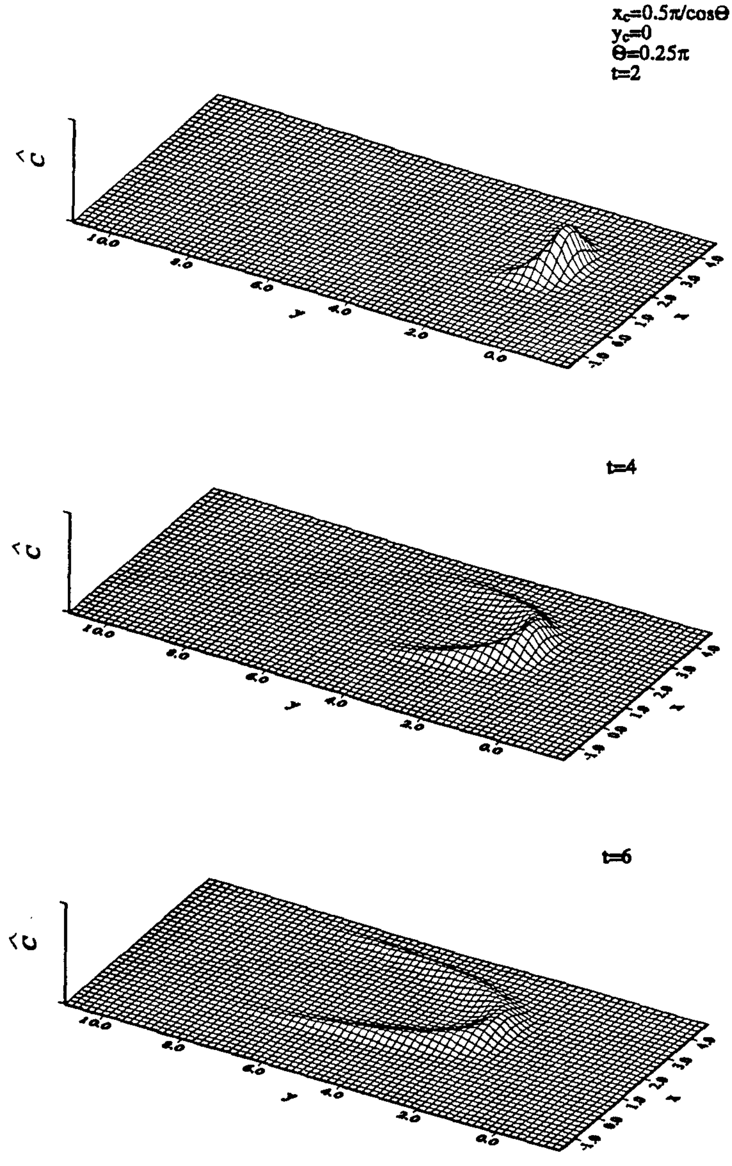


**Figure 2:** Evolution of a concentration released in a bidirectional wave. Initial position at $x_c/\sqrt{2} = \pi/2$.

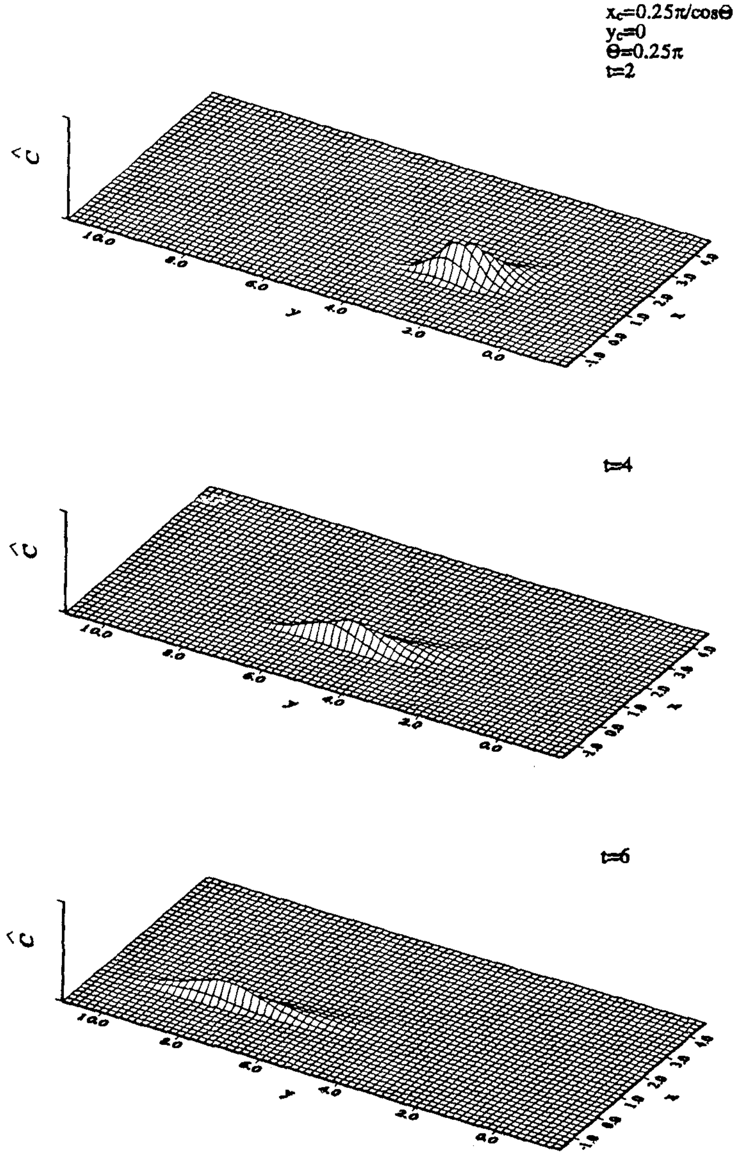


**Figure 3:** Evolution of a concentration released in a bidirectional wave. Initial position at $x_c/\sqrt{2} = \pi/4$.