A NUMERICAL MODEL OF BEACH CHANGE DUE TO SHEET-FLOW

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ABSTRACT

First a discussion is made on the relation between the time-average of a Boussinesq-type equation and the mean flow equation. Then nonlinear wave transformation on a sloping bed is computed by a set of Boussinesq equations including a breaker-induced energy dissipation term. Comparisons are made between the computations and laboratory measurements for cross-shore distributions of the wave height and mean water elevation, and for time-histories of the near-bottom velocity near and after breaking. Undertow current velocity in the nearshore zone is calculated by a semi-empirical formula and is also compared with measurement data. A beach profile change model is set up by combining the Boussinesq-type equations, a sediment transport rate formula for the sheet-flow proposed by Dibajnia and Watanabe, which incorporates the asymmetric orbital velocity due to wave nonlinearity as well as the undertow current, and a sediment mass conservation equation proposed by Watanabe *et al.*, which includes the effect of local bottom slope. The validity of the model is examined through the comparisons of the computed transport rate distributions and beach profiles with the laboratory data obtained in large wave flume experiments.

INTRODUCTION

Prediction of beach processes is generally inevitable to develop or protect coastal zones properly. Short-term beach topography changes have usually been predicted in Japan by the numerical model proposed by Watanabe *et al.* (1986) or its improved versions. In this model, however, since the wave computation is performed with linear time-dependent mild slope equations involving breaker-induced energy dissipation, the calculations of the nearshore currents and the mean water elevation should be carried out only through the evaluation of radiation stresses, and hence the wave-current interaction can be treated by the iteration of the computations of waves and of currents. In addition, the effect of

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wave nonlinearity on the sediment transport rate has been incorporated on the basis of the simplified semi-empirical formulas.

Dibajnia *et al.* (1992) have performed the prediction of beach profile change applying the formula by Dibajnia and Watanabe (1992) for the sheet-flow transport rate under nonlinear waves. However, since they have employed the linear wave model for the wave computation, a sort of inconsistency has been existing from a theoretical viewpoint.

Prediction of beach profile change under sheet-flow condition is important, because significant short-term beach deformation is caused by large waves, under which the sheet-flow sediment transport is usually predominant. In addition, under such conditions, one cannot discard the asymmetry of near-bottom orbital motion and the presence of undertow particularly in the surf zone. This paper presents a numerical model of beach profile change based on a sheet-flow transport rate formula mentioned above and on Boussinesq-type wave equations, incorporating the effects of asymmetric velocity and the undertow. Comparisons will be made between the numerical computations and measurements.

BOUSSINESQ EQUATION AND NEARSHORE CURRENT EQUATION

Boussinesq-type equations include nonlinear terms to the order of $O(a^2)$, where a is the wave amplitude. Since nearshore currents (wave-induced mean flow) and wave setup/down are the phenomena of the order of $O(a^2)$, they will be evaluated with a Boussinesq-type equation. In this section, we will show that the time-average of the Boussinesq equation that involves vertical velocity distributions becomes equivalent to the governing equation for the nearshore currents, and that thus the concurrent computation of waves and currents becomes possible.

Abbott (1979) has presented the Boussinesq equation involving vertical velocity distributions as follows:

$$\frac{\partial \rho u_\alpha}{\partial t} + \frac{\partial}{\partial x_\beta}(\rho u_\alpha u_\beta) + \frac{\partial}{\partial z}(\rho w u_\alpha) + \rho g \frac{\partial \eta}{\partial x_\alpha} = -\rho \frac{\partial^3 \eta}{\partial x_\alpha \partial t^2} \left\{ \frac{(h+\eta)^2 - (h+z)^2}{2(h+\eta)} \right\} \tag{1}$$

where h is the still water depth, η the instantaneous surface displacement, u the horizontal velocity, w the vertical velocity, t the time, and x and z are the horizontal and vertical coordinates, respectively. The subscripts α and β take a value of 1 or 2, and the summation convention is adopted.

Boundary conditions on the free surface and the seabed are as follows:

$$\frac{\partial \eta}{\partial t} - w + u_\alpha \frac{\partial \eta}{\partial x_\alpha} = 0 \quad (z = \eta) \tag{2}$$

$$w + u_\alpha \frac{\partial h}{\partial x_\alpha} = 0 \quad (z = -h) \tag{3}$$

Integrating Eq. (1) from the bottom to the free surface and using the boundary conditions, we obtain

$$\frac{\partial}{\partial t}\int_{-h}^{\eta}\rho u_\alpha dz+\frac{\partial}{\partial x_\beta}\int_{-h}^{\eta}\rho u_\alpha u_\beta dz+\rho g(h+\eta)\frac{\partial\eta}{\partial x_\alpha} = -\rho\frac{\partial^3\eta}{\partial x_\alpha\partial t^2}\frac{(h+\eta)^2}{3} \tag{4}$$

Here we will decompose the horizontal velocity u_α into the time-average (steady) component U_α and the periodic (wave) component u'_α:

$$u_\alpha = U_\alpha + u'_\alpha \tag{5}$$

and define three kinds of mass transport as follows:

$$\overline{\int_{-h}^{\eta}\rho u'_\alpha dz} = M'_\alpha \tag{6}$$

$$\overline{\int_{-h}^{\eta}\rho U_\alpha dz} = \rho U_\alpha(h+\overline{\eta}) = M_\alpha \tag{7}$$

$$\overline{\int_{-h}^{\eta}\rho u_\alpha dz} = \tilde{M}_\alpha = M_\alpha + M'_\alpha = \rho\tilde{U}_\alpha(h+\overline{\eta}) \tag{8}$$

where $\overline{\eta}$ is the mean water displacement, M'_α and M_α are the mass transport due to the orbital velocity and to the mean flow, respectively, and $\tilde{M}_\alpha$ is their summation.

Substitution of Eqs. (6) to (8) into the time-average of Eq. (4) yields

$$\frac{\partial\tilde{M}_\alpha}{\partial t}+\frac{\partial}{\partial x_\beta}\left\{\tilde{U}_\beta\tilde{M}_\alpha+\overline{\int_{-h}^{\eta}\rho u'_\alpha u'_\beta dz}-\frac{M'_\alpha M'_\beta}{\rho(h+\overline{\eta})}\right\} + \rho g(h+\overline{\eta})\frac{\partial\overline{\eta}}{\partial x_\alpha} = -\rho\frac{\partial^3\overline{\eta}}{\partial x_\alpha\partial t^2}\frac{(h+\overline{\eta})^2}{3} \tag{9}$$

On the other hand, Eq. (10) gives the governing equation for the nearshore current and the mean water elevation, in which the bottom friction and lateral mixing terms are neglected.

$$\frac{\partial\tilde{M}_\alpha}{\partial t}+\frac{\partial}{\partial x_\beta}[\tilde{U}_\beta\tilde{M}_\alpha+S_{\alpha\beta}]+\rho g(h+\overline{\eta})\frac{\partial\overline{\eta}}{\partial x_\alpha}=0 \tag{10}$$

where $S_{\alpha\beta}$ is the radiation stress expressed as:

$$S_{\alpha\beta}=\overline{\int_{-h}^{\eta}(\rho u'_\alpha u'_\beta+p\delta_{\alpha\beta})dz}-\frac{1}{2}\rho g(h+\overline{\eta})^2\delta_{\alpha\beta}-\frac{M'_\alpha M'_\beta}{\rho(h+\overline{\eta})} \tag{11}$$

Comparing Eqs. (9) and (10), we can readily find that their differences exist only in the radiation stress term and on the-right hand side. Since the Boussinesq equation holds for shallow water waves, we apply a long wave approximation to the pressure p in Eq. (11). Abbott (1979) has obtained the following relation for the pressure, in the process of his derivation of Eq. (1):

$$\frac{p(z)}{\rho} = g(\eta - z) + \frac{\partial^2 \eta}{\partial t^2} \frac{(h+\eta)^2 - (h+z)^2}{2(h+\eta)} \tag{12}$$

Substitution of Eq. (12) into Eq. (11) and some manipulations make Eq. (10) become identical to Eq. (9).

It has thus been shown that, as far as the long wave approximation holds good, the time-average of the Boussinesq equation becomes equivalent to the mean flow equation. In other words, solving the Boussinesq equation, we can compute not only the wave field but also the nearshore current and mean water level field, which are obtained by taking the time-average of the solutions for the velocity u_α and for the surface displacement η, respectively. This means that the wave-current interaction is automatically incorporated in computations.

COMPUTATION OF WAVES AND MEAN WATER ELEVATION

A set of one-dimensional Boussinesq equations involving a breaker-induced energy dissipation term is expressed in terms of the surface displacement η and the flow rate Q as follows:

$$\frac{\partial \eta}{\partial t} + \frac{\partial Q}{\partial x} = 0 \tag{13}$$

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{Q^2}{D}\right) + gD\frac{\partial \eta}{\partial x} - \frac{h^2}{3}\frac{\partial^3 Q}{\partial t \partial x^2} + M_D = 0 \tag{14}$$

where

$$D = h + \eta, \quad Q = \int_{-h}^{\eta} u\, dz,$$

u is the horizontal velocity and M_D corresponds to the breaker-induced energy dissipation and is given by (Sato and Suzuki, 1990)

$$M_D = \frac{gD}{\sigma^2} f_D \frac{\partial^2 Q}{\partial x^2} \tag{15}$$

The quantity σ is the angular frequency, and f_D is the energy dissipation coefficient expressed as (Watanabe & Dibajnia, 1987):

$$f_D = \alpha_D \tan\beta \sqrt{\frac{g}{h}} \sqrt{\frac{\hat{Q} - Q_r}{Q_s - Q_r}} \tag{16}$$

in which the coefficient $\alpha_D = 2.5$, $\tan\beta$ is the bottom slope around the breaking point, $\hat{Q}$ is the amplitude of Q, and Q_s and Q_r correspond to $\hat{Q}$ in the dissipation zone on a uniform slope and in the recovery zone in the constant depth water, respectively, given by

$$Q_s = \gamma_s C h \tag{17}$$

$$Q_r = \gamma_r C h \tag{18}$$

$$\gamma_s = 0.4\,(0.57 + 5.3 \tan\beta) \tag{19}$$

$$\gamma_r = 0.4 \left(\frac{a}{h}\right)_{\mathrm{B}} \tag{20}$$

where C is the wave celerity, and $(a/h)_{\mathrm{B}}$ is the ratio of the wave amplitude to the water depth at a breaking point.

In order to avoid re-reflection of outgoing waves (reflected from the slope or structures, if any), we should impose such a condition that they can freely pass through the offshore boundary of a computation domain. For this, we express the flow rate Q on the offshore boundary as the summation of that of the incident waves Q_{in} and of the outgoing waves Q_{out}:

$$Q = Q_{\mathrm{in}} + Q_{\mathrm{out}} \tag{21}$$

The corresponding horizontal velocities are expressed, under the long wave approximation, as:

$$\overline{u}_{\mathrm{in}} = \sqrt{g/h}\;\eta_{\mathrm{in}}, \quad \overline{u}_{\mathrm{out}} = -\sqrt{g/h}\;\eta_{\mathrm{out}} \tag{22}$$

Using $Q = \overline{u}\,(h+\eta)$ and neglecting terms with η^2, we obtain the following relation from Eq. (21):

$$Q = 2\,C\,\eta_{\mathrm{in}} - C\,\eta \tag{23}$$

On the other hand, we impose the condition given by Eq. (24) on the shoreward boundary, where r is the reflection coefficient.

$$Q = (1 - r) \cdot C\eta \tag{24}$$

Using these boundary conditions, numerical computations for Eqs. (13) and (14) will be conducted by the finite difference method with the central difference-staggered mesh scheme.

Computations of wave transformation on a slope have been made and the results have been compared with experimental data obtained by Sato *et al.* (1987). In the experiment, the water surface displacement and the near-bottom orbital velocity were recorded under the conditions that the bottom slope was 1/20 and the incident wave height and period were 6.1cm and 1.18s.

Figures 1 and 2, respectively, compare the cross-shore distributions of the wave height and the mean water surface elevation between the computations and the measurements. Agreement is very good for the wave height even near the breaking point. The mean water level $\overline{\eta}$ that has been computed by taking the time average of η, without the computation of the radiation stress, also shows a fairly good agreement with the measurements.

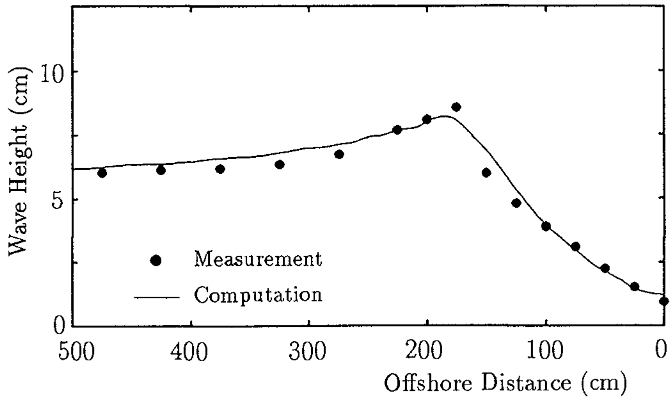

Fig. 1 Comparison of cross-shore distributions of wave height.

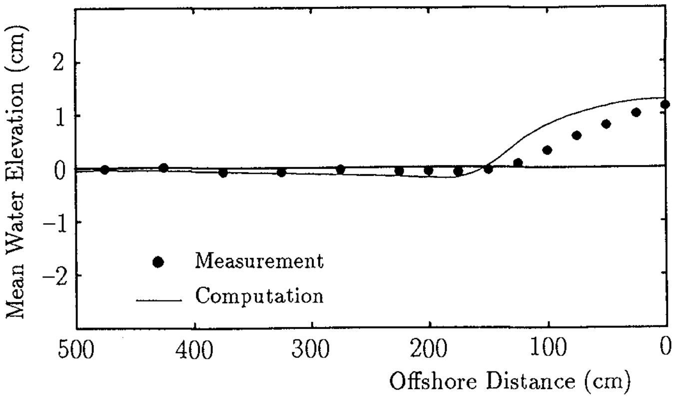

Fig. 2 Comparison of cross-shore distributions of mean water surface elevation.

CALCULATION OF NEAR-BOTTOM ORBITAL VELOCITY

The orbital velocities were measured at several points 5mm in height above the bottom. By considering the finite water depths in the region, the near-bottom orbital velocity u_b has been computed as a function of time at each location by the following equation:

$$u_b = \frac{Q}{h+\eta} \cdot \frac{\cosh(2\pi z'/L)}{\cosh(2\pi h/L)} \tag{25}$$

where z' is the height of a point above the bottom, and L is the local wavelength. Figures 3 (a), (b) and (c) show the comparisons of u_b between the computations and the measurements, for three locations: (a) outside the surf zone, (b) near the breaking point, and (c) in the surf zone.

In previous computations of the orbital velocities based on the small-amplitude wave theory, the method proposed by Isobe and Horikawa (1981) has given fairly good estimates up to the neighborhood of breaking points, but it has failed to evaluate them in the surf zone after breaking. As shown in Figs. 3 (a) to (c), the present model can directly reproduce the near-bottom orbital velocities with a remarkably high accuracy. The above comparisons are for a case of plunging breaker, and it has been found that the present wave model works for cases of spilling breaker as well.

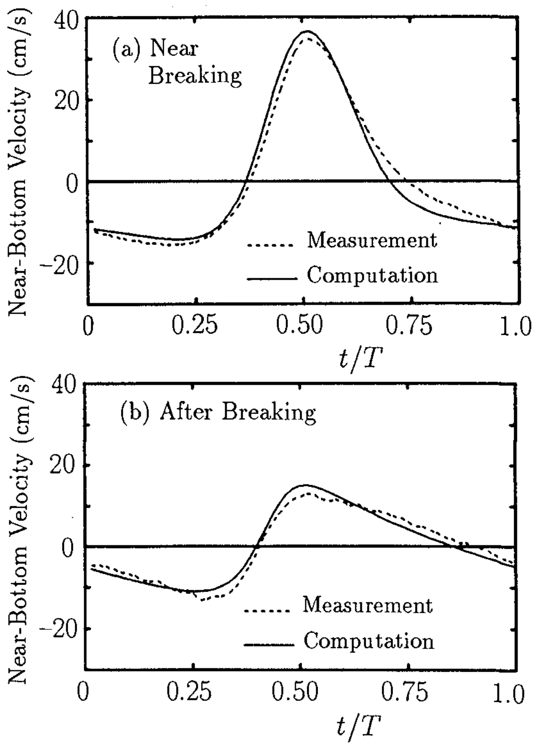

Fig. 3 Comparisons of near-bottom orbital velocities.

EVALUATION OF UNDERTOW VELOCITY

In the surf zone, the undertow (return flow near the bed) usually develops to compensate the shoreward mass transport accompanying the breakers. Since the concentration of suspended sediment is very high near the bottom, the effect of the undertow on the sediment transport should inevitably be taken into consideration. However, highly reliable models have not yet been established for the prediction of the undertow velocity under general conditions.

In the present study, following Dibajnia *et al.* (1992), we have evaluated the cross-shore distribution of steady flow velocity U due to the undertow by decomposing it into three components: the return flow velocity U_{B} caused by wave breaking, the seaward velocity U_{W} compensating the mass transport, and the Eulerian mass transport velocity U_{E} at the outer edge of the bottom boundary layer. According to previous studies, the undertow velocity becomes nearly zero near the breaking point. Hence, for simplicity, as a parameter representing the degree of intensity of the breaker-induced large vortex, we have introduced the following function K_U, whose value is zero before the plunging point $x \leq X_p$, being unity after the large vortex development point X_i and changing linearly between these two points, *i.e.*,

$$K_U = \begin{cases} 0 & (x \leq X_p) \\ \dfrac{X_p - x}{X_p - X_i} & (X_p < x \leq X_i) \\ 1 & (X_i < x) \end{cases} \tag{26}$$

Then the resultant total undertow velocity U has been calculated by

$$U = K_U \left(U_{\mathrm{B}} + U_{\mathrm{W}} + U_{\mathrm{E}} \right) \tag{27}$$

The first component U_{B} has been estimated by the following formula proposed by Sato *et al.* (1987):

$$U_{\mathrm{B}} = -A \frac{H^2}{h \cdot T} \tag{28}$$

where H and h are the local wave height and water depth, respectively, and A is a dimensionless coefficient of the order of unity. In the present study, we have used $A = 1$.

Figure 4 shows the comparison of the undertow velocity between the computation and the measurements, indicating a considerably good agreement. However, this only demonstrates the validity of the present method, Eqs. (26) to (28), for a constant slope bed in a wave flume, and its applicability to more general conditions (prototype scale and complicated bottom topography) is still questionable.

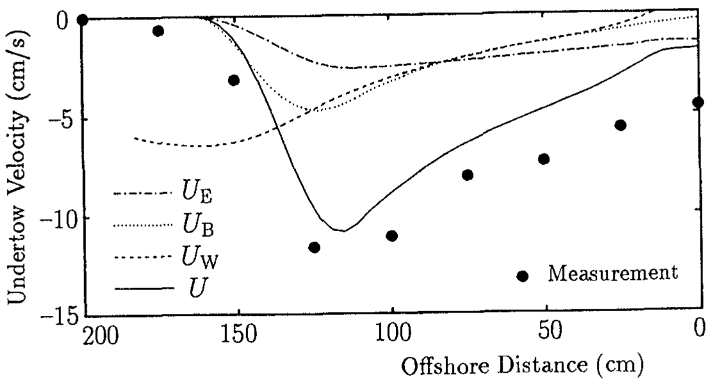

Fig. 4 Comparisons of cross-shore distributions of undertow velocity.

COMPUTATION OF SEDIMENT TRANSPORT RATE AND BEACH PROFILE CHANGE

Dibajnia and Watanabe (1992) have proposed a sheet-flow sand transport rate formula based on extensive laboratory data obtained in oscillatory flow tank experiments, considering the entrainment and settling processes of suspended sediment during the two successive half wave periods. According to this formula, the net rate of sheet flow transport $q_{\rm net}$ can be calculated as a function of the root-mean-square amplitudes, u_c and u_t and the time durations, T_c and T_t, of the onshore and offshore velocity, respectively, and the grain diameter d, settling velocity w_0, and porosity λ_v of the sediment, as follows:

$$q_{\rm net} = \text{sign}\,(\Gamma) \cdot 0.001|\Gamma|^{0.55} \cdot w_o d/(1 - \lambda_v) \tag{29}$$

$$\Gamma = \frac{u_c T_c\,(\Omega_c^3 + \Omega_t'^3) - u_t T_t\,(\Omega_t^3 + \Omega_c'^3)}{(u_c + u_t)\,T} \tag{30}$$

where Ω_c, Ω_c', Ω_t and Ω_t' are functions of u_c, u_t, T_c, T_t, d, w_0 and the specific density of the sediment (For details, see Dibajnia and Watanabe, 1992).

Dibajnia, Shimizu and Watanabe (1992) have presented a numerical model for a profile change of a sheet flow predominant beach using the above transport rate formula and the linear time-dependent mild slope equations. In the present study, we have incorporated the effect of wave nonlinearity in the prediction of beach profile change by combining the wave model based on the Boussinesq equation and the undertow model mentioned above as well as the sediment transport rate formula, Eq. (29). The change of a beach profile has been computed from the cross-shore distribution of $q_{\rm net}$ by the sediment mass conservation equation including the effect of the local bottom slope (Watanabe *et al.*, 1984):

$$\frac{\partial z_b}{\partial t} = -\frac{\partial h}{\partial t} = -\frac{\partial}{\partial x}\left(q_{\rm net} - \varepsilon_s |q_{\rm net}| \frac{\partial z_b}{\partial x}\right) \tag{31}$$

where z_b is the bottom elevation, and a value of 2.0 has been used for the coefficient ε_s.

In order to examine the validity of the model, we have performed comparisons of the computations with the measurements obtained by Shimizu *et al.* (1985) in their large scale experiments. Table 1 indicates the experimental conditions for the adopted three cases, in which d_{50} is the median diameter of the sand, $\tan\beta$ is the initial beach slope, T and H_0 are the period and height of incident waves.

Table 1 Experimental conditions.

Case	d_{50}(mm)	$\tan\beta$	T(s)	H_0(m)	Breaker type
3–2	0.27	1/20.0	6.0	1.05	Plunging
3–4	0.27	1/20.0	3.1	1.54	Spilling
4–2	0.27	1/33.3	4.5	0.97	Plunging

Although the measurements were made for a total time duration of 30 hours, the computations have been conducted for a duration of 5 or 7 hours only, because the accuracy of the undertow computation may become unsatisfactory as the beach profiles get complicated with time. Figures 5, 6 and 7 show the comparisons between the computations and the measurements of Cases 3-2, 3-4 and 4-2, respectively, for the wave height distributions, the net rates of the sediment transport, and the beach profiles.

These figures demonstrate considerably good agreement between the computations and the measurements, indicating the overall validity of the present model that includes the nonlinear wave equation, the formulas for the near-bottom orbital velocity and for the undertow velocity, and the sheet flow transport rate formula.

CONCLUDING REMARKS

This paper has presented a numerical model for beach profile change, which incorporates the asymmetric orbital velocity due to wave nonlinearity as well as the undertow current. By using the Boussinesq equation, the iteration of computations for the waves and for the mean flow and mean water level has become unnecessary, and the computational accuracy of the near-bottom orbital velocities has been highly improved. A beach profile change model has been established by the combination of the wave model based on the Boussinesq equation and the sediment transport rate formula for the sheet flow proposed by Dibajnia and Watanabe (1992). An overall validity of the present numerical model has been examined through the comparisons of the computed sediment transport rate distributions and beach profiles with the laboratory data obtained in large wave flume experiments by Shimizu *et al.* (1985). A better method will be required for the prediction of the undertow velocity distribution under general conditions so as to make the present model more practical and useful.

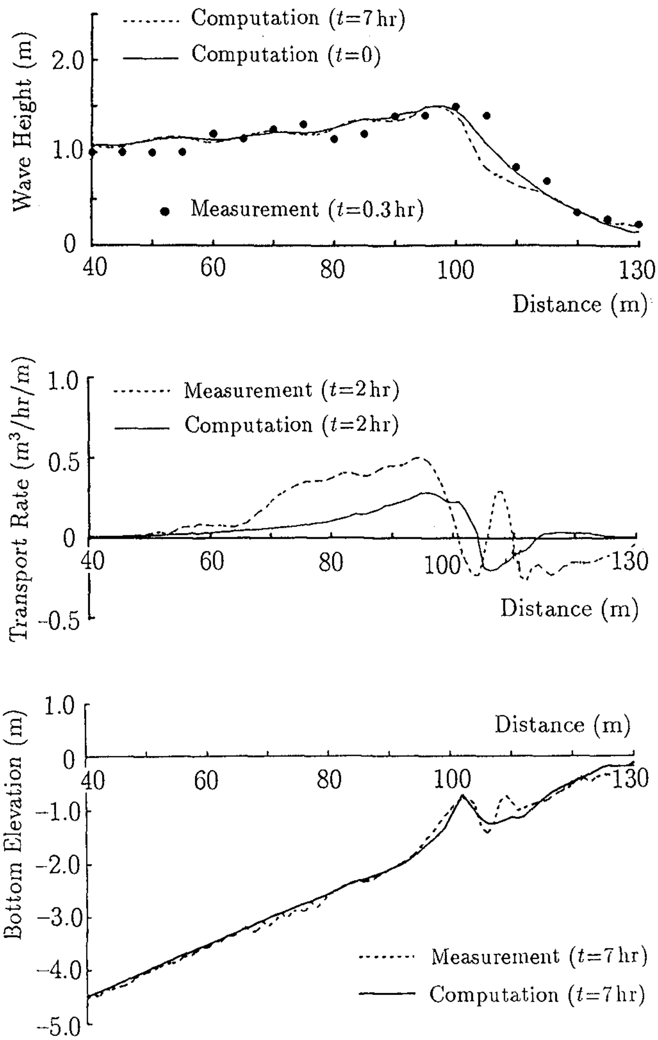

Fig. 5 Comparisons of wave height, net transport rate and beach profile. (d_{50} = 0.27mm, $\tan\beta$ = 1/20, H_0 = 1.05m, T = 6.0s, Plunging breaker)

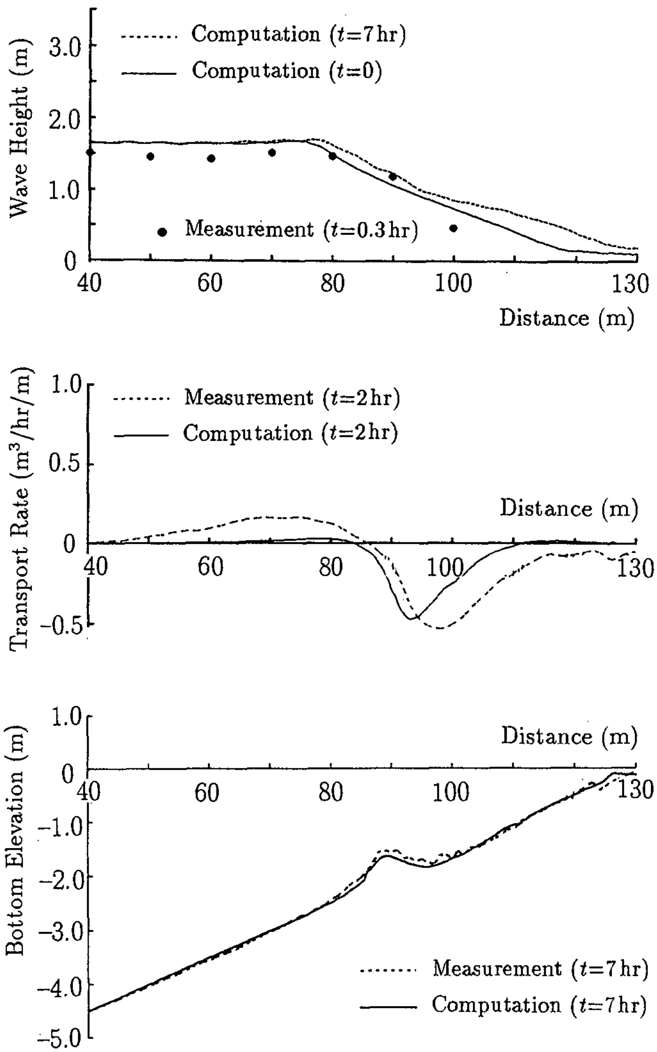

Fig. 6 Comparisons of wave height, net transport rate and beach profile. (d_{50} = 0.27mm, $\tan\beta = 1/20.0$, H_0 = 1.54m, T = 3.1s, Spilling breaker)

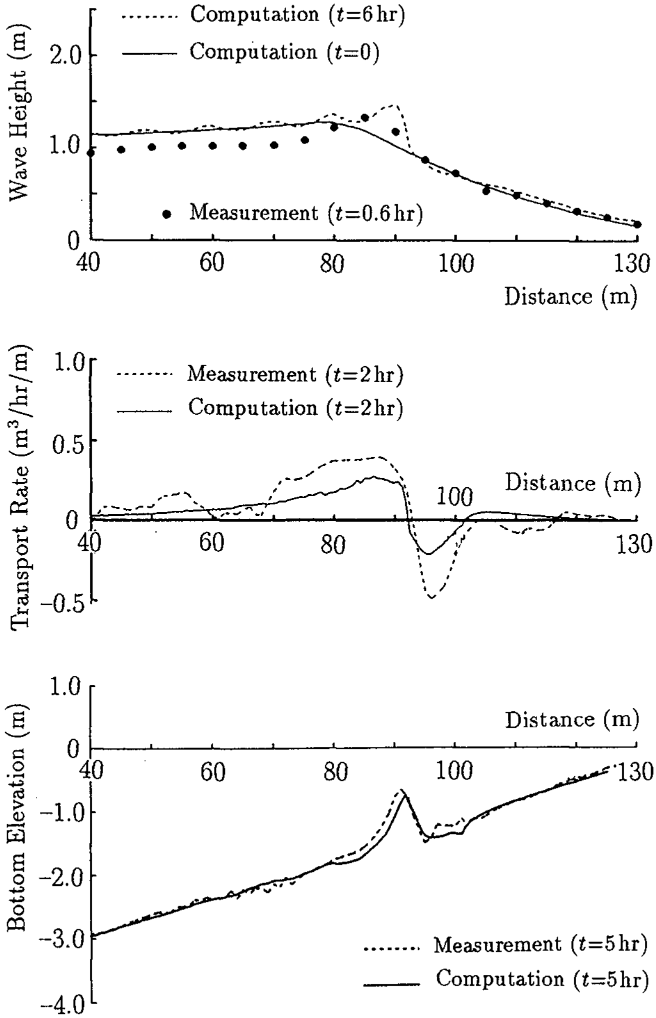

Fig. 7 Comparisons of wave height, net transport rate and beach profile. (d_{50} = 0.27mm, $\tan\beta$ = 1/33.3, H_0 = 0.97m, T = 4.5s, Plunging breaker)

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