CHAPTER 67

Weakly non-Gaussian model of wave height distribution for nonlinear random waves

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ABSTRACT

The wave height distribution with Edgeworth's form of a cumulative expansion of probability density function(PDF) of surface elevation are investigated. The results show that a non-Gaussian model of wave height distribution reasonably agrees with experimental data. It is discussed that the fourth order moment(kurtosis) of water surface elevation corresponds to the first order nonlinear correction of wave heights and is related with wave grouping.

INTRODUCTION

Wave height statistics(*e.g.* wave height distribution, run length and etc) of random waves play important roles in designing coastal and ocean structures. The Rayleigh distribution is regarded as the distribution of wave heights in stochastic processes with a linear and narrow banded spectrum. Over a few decades, a considerable number of studies have been made on the validity of the Rayleigh distribution. It is commonly known that large wave heights in field do not necessarily obey the Rayleigh distribution. For example, Haring(1976) shows that large wave heights observed in storms are on the order of 10 percent less than those predicted by the Rayleigh distribution. After that, Forristall(1984), and Myrhaug and Kjeldsen(1987) also reported that occurrence probabilities of large wave heights in field are smaller than the predicted value of the Rayleigh distribution.

On the contrary, Yasuda *et al.*(1992,1994) numerically investigated that the third order nonlinear interactions have significant effects on the statistical

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properties of random wave train. That is, the third order nonlinear solution in deep water increases the occurrence probabilities of large wave heights more than the linear and second order one do. Stansburg(1993) also found the same results in his experimental work. However, there is no theoretical distribution which agrees with the data, although many studies have attempted to establish the wave height distribution without a linear or narrow banded spectrum assumption.

The Rayleigh distribution is put to practical use under the assumption that water surface elevations are regarded as independent stochastic processes, since the nonlinear wave-wave interactions are weak in deep water. Thus, the probability density function of the surface elevation had been assumed to be the Gaussian on the basis of the central-limit theorem. For the statistical point of view, the fourth order moment of the surface elevation is directly related to the third order nonlinear interaction(Longuet-Higgins 1963). It is therefore necessary to include the effects of the fourth order moment of the surface elevation for the wave height distribution to consider the influences of the third order nonlinear interaction.

In this study, a wave height theory is extended for a weakly nonlinear random waves with cumulative expansion of surface elevation including the fourth order moment and then its validity is checked with experimental data.

PDF OF WAVE HEIGHTS

Probability density function of surface elevation

According to statistical theory, the probability density function(PDF) $p^{(l)}(x)dx$ of the l-independent variables x_i $\{i \in Z\}$(subscript i is dropped hereafter for simplicity) can be described as

$$p^{(l)}(x)dx = \sum_{r=0}^{\infty} c_r H_r(x) G(x)\, dx, \qquad (r \in Z) \tag{1}$$

$$G(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), \tag{2}$$

where $H_r(x)$ is the Chebyshev-Hermite polynomial, $G(x)$ the Gaussian and c_r the rth order coefficient of the Gram-Charlier expansion. The convergence of the Gram-Charlier expansion is not monotonic with the order r(*e.g.* c_4=$O(l^{-1})$, c_5=$O(l^{-3/2})$), although the convergence of the rth order cumulant $\kappa_r^{(l)}$ is independent of l. We hence choose the Edgeworth asymptotic expansion to describe the PDF of surface elevation.

Introducing the characteristic function and collecting the terms for l, the Edgeworth expansion of type A is formally given by (*e.g.* Kendall and Stuart

1963)

$$p^{(l)}(x)dx = G(x)\left\{1 + \frac{1}{\sqrt{l}}\frac{\kappa_3}{6}H_3(x) + \frac{1}{l}\left[\frac{\kappa_4}{24}H_4(x) + \frac{\kappa_3^2}{72}H_6(x)\right] + \frac{1}{l\sqrt{l}}\left[\frac{\kappa_5}{120}H_5(x) + \frac{\kappa_3\kappa_4}{144}H_7(x)\right] + \cdots\right\}dx, \tag{3}$$

where κ_r is the rth order cumulant. The rth order cumulant has the relationship to the rth order moment μ_r:

$$\left.\begin{aligned}
&\kappa_1 = 0\\
&\kappa_2 = 1\\
&\kappa_3 = \mu_3\\
&\kappa_4 = \mu_4 - 3\\
&\kappa_5 = \mu_5 - 10\mu_3\\
&\kappa_6 = \mu_6 - 15\mu_4 - 10\mu_3^2 + 30\\
&\kappa_7 = \mu_7 - 21\mu_2\mu_5 - 35\mu_3\mu_4 + 210\mu_2^2\mu_3
\end{aligned}\right\}. \tag{4}$$

We set the mean value μ_1 so as equal to zero and normalize all the variables by the standard deviation. Therefore, μ_3 is skewness and μ_4 is kurtosis. Each component within the bracket $[\cdot]$ of eq.(3) has monotonic convergence for l(the typical notation for l is dropped hereafter for simplicity).

It must be noted that an asymptotic expansion does not have monotonic convergence for higher order corrections, although it sometimes gives good agreement for a first few terms. Moreover, higher order moments and cumulants are influenced by sampling frequencies of data. We, therefore, use first three terms of eq.(3) to describe the PDF of the surface elevation. The influences of truncation of eq.(3) are already discussed in detail by Mori(1996). We truncate here higher than $1/l\sqrt{l}$ terms of eq.(3) following Mori(1996). This truncation gives the following relationship to the moments

$$\kappa_5 = \mu_5 - 10\mu_3 = 0, \tag{5}$$

$$\kappa_6 = \mu_6 - 15\mu_4 + 30 = 0. \tag{6}$$

The validity of these assumptions will be examined in next section.

Distribution of wave height

We assume that waves to be analyzed here are unidirectional with narrow banded spectrum and satisfy the stationarity and ergodic hypothesis. The surface elevation hence can be evaluated by the characteristic frequency $\bar{\omega}$:

$$I_c(t) = \sum_{n=1}^{\infty} a_n \cos[(\omega_n - \bar{\omega})t + \varepsilon_n] \tag{7a}$$

$$I_s(t) = \sum_{n=1}^{\infty} a_n \sin[(\omega_n - \bar{\omega})t + \varepsilon_n] \tag{7b}$$

where a_n is the amplitude of the nth mode, ω_n the angular frequency and ε_n the phase function. If ε_n is distributed uniformally and is a temporally independent variables, eq.(7) give a linear random wave field. The surface elevation $\eta(t)$ is rewritten into the amplitude of wave envelope $R(t)$ and phase angle $\theta(t)$ with I_c and I_s:

$$\eta(t) = R(t)\cos[\bar{\omega}t + \theta(t)] \tag{8a}$$

$$R(t) = \sqrt{I_c^2 + I_s^2} \tag{8b}$$

$$\theta(t) = \tan^{-1}\left[\frac{I_s(t)}{I_c(t)}\right] \tag{8c}$$

Under the assumption that the PDF of I_c and I_s are described by eq.(3) up to $1/l$ terms, I_c and I_s are independent statistical variables. Integration of $\theta(t)$ over the range from 0 to 2π results in the following PDF of wave amplitude

$$p(R)\,dR = \frac{1}{2\pi}\exp\left(-\frac{R^2}{2}\right)\left[1 + \sum_{i,j}\alpha_{i,j}A_{i,j}(R)\right]\,dR, \tag{9}$$

where $\sum_{i,j}$ is a special double summation for $i,j(i$=4,6 and j=$i/2)$, $A_{i,j}(R)$ is polynomial for R(see Appendix) and $\alpha_{i,j}$ is the coefficient with μ_3 and μ_4:

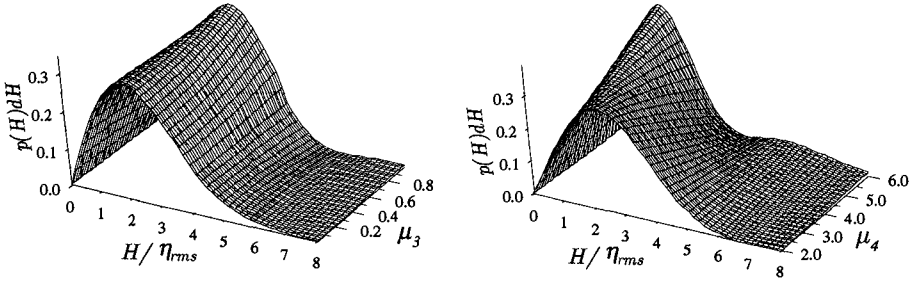
$$\left.\begin{aligned}
\alpha_{4,1} &= \tfrac{1}{2^5}(\mu_4 - 3)\\
\alpha_{4,2} &= \tfrac{1}{2^{13}\times 3}(\mu_4 - 3)^2\\
\alpha_{6,1} &= \tfrac{5}{2^6\times 3^2}\,\mu_3^2\\
\alpha_{6,2} &= \tfrac{1}{2^{13}\times 3^2}\mu_3^2(\mu_4 - 3)\\
\alpha_{6,3} &= \tfrac{5}{2^{16}\times 3^4}\,\mu_3^4
\end{aligned}\right\} \tag{10}$$

The assumption that I_c and I_s are mutually independent is inadequate for strong nonlinear waves($\mu_3 \geq 0.30$, see Mori 1996), but we should keep this assumption in this study.

By assuming a narrow banded spectrum for a wave field, a wave height H is regarded by two times of its amplitude A (H=$2A$). This assumption is inadequate when the vertical asymmetry of surface profile is not negligible(*i.e.* $\mu_3 \rightarrow$ large). Therefore, eq.(10) is valid for a weakly narrow banded process($\mu_3 \ll 1$).

The assumptions of a weakly nonlinear and narrow banded spectrum give the wave height distribution as

$$p(H)\,dH = \frac{H}{4}\exp\left(-\frac{H^2}{8}\right)\left[1 + \sum_{i,j}\beta_{i,j}B_{i,j}(H)\right]\,dH, \tag{11}$$



(a) For the case of μ_4=3 **(b)** For the case of μ_3=0

Figure 1 The variation of PDF of wave heights for the fixed value of μ_3 and μ_4.

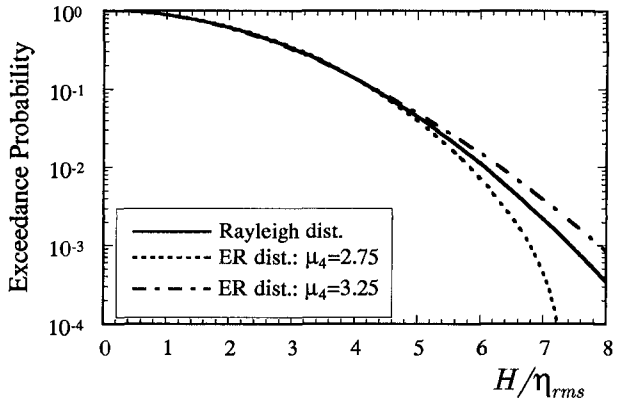


Figure 2 The exceedance probability of wave heights for μ_4=2.75 and 3.25 with μ_3=0.

where,

$$\left.\begin{aligned}
\beta_{41} &= \tfrac{1}{2^9}(\mu_4 - 3) \\
\beta_{42} &= \tfrac{1}{3\times 2^{21}}(\mu_4 - 3)^2 \\
\beta_{61} &= \tfrac{5}{2^{12}\times 3^2}\,\mu_3^2 \\
\beta_{62} &= \tfrac{1}{2^{23}\times 3^2}\,\mu_3^2\,(\mu_4 - 3) \\
\beta_{63} &= \tfrac{5}{2^{28}\times 3^4}\,\mu_3^4
\end{aligned}\right\}, \tag{12}$$

and $B_{i,j}(H)$ is polynomial for H(see Appendix).

The exceedance probability of wave heights is given by integrating eq.(11)

Table 1 Wave statistics of typical two cases.

Case	m	$k_p a_{1/3}$ at P1	breaking ratio
1	10	0.10	0%
2	10	0.20	10%

between $[H, \infty)$ as

$$P(H) = \exp\left(-\frac{H^2}{8}\right)\left[1 + \sum_{i,j} \beta_{i,j} E_{i,j}(H)\right], \tag{13}$$

where $E_{i,j}(H)$ is polynomial for H(see Appendix).

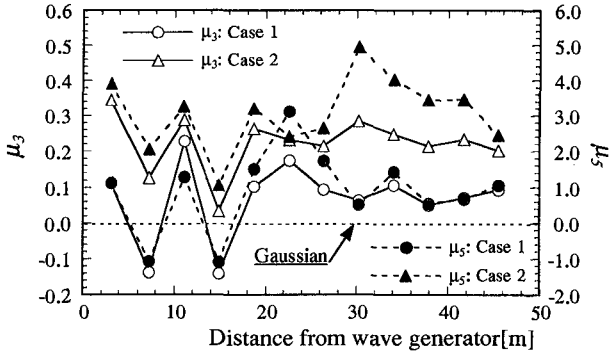
The first point to be noticed is that the additional terms within brackets[·] in eqs.(9), (11) and (13), which are related to non-Gaussian properties of the PDF of surface elevation, are equal to zero when μ_3=0 and μ_4=3(*i.e.* Gaussian). Therefore eqs.(9) and (11) with μ_3=0 and μ_4=3 are the same the Rayleigh distribution. We shall, hence, call with distribution of this type as the Edgeworth-Rayleigh distribution. The second important point to be noted is that the first order correction to the wave amplitudes or wave heights is kurtosis($\alpha_{4,j}$ or $\beta_{4,j}$).

Figure 1 show the variation of the PDF of the wave heights on the values of μ_3 and μ_4. For the case of μ_4=3 case, Fig.1(a), the shape of the PDF of the wave heights is not varied as increasing the value of μ_3. However, Fig.1(b) shows that the peak of the PDF is shifted to gently as increasing of μ_4, because μ_4 is the first order correction of the wave height distribution. Figure 2 shows the exceedance probability of the wave heights for the case of μ_4=2.75 and 3.25 with μ_3=0. The occurrence probability of the larger wave heights exceeds that of the Rayleigh distribution is increased when the value of μ_4 is larger than 3. We can summarize that the value of kurtosis is dominated parameter for the PDF of wave heights.

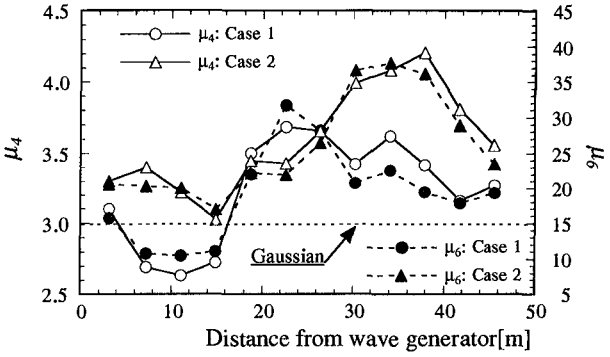
RESULTS

The laboratory experiment was conducted in the glass channel and is 65m long, 1m wide, 2m high and was filled to a depth of about 1.0m. Waves were generated by a computer-controlled piston type wave paddle. Water surface displacements were measured with twelve capacitance type wave gages. The measurements with a sampling frequency of 32Hz were performed for over 330s. No corrections were applied for filter response of the wire.

The initial spectra were given using the Wallops type spectra with band widths of m=5, 10, 30, 60 and 100, and peak frequency of f_p=1Hz which gives a spectral peak wavenumber k_p=4.072m^{-1} and a characteristic water depth of



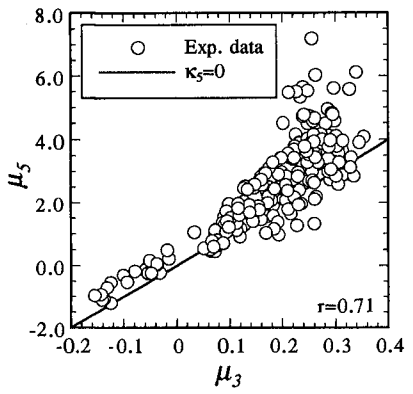
(a) Spatial variation of μ_3 and μ_5



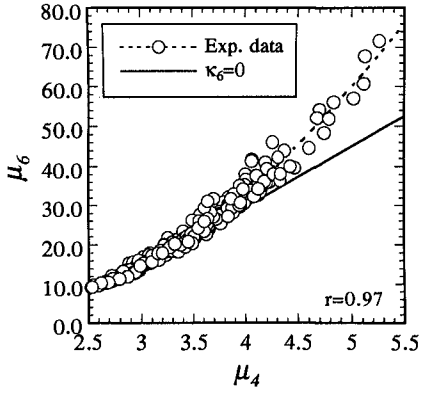
(b) Spatial variation of μ_4 and μ_6

Figure 3 The spatial variations of the higher order moments of surface elevations of the case 1 and 2.

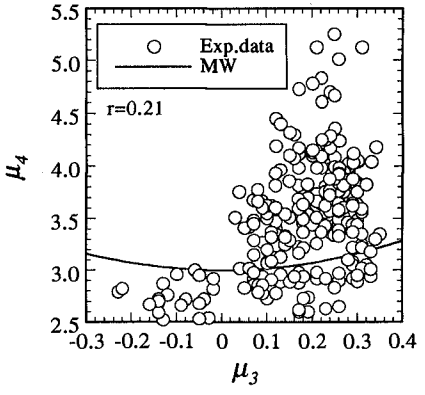
$k_p h$=3.99, so that the waves were deep water waves. Here, h is the water depth. The number of waves were about 350-450. The maximum frequency of generated waves were 2Hz. Therefore, higher frequency components of generated waves were generated by nonlinear interaction. Initial phases of the waves were given by uniformly distributed random numbers. The initial characteristic wave steepnesses $k_p a_{1/3}$ were set about 0.1 to 0.25. Here, $a_{1/3}$ is a half of the significant wave height. Breaking waves were observed for higher waves with the steepness that the value of $k_p a_{1/3}$ is larger than 0.13. For example, the visually observed breaking ratio is about 10% for waves with the initial steepness of $k_p a_{1/3} = 0.20$ and 20% for 0.14. Consequently, waves of 24 cases were generated under the combination of the spectrum bandwidth parameter m and the wave steepness.



(a) μ_3 vs. μ_5



(b) μ_4 vs. μ_6



(c) μ_3 vs. μ_7

Figure 4 Relationships among the higher order moments, μ_3, μ_4, μ_5, μ_6 and μ_7.

Typical measured two cases are shown in Table 1 for breaking and non-breaking cases. The spatial variations of higher order moments for case 1 and 2 are shown in Fig.3. These show that the higher order moments are fluctuated until the location 20m distant from the wave generator(that is about 13 wave lengths of peak frequency). There are marked increase in the moments more than 20m away from the wave generator. All of the higher order moments are not equal to the Gaussian, more and less. The odd order moments μ_3 and μ_5 seem to level out 20m away from wave generator, although the even order moments μ_4 and μ_6 are still increased. The higher order moments are generally influenced

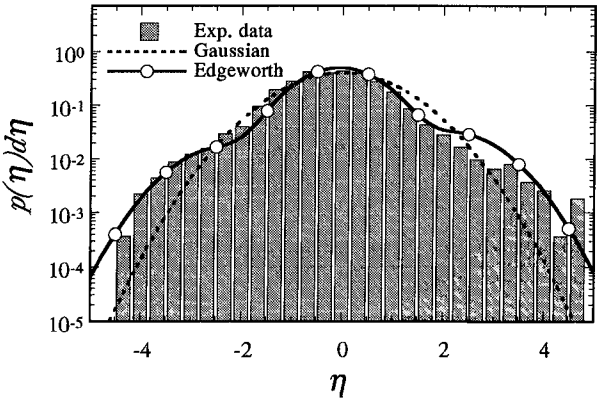


Figure 5 PDF of the surface elevation of experimental data of case 1 at location 8.

by the sampling frequency, but these trends do not depend on the sampling of data in our experiments. In addition, the spatial variations of μ_5 and μ_6 are similar to those of μ_3 and μ_4, respectively. These indicates the non-Gaussian properties of surface elevations.

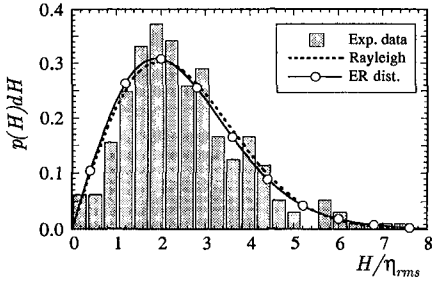
The experimental data show that the higher order moments are not constant. It should be examined the relationship between the higher order moments. The $(n+2)$th order cumulant is related to the nth order cumulant at the lowest order correction. Hence, the higher order cumulants than the 5th one are assumed zero to formulate the Edgeworth-Rayleigh distribution:

$$\kappa_5 = \mu_5 - 10\mu_3 = 0$$
$$\kappa_6 = \mu_6 - 15\mu_4 + 30 = 0$$

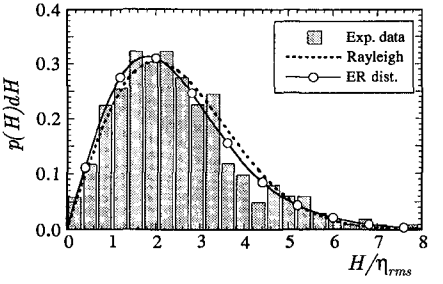
In order to check the validity of these assumptions, the relationships between μ_3 and μ_5, μ_4 and μ_6, and μ_3 and μ_4 are examined, respectively and are shown in Fig.4. Circles denote experimental data and solid lines in Fig.4(a) and Fig.4(b) are given by the eq.(5) and eq.(6), respectively. The correlation coefficient between μ_5 and μ_3 is 0.71, and therefore μ_5 could be regarded as a linear dependent variable on μ_3. μ_6 is also strongly related with μ_4(the correlation coefficient is 0.97). The experimental data show quite good agreement with eq.(6) in which the value of μ_4 is less than 4. Marthinsen and Winterstein(1992) derived the relationship between μ_3 and μ_4 from the second order kernel functions:

$$\mu_4 = 3 + \left(\frac{4}{3}\mu_3\right)^2 \tag{14}$$

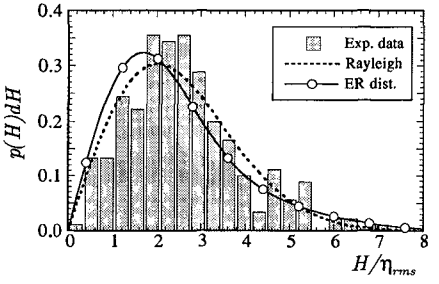
The solid line in Fig.4(c) indicates eq.(14). However, there is no obvious relation between μ_3 and μ_4 from the data(the correlation coefficient is 0.21). Conse-



(a) Case 2 at location 5



(b) Case 1 at location 5



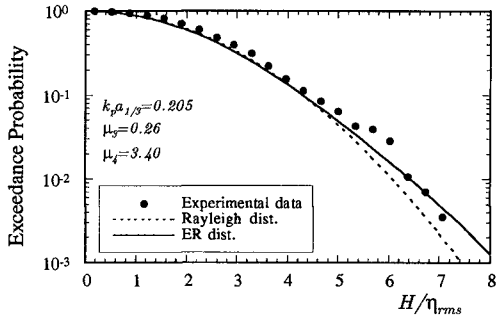
(c) Case 1 at location 8

Figure 6 The PDF of wave heights at several locations.

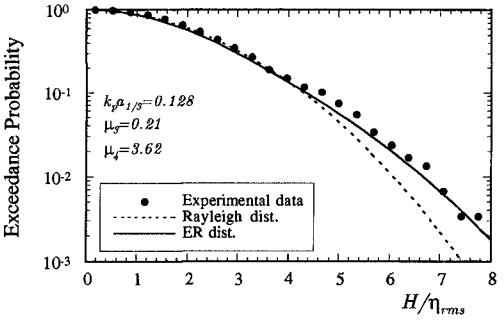
quently, it follows from what has been said that the 5th and 6th order cumulants can be regarded as zero(*i.e.* eqs.(5) and (6) are valid), if the value of μ_3 is smaller than 0.3 and μ_4 is smaller than 4. And the value of μ_4 is independent on μ_3.

The PDF of surface elevation of case 1 at location 8 is shown in Fig.5. The histogram is experimental data, dotted line is the Gaussian and solid line with circle is eq.(3) up to $1/l$ terms. The Edgeworth expansion shows agreement with experimental data in comparison with the Gaussian.

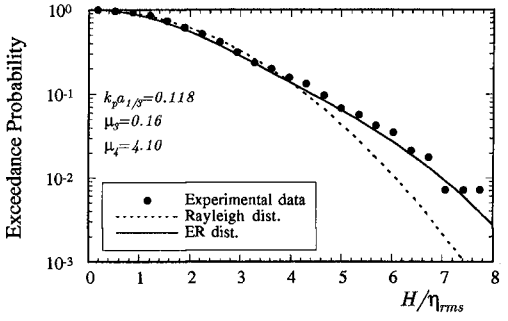
The PDF and exceedance probability of wave heights are shown in Figs.6 and 7, respectively. The histogram and filled circles • denotes experimental data, dotted line the Rayleigh distribution and solid line the Edgeworth-Rayleigh distribution. There are no significant difference between the Rayleigh and the Edgeworth-Rayleigh distribution for the PDF of wave heights. However, the Edgeworth-Rayleigh distribution for the exceedance probability of wave heights show good agreement with the experimental data in comparison with the



(a) Case 2 at location 5

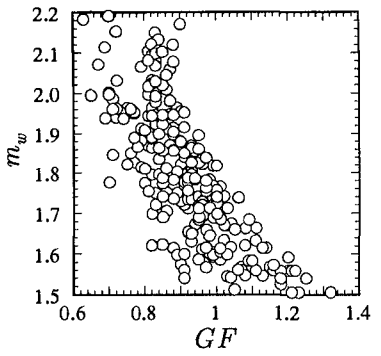


(b) Case 1 at location 5

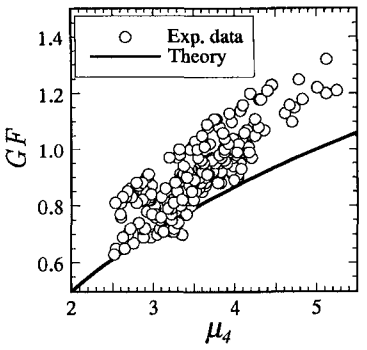


(c) Case 1 at location 8

Figure 7 The exceedance probability of wave heights at several cases and locations.



(a) GF vs. Weibull module m_w



(b) μ_4 vs. GF

Figure 8 Relationships between μ_4, GF and the Weibull modulus m_w, and GF and μ_4.

Rayleigh distribution. The number of waves decrease as wave heights getting larger, so the number of waves are quite few in the range $H/\eta_{rms} \geq 5$ in Fig.7. The Edgeworth-Rayleigh distribution agrees with the experimental data, even if the number of waves are not so many. Moreover, the Edgeworth-Rayleigh distribution agrees with the experimental data for larger value of $\mu_4(\mu_4 \geq 4)$, such as Fig.7(c).

RELATIONSHIP BETWEEN WAVE HEIGHT AND WAVE GROUPING

Before move to conclusion, it should be added that the relationship between the PDF of wave heights and wave grouping. Mase(1989) investigated an empirical relationship between the groupiness factor(GF) and the PDF of wave height with the Weibull modulus m_w of single parameter of the Weibull distribution.

GF is defined as

$$GF = \sqrt{\frac{1}{T_0}\int_0^{T_0}[E(t) - \bar{E}]^2 dt} \,/\, \bar{E}, \tag{15}$$

$$E(t) = \int_{-T_p}^{T_p} \eta^2(t+\tau)(1 - |\tau|/T_p)d\tau, \tag{16}$$

where $E(t)$ is SIWEH, $\bar{E}$ a mean value of $E(t)$, T_0 an observation period and T_p the spectral peak period. If we substitute the delta function $\delta(t-\tau)$ into the numerical trigonometric filter $1 - |\tau|/T_p$ of eq.(16), we obtain the following simple relation

$$GF' = \sqrt{\mu_4 - 1}. \tag{17}$$

This means that if we do nit use the numerical filter to calculate GF, there is a direct relation between GF and kurtosis μ_4. We already know that kurtosis μ_4 is the parameter which controls the wave height distribution. That is, both the parameters the Weibull modulus and μ_4 govern the wave height distribution. Therefore, there is a obvious relation between kurtosis and GF. That is the reason why the Weibull modulus m_w governing the wave height distribution depends on GF. The relationship between GF and the Weibull modulus, and GF and μ_4 are shown in Fig.8. These relations can be summarized as

$$\mathrm{GF} \sim \mu_4 \sim m_w. \tag{18}$$

The relationship between the Weibull modulus and GF has not a physical meaning so that we suggest to use kurtosis instead of GF to represent the wave grouping. Moreover, GF suffers from the influence of the numerical filter to obtain SIWEH. Therefore, the value of GF is influenced by two characteristics of random waves as a shape of wave height distribution and a spectrum band width. In other words, GF is insufficient as the fundamental statistical parameter to represent properties of the random wave.

CONCLUSION

A weakly non-Gaussian model of wave height distribution referred here as to the Edgeworth-Rayleigh distribution is suggested for waves with narrow banded spectra. It is found that the first order correction of the wave height distribution is equal to the fourth order moment of the surface elevation. It is also made clear that the occurrence probability of larger wave heights increases with the increasing of the value of kurtosis. The experimental data show good agreement with the Edgeworth-Rayleigh distribution within $\mu_3 \ll 1$ and $\mu_4 \leq 4$.

ACKNOWLEDGMENT

The authors are grateful to Dr.Tada at Technical Research & Development Institute of Nishimatsu Construction Co., Ltd. for his cooperation.

APPENDIX

$$\left.\begin{aligned}
A_{4,1}(R) &= R^4 - 8R^2 + 8 \\
A_{4,2}(R) &= R^8 - 32R^6 + 288R^4 - 768R^2 + 384 \\
A_{6,1}(R) &= R^6 - 18R^4 + 72R^2 - 48 \\
A_{6,2}(R) &= R^{10} - 50R^8 + 800R^6 - 4800R^4 + 9600R^2 - 3840 \\
A_{6,3}(R) &= R^{12} - 72R^{10} + 1800R^8 - 19200R^6 + 86400R^4 \\
&\quad -138240R^2 + 46080
\end{aligned}\right\} \tag{19}$$

$$\left.\begin{aligned}
B_{4,1}(H) &= H^4 - 32H^2 + 128 \\
B_{4,2}(H) &= H^8 - 128H^6 + 4608H^4 - 49152H^2 + 98304 \\
B_{6,1}(H) &= H^6 - 72R^4 + 115H^2 - 48 \\
B_{6,2}(H) &= H^{10} - 200H^8 + 12800H^6 - 307200H^4 + 2457600H^2 \\
&\quad -3932160 \\
B_{6,3}(H) &= H^{12} - 288H^{10} + 28800H^8 - 1228800H^6 \\
&\quad +2211840H^4 - 141557760H^2 + 188743680
\end{aligned}\right\} \tag{20}$$

$$\left.\begin{aligned}
E_{4,1}(H) &= H^2(H^2 - 16) \\
E_{4,2}(H) &= H^2(H^6 - 96H^4 + 2304H^2 - 12288) \\
E_{6,1}(H) &= H^2(H^4 - 48H^2 + 384) \\
E_{6,2}(H) &= H^2(H^8 - 160H^6 + 7680H^4 - 1228800H^2 + 4915200) \\
E_{6,3}(H) &= H^2(H^{10} - 240H^8 + 19200H^6 - 614400H^4 \\
&\quad +7372800H^2 + 23592960)
\end{aligned}\right\} \tag{21}$$

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