CHAPTER 90

A HAMILTONIAN MODEL FOR NONLINEAR WATER WAVES AND ITS APPLICATIONS

A.K. Otta[1], M.W. Dingemans[2], A.C. Radder[3]

Abstract

Evolution equations for nonlinear long waves are considered from an approximation to the exact Hamiltonian (total energy) for the water waves. The approximation which is used here has two distinct advantages over many other formulations which are commonly used for the same purpose. Further, a variation of these evolution equations is considered in order to incorporate higher-order nonlinearity. Numerical solutions of the evolution equations have been carried out for both the systems. Application of these models is illustrated in some practical cases. Comparisons between experimental measurements and computed results show that the model can be used for satisfactory prediction of nonlinear transformation of non-breaking waves over varying depth. Two features for further investigation are: (i) inclusion of both short-wave and long-wave nonlinearity so that the model can be used with uniform validity from deep to shallow water and (ii) modifications of the evolution equations so that they can be applied to propagation of breaking waves in a robust way.

Introduction

Commonly used nonlinear equations for propagation of water waves can be categorised into two main groups: Stokes-type valid in deep water and Boussinesq-type valid for fairly long waves. Several modifications to classical Boussinesq equations have been presented in an attempt to increase the validity of the model equations with respect to frequency dispersion and varying depth (e.g., Madsen *et al.*, 1991; Dingemans & Merckelbach, 1996). In this article, we discuss an alternative approach based on an explicit Hamiltonian formulation (Radder, 1992) for modelling of nonlinear waves over varying depth. The explicit expression for

[1][Formerly at Delft Hydraulics], ICCH, Agern Alle 5, DK-2970 Hørsholm, Denmark

[2]Senior Researcher and corresponding author, Delft Hydraulics, P.O. Box 152, 8300 AD Emmeloord, The Netherlands

[3]Senior Researcher, Rijkswaterstaat-RIKZ, P.O. Box 20907, 2500 EX The Hague, The Netherlands

the kinetic energy in the form of a surface integral involving the free surface potential φ and the surface elevation ζ is derived by using a conformal mapping in the complex plane. Therefore, the formulations are limited in a strict sense to unidirectional problems in the horizontal space.

Formulations

We use the symbols $\zeta(x,t)$ and $\varphi(x,t) = \Phi\{x,\zeta(x,t),t\}$ to denote respectively the free surface elevation and potential where Φ is the usual velocity potential. The evolution equations for ζ and φ follow the Hamiltonian structure (see, for example, Benjamin and Olver, 1982):

$$\frac{\partial\zeta}{\partial t} = \frac{\delta\mathcal{H}}{\delta\varphi} \quad , \quad \frac{\partial\varphi}{\partial t} = -\frac{\delta\mathcal{H}}{\delta\zeta} \tag{1}$$

where $\mathcal{H}$ is the total energy (sum of potential and kinetic energy) of the water mass which reads as

$$\begin{aligned} \mathcal{H} &= \frac{1}{2}\rho g\int_{-\infty}^{\infty}\zeta^2 dx + \frac{1}{2}\rho\int_{-\infty}^{\infty}dx\int_{-h}^{\zeta}(\nabla\Phi\cdot\nabla\Phi)dz \\ &= \int_{-\infty}^{\infty}\frac{1}{2}\rho g\zeta^2 dx + \frac{1}{2}\rho\int_{-\infty}^{\infty}dx\sqrt{1+(\zeta_x)^2}\varphi\Phi_n\ . \end{aligned} \tag{2}$$

The essential difficulty in deriving explicit evolution equations from (1) and (2) lies in the vertical integral or the Neumann operator of the potential Φ in the expression for kinetic energy density. This is where different procedures differ in the way that the simplifications are made. Several of the more commonly known weakly nonlinear formulations (either of the Stokes or Boussinesq-like approximations) correspond to some of these variations. Recently, Craig and Groves (1994) have presented a comprehensive discussion of these variations.

The basis for our approach is an explicit formulation of the Hamiltonian for two-dimensional water wave problems (unidirectional propagation) presented by Radder (1992). All details of the derivations are not included here. These may be found in Radder (1992), Otta and Dingemans (1994a) and Dingemans and Otta (1996). Radder has shown that an exact expression for the kinetic energy $\mathcal{T}$ as a function of the free-surface variables ζ and φ can be obtained in an explicit form using a conformal mapping, namely the Woods' transformation from the physical space (horizontal and vertical coordinates x and z) to $(-\infty < \chi < \infty,\ 0 \leq \xi \leq 1)$. This expression for the kinetic energy in the χ space along the free surface ($\xi = 1$) is given by

$$\mathcal{T} = -\frac{\rho}{2\pi}\int_{-\infty}^{\infty}d\chi\frac{\partial\varphi}{\partial\chi}\int_{-\infty}^{\infty}d\chi'\frac{\partial\varphi}{\partial\chi'}\log\tanh\frac{\pi}{4}|\chi-\chi'| \tag{3}$$

where log is the natural logarithm and the variable χ is related to x along the free surface through

$$\frac{dx}{d\chi} = \left(\frac{d/d\chi}{\tan d/d\chi}\right)\zeta(\chi) + \left(\frac{d/d\chi}{\sin d/d\chi}\right)h(\chi)\ . \tag{4}$$

The mapping (4) is valid exactly for arbitrary variation of depth h and surface elevation ζ.

We need to inverse the Jacobian, given by (4), in order to express $\mathcal{T}$ in the physical space. It is, however, difficult to obtain an exact inversion. Consequently, we shall use an approximation to the exact Jacobian leading to an approximation of the Hamiltonian ($\mathcal{H}^a$) in the physical space. A notable distinction of the method is that the positive definiteness of the kinetic energy density can be ensured for each approximation if one selects the inversion such that $\frac{dx}{d\chi} > 0$.

The simplest approximation which accounts for long-wave nonlinearity (it is assumed that $kh{\cdot}ka$ is small) for moderately high waves is arrived at by $dx/d\chi \approx \eta$ where η is the total water depth $(h + \zeta)$. The resulting Hamiltonian $\mathcal{H}^a$ is

$$\frac{\mathcal{H}^a}{\rho} = \frac{1}{2}\int_{-\infty}^{\infty} dx\, g\zeta^2 - \frac{1}{2\pi}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dx' \phi_x \phi_{x'} \log\tanh\frac{\pi}{4}\left|\int_x^{x'} \frac{dx''}{\eta}\right| . \tag{5}$$

With regards to the approximate inversion we introduce the variable p such that

$$\frac{dx}{dp} = h + \zeta \tag{6}$$

and the resulting evolution equations are

$$\frac{\partial\zeta}{\partial t} = \frac{1}{\pi}\frac{\partial}{\partial x}\int_{-\infty}^{\infty} dx'\varphi_{x'} \log\tanh\frac{\pi}{4}\left|\int_x^{x'}\frac{dr}{\eta}\right| \tag{7}$$

$$\frac{\partial\varphi}{\partial t} = -g\zeta - \frac{1}{2\eta^2}\int_{-\infty}^{x} dx' \int_x^{\infty} dx'' \frac{\varphi_{x'}\varphi_{x''}}{\sinh\left(\frac{\pi}{2}\int_{x'}^{x''}\frac{dr}{\eta}\right)} . \tag{8}$$

We note that for this approximation, positive definiteness is maintained as long as the total water depth $(h+\zeta)$ remains greater than zero. This feature of positive definiteness is not guaranteed in the entire wave spectrum in many other approximate Hamiltonian system. Another significant advantage is that in the limiting case of infinitesimal waves the dispersion relationship of the evolution equations derived from (5) is identical to that from the classical linear wave theory over uniform depth. This has the positive consequence that linear propagation and shoaling are properly predicted for all free components over the entire wave spectrum in transferring from deep to shallow water although nonlinear interactions of the short-wave type may suffer from the same limitation as when Boussinesq formulations are used.

Higher-order Description

The approximation (5) and the evolution equations derived therefrom do very well in reproducing long-wave nonlinearity as will be shown later. However, for better description of the form near the crests during wave steepening and propagation of higher waves, higher-order nonlinearity needs to be incorporated in the evolution equations. Both short-wave and long-wave nonlinearity can be accounted for in the approximate Hamiltonian in a hierarchial way by seeking higher-order inversion to (4) than used in (6). This is, however, not very straightforward. Instead, we choose an intermediate appoach where the evolution equation for the surface elevation is obtained by replacing χ by p in the exact expression for the kinetic energy; *i.e.*,

$$\frac{\partial m}{\partial t} = \frac{1}{\pi}\int_{-\infty}^{\infty} dp\varphi_{p'} \log\tanh\frac{\pi}{4}\left|p-p'\right| \tag{9}$$

with $\zeta(x,t) = \partial m/\partial x$ in combination with the dynamic free surface boundary condition used directly for the free surface potential, *i.e.*

$$\frac{\partial \varphi}{\partial t} = -g\zeta - \frac{1}{2}\varphi_x^2 + \frac{1}{2}\frac{(\zeta_t+\varphi_x\zeta_x)^2}{1+(\zeta_x)^2}\,. \tag{10}$$

Numerical Solution

The evolution equations (7) and (8) are not convenient for direct numerical treatment due to the singular kernels and the double integral in the latter. The integrands are first regularised to make them more amenable for numerical solution. The modified equations (Otta & Dingemans, 1994a) are presented in the following:

$$\zeta_t = \frac{1}{\pi}\frac{d}{dx}\left[-\pi v\eta + \int_{-\infty}^{\infty} dx'\lambda(x',x)\log\tanh\frac{\pi}{4}\left|\int_x^{x'}\frac{dx''}{\eta}\right|\right] \tag{11}$$

$$v_t = -g\zeta_x + \frac{d}{dx}\left[\frac{1}{\eta^2}\int_{-\infty}^{x} dx'\, v\eta\zeta_t\right] \tag{12}$$

where

$$v = \varphi_x, \qquad \lambda(x',x) = v(x') - \frac{\eta(x)}{\eta(x')}v(x)\,. \tag{13}$$

The equations, given by (11) and (12), are numerically solved using an predictor-corrector scheme, namely the Adams-Bashforth-Moulton method. The spatial discretization of the right hand sides of the equations is carried out using the **sinc**-series approximation (Lund & Bowers, 1992). More about the characteristics and advantages of using the **sinc**-series and the high order of accuracy of the numerical scheme used is described in a separate article under preparation.

Certain advantages are gained by carrying out the numerical evaluation of the evolution equations, given by (9) and (10), in a uniformly spaced p-space. Using the sinc approximation to express $\mathcal{V}(p')$ ($\mathcal{V} = \varphi_p$), *i.e.*,

$$\mathcal{V}(p') = \sum_{l=-\infty}^{\infty} \mathcal{V}_l \operatorname{sinc}\frac{p'-p_l}{\Delta p} \tag{14}$$

with l denoting the grid index and Δp the uniform interval between two consecutive grids in the p-space we obtain the discretised form of the evolution equation (9):

$$\left.\frac{\partial m}{\partial t}\right|_j = \sum_l \mathcal{V}_l I(|l-j|\,;\Delta p) \tag{15}$$

where

$$I(|l-j|\,;\Delta p) = -\frac{1}{\pi}\int_{-\infty}^{\infty} dp' \operatorname{sinc}\frac{p'-p_l}{\Delta p}\log\tanh\frac{\pi}{4}\left|p_j - p'\right| \ . \tag{16}$$

For a time-invariant value of Δp, the terms $I(|l-j|\,;\Delta p)$'s remain constant and need to be evaluated only once. We note, however, that the x-coordinates corresponding to the fixed values of p grids are time-variant. The local evolution rates $[\zeta_t(x,t),\ \varphi_t(x,t)]$ then need to be transferred to the constant p-grids through the relation

$$\begin{aligned} \frac{d\psi}{dt}(x^p,\, t) &= \frac{\partial\psi}{\partial t} + \frac{dx^p}{dt}\frac{\partial\psi}{\partial x} \\ &= \frac{\partial\psi}{\partial t} + \frac{dx^p}{dt}\frac{\partial p}{\partial x}\frac{\partial\psi}{\partial p} \end{aligned} \tag{17}$$

where ψ is used to denote either ζ or φ and x^p denotes the x-coordindate of a constant p-value. The value of x^p itself is updated with time.

Boundary Conditions

The evolution equations (with the exception of (10)) described so far include an integral over the infinite span. This has its origin to our deliberate choice of excluding the lateral boundary from the formulations in order that the Hamiltonian can be expressed as a function of ζ and φ. With no further modifications, these evolution equations can be used to study an initial value problem. This means that the span in the x-interval for the numerical solution has to be much longer than the physical interval of interest with the consequence that the required computational time increases tremendously. This is, however, not necessary.

Consider for example the evolution equation (9). We rewrite this equation as

$$\begin{aligned} \frac{\partial m}{\partial t} &= \frac{1}{\pi}\int_{-\infty}^{p_0} dp\varphi_{p'}\log\tanh\frac{\pi}{4}\left|p-p'\right| + \frac{1}{\pi}\int_{p_0}^{p_n} dp\varphi_{p'}\log\tanh\frac{\pi}{4}\left|p-p'\right| \\ &\quad + \frac{1}{\pi}\int_{p_n}^{\infty} dp\varphi_{p'}\log\tanh\frac{\pi}{4}\left|p-p'\right| \end{aligned} \tag{18}$$

where p_0 and p_n denote respectively the begin and end point of the computation interval. Though the intervals of the first and third integrals in (18) outside the computational interval are semi-infinite, the rapid decay of the function log tanh() appearing in the kernel of the integral allows reduction of the interval to usually a few (local) water depths without any loss of accuracy. We assume for simplicity that the bottom stretches uniformly to the left of p_0 and the right of p_n. For generating incoming waves in the computational interval, we use an appropriate theory to translate the time record of surface elevation $\zeta(t)$ at a given location (say p_0) to a space and time record for both ζ and φ outside of the computational interval. In the simplest way this is done by using linear theory to specify the incident field in the offshore side. On the shallower part a sponge layer can be introduced to absorb the radiating waves. For this purpose the evolution equations are modified; *i.e.*,

$$\zeta_t = R_\zeta(\zeta, \varphi) - \mu(x)\zeta \tag{19}$$

where $R_\zeta(\zeta, \varphi)$ represents the unmodified part of the equation and $\mu(x)$ a damping coefficient. In many cases, the introduction of a sponge layer over a reasonable length of the computational domain allows absorbtion of the radiating waves such that the third integral in (18) becomes non-significant. In a more general approach, a combination of the sponge layer and a Sommerfeld type of radiation condition can be implemented to specify the space and time record of ζ and φ outside of p_n to compute the third integral in (18). Details of the types of boundary conditions and the procedures are described in Otta & Dingemans (1994b) and Dingemans & Otta (1996).

Examples of Applications

In this section we discuss the application of the models to some critical cases. An illustrative example is the study of the propagation of a solitary wave over a bottom of constant depth. The initial surface profile $\zeta(x, t = 0)$ and the potential $\varphi(x, t = 0)$ are specified from Tanaka's (1986) solution which may be considered exact. Fig. 1 shows that this profile changes considerably with a decrease in the crest height and the formation of a dispersive tail during the propagation according to (7) and (8). The change in crest height and the dispersive tail is much less prominent when the set of evolution equations with the higher order of nonlinearity is used.

An example of practical interest is the propagation of waves over varying depth. To test the validity of our two models for such cases we consider an experimental setup (Fig. 2) as described in Luth *et al.* (1994) and used in Dingemans (1994). Transformation of an incident train of periodic waves is shown at different locations along the bar in Fig. 3. The incident waves are of mild amplitude such that they do not break over the bar. The computed profiles shown here are obtained by using (9) and (10). The nonlinear interactions on top of the bar (at location $x = 13.5$ m) and the propagation of the free waves behind the bar (location $x = 19$ m) are reproduced in a satisfactory manner. Though not shown here, results obtained from (7) and (8) for this case have been found to be be nearly as good with only slightly differences near the crest. In a previous study (Dingemans, 1994), the model based on (7) and (8) has been found to yield better agreement with experimental measurements than many Boussinesq-like models.

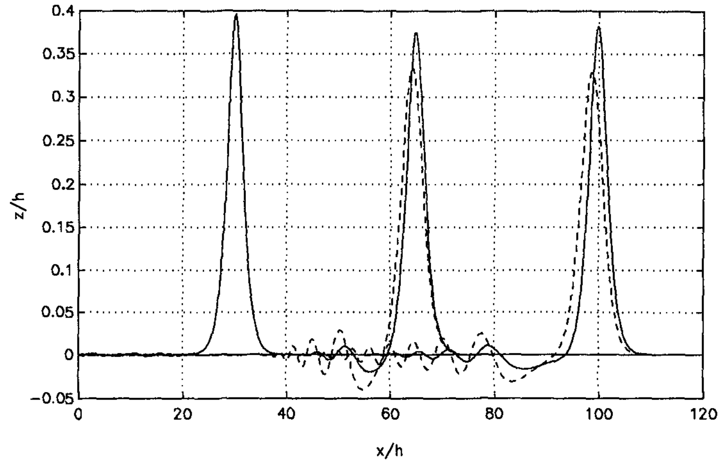

Figure 1: Computed profiles of a solitary wave (initial height/depth= 0.4) during propagation over a bottom of constant depth. Solid line: based on (9) and (10); dashed line: based on (7) and (8).

Wave Breaking

The evolution equations, given by (7) and (8) or by (9) and (10), are valid for waves which do not break. Wave-breaking is, however, of great practical significance in coastal areas. In field conditions, wave breaking in shallow water is identified by overturning of the water surface and plunging on to itself (plunging breaker) or the formation of a thin jet near the crest that spills over (spilling breaker). There are several difficulties, both physical and numerical, in modelling this process of breaking reasonably. Although a proper description of the water motion in the neighbourhood of breaking is essential for some purposes, it is sufficient to incorporate the main effects of breaking in the post-breaking behaviour of wave propagations for many a problem. To incorporate this effect in weakly nonlinear equations two steps are essential. Since either overturning of the surface and the spilling jet may not be manifested in the model behaviour or the numerical limitations of an adopted scheme may not permit computations advancing to the desired point, a criterion has to be used to indicate the incipience of breaking. Secondly, the evolution equations (derived originally for non-breaking waves) have to be modified so that the effects of wave-breaking (*e.g.*, energy dissipation) are incorporated within the framework.

Several criteria have been used in different models for the definition of the onset of breaking. For example, Schäffer *et al.* (1993) found it necessary to use a

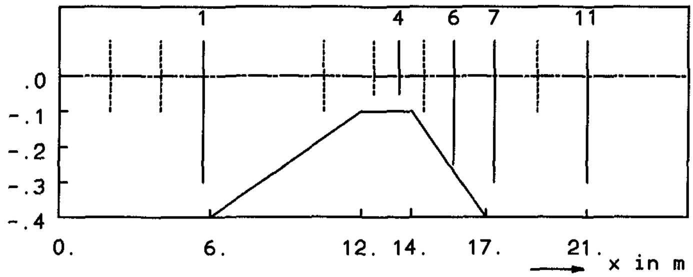

Figure 2: Geometry of the experimental setup.

criterion of the maximum of the local surface slope reaching a value of 20° in their time-domain Boussinesq model to a mark the onset of breaking. We consider here a criterion which is based on the idea of the Jacobian of the conformal mapping $d\chi/dx$ becoming zero. This condition has been mentioned for horizontal bottom by Dingemans and Radder (1991) using a first-order expression for the Jacobian including both short-wave and long-wave nonlinearity. For varying depth, the criterion reads as

$$\eta(p) + \int_0^\infty \frac{d\left[\zeta(p+q) + \zeta(p-q)\right]}{\exp(\pi q) - 1} - \int_0^\infty \frac{d\left[h(p+q) + h(p-q)\right]}{\exp(\pi q) + 1} = 0\ . \qquad (20)$$

The criterion (20) has been found to be a good indicator of the onset of breaking for waves of symmetric permanent form and some cases of asymmetric forms before plunging (Radder, 1995; personal communication) although a comprehensive analysis of the criterion for the onset of breaking in dynamical situations is still very much needed.

During numerical experimentations with the model equations (7) and (8) we have observed that the condition (20) was never reached in the computed surface form while breaking did occur in reality. Though this observation is not completely surprising, it was difficult to predict *a priori*. It is our conjecture that this is caused by the inadequacy in the representation of nonlinearity in the equations (7) and (8). This conjecture is in fact the primary motivation behind the set-up for a higher-order description. On application of the model, based on (9) and (10), the breaking criterion is met in several cases followed by numerical instabilities. An example of such a computation is shown in Fig. 4 where a train of periodic waves is allowed to propagate onshore over a sea-bed profile near Egmond on the Dutch coast. In a preliminary approach to model post-breaking behaviour, we have considered modifying the surface elevation at each evolution step following two criteria: (a) the surface form remains within the limit of the breaker criterion and (b) the modification does not lead to a change in the mass balance. The effect is to reduce the crest height and the surface steepness with an associated dissipation of energy. This approach has shown some qualitative success in a limited number of cases, *e.g.*, during the propagation of a solitary wave

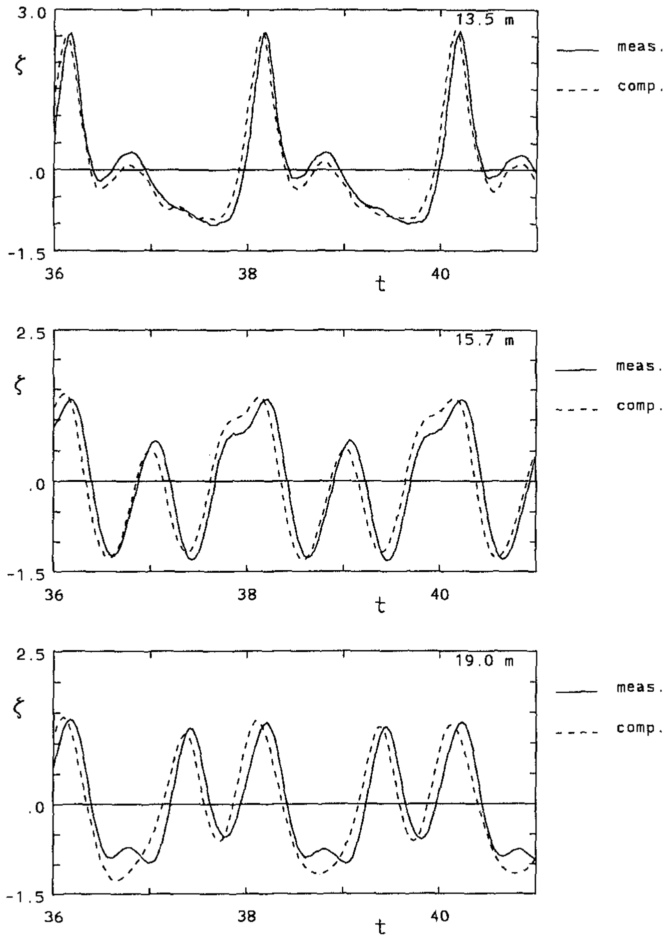

Figure 3: Transformation of a train of periodic sine waves due to propagation over a bar (Fig. 2), Water depth= 40 cm (offshore), 10 cm (over the bar); Incident wave height= 2 cm, period= 2.02 s.

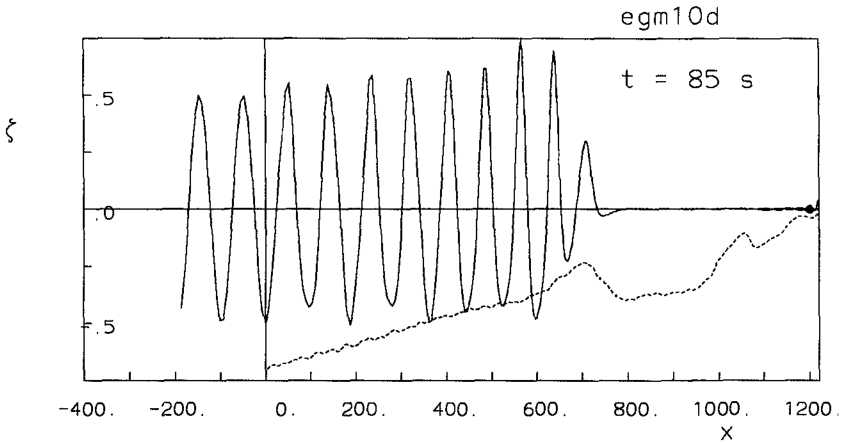

Figure 4: Propagation of a regular wave train towards the shore over the barred bathymetry near Egmond. Incident wave height= 1 m, period= 10 s. The bed profile is not shown to scale. Still-water depth at $x = 0$ is 11.4 m.

on a slope. However, much remains to be done in the formulation of a proper beaking mechanism in the model and making the model practically applicable to breaking waves.

Summary

In the present article, we have discussed two sets of evolution equations. One set of evolution equations, given by (7) and (8), is derived from an approximate Hamiltonian with two important properties: the kinetic-energy density is guaranteed to be positive definite for all wave heights across the entire wave spectrum as long as the local water depth is greater than zero and secondly the dispersion relationship in the limiting case of infinitesimal amplitude is exactly identical to that obtained from the classical linear theory of water waves. These two features are of great practical significance in ensuring stability of the model equations during propagations of an irregular train or of the generated higher harmonics and in representation of proper phase speed of all components. Numerical experiments with some critical test cases show that the model describes the nonlinear transformation during propagation of non-breaking waves over a varying depth satisfactorily. The performance of the model has been found to be superior to that of many Boussinesq-like models during an intercomparison study completed recently. We have experimented with including a higher-order nonlinear representation in the model equations. The experiments indicate that higher-order nonlinearity can improve the result of the Boussinesq-like approximations of fairly long, fairly low waves especially near the breaking point. However, a more systematic approach is necessary than what has been presented in order to ensure stability and conservation of energy of the system.

Acknowledgements

This work was commissioned by the National Insitute for Coastal and Marine Management/RIKZ of the Dutch Public Works Department. The work done by the first author as reported here was completed during his employment in Delft Hydraulics of the Netherlands.

References

Benjamin, T.B. & P.J. Olver (1982). Hamiltonian structure, symmetries and conservation laws for water waves, *J. Fluid Mech.*, 125: 137-185.

Craig, W. & M.D. Groves (1994). Hamiltonian long-wave approximations to the water-wave problem, *Wave Motion*, 19: 367-389.

Dingemans, M.W. (1994). Comparison of computations with Boussinesq-like models and laboratory measurements, MAST G8-M Note, H1684, 32 pp.

Dingemans, M.W. and A.C. Radder (1991). Use of Hamiltonian techniques in water wave propagation problems. In: *Nonlinear Water Waves Workshop*, University of Bristol, ed. D.H. Peregrine, pp 24-31.

Dingemans, M.W. & L.M. Merckelbach (1996). Boussinesq-like equations with higher-order frequency dispersion, in preparation.

Dingemans, M.W. & A.K. Otta (1996). Hamiltonian formulation of water waves: higher-order description, wave breaking formulation and numerical implementations. Report H782, Part 3, *Delft Hydraulics*, 69 pp.

Lund, J. and K.L. Bowers (1992). *Sinc Methods for Quadrature and Differential Equations*, SIAM, Philadelphia, 304 pp.

Luth, H.R., G. Klopman and N. Kitou (1994). Project 13G: Kinematics of waves breaking partially on an offshore bar; LDV measurements for waves with and without a net onshore current, Report H1573, *Delft Hydraulics*, 40 pp.

Madsen, P.A., R. Murray and O.R. Sorensen (1991). A new form of the Boussinesq equations with improved linear dispersion characteristics, *Coastal Engineering*, 15: 371-388.

Otta, A.K. and M.W. Dingemans (1994a). Hamiltonian formulation of water waves: 1D-formulation, numerical evaluations and examples. Report H782, *Delft Hydraulics*, 63pp.

Otta, A.K. and M.W. Dingemans (1994b). Hamiltonian formulation of water waves: formulation for weakly-reflecting boundary conditions, Report H782, part 2, *Delft Hydraulics*, 18pp.

Radder, A.C. (1992). An explicit Hamiltonian formulation of surface waves in water of finite depth, *J. Fluid Mech.*, 237: 435-455.

Radder, A.C. (1993). An integral criterion for wave breaking in deep and shallow water, *Personal note*, 5 pp.

Schäffer , H.A., P.A. Madsen and R. Deigaard (1993). A Boussinesq

model for waves breaking in shallow water, *Coastal Engineering*, 20: 185-202.
Tanaka, M. (1986). The stability of solitary waves, *Physics of Fluids*, 29: 650-655.