A new formulation of deterministic and stochastic evolution equations for three-wave interactions involving fully dispersive waves

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Abstract

This paper presents a new and more accurate set of deterministic evolution equations for three-wave interactions involving fully dispersive, weakly nonlinear, irregular, unidirectional waves. The equations are derived directly from the Laplace equation with leading order nonlinearity in the surface boundary conditions. It is demonstrated that previous fully dispersive formulations from the literature have used an inconsistent linear relation between the velocity potential and the surface elevation. As a consequence these formulations are accurate only in shallow water, while nonlinear transfer of energy is significantly underestimated for larger wave numbers. In the present work we correct this inconsistency. In addition to the improved deterministic formulation, we present improved stochastic evolution equations in terms of the energy spectrum and the bispectrum for unidirectional waves.

1. Introduction

Three-wave interactions (or triad interactions) generally play an important role in the nonlinear transformation of irregular waves in shallow or intermediate depth waters. Very often these phenomena can be described quite accurately by Boussinesq-type formulations either in terms of time-domain equations (see e.g. Madsen & Sørensen, 1993) or in terms of evolution equations for the spatial variation of the complex amplitudes at discrete frequencies (see e.g. Freilich & Guza, 1984; Madsen & Sørensen, 1993). In both cases the phase information is retained and we talk about deterministic formulations.

Recently, Herbers & Burton (1997) and Kofoed-Hansen & Rasmussen (1998) presented stochastic formulations derived from deterministic Boussinesq-type evolution equations. In their formulations the second- and third-order statistics of random, shoaling waves are described by a coupled set of evolution equations for the energy spectrum and the bispectrum.

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While Boussinesq-type formulations are generally more or less restricted by weak dispersion, the approach by Agnon et al. (1993) and Kaihatu & Kirby (1995) retains full dispersion. They presented deterministic evolution equations derived directly from the Laplace equation with leading order nonlinearity in the surface boundary conditions. On this basis Agnon & Sheremet (1997) derived stochastic evolution equations for the energy spectrum and the bispectrum.

One of the key arguments by Agnon et al. (1993) and Kaihatu & Kirby (1995) for deriving equations with full dispersion was the need for a nonlinear evolution equation describing the interaction processes all the way from deep to shallow water. Indeed, triad interaction models based on fully dispersive equations can be expected to be superior to those based on Boussinesq-type formulations for large wave numbers not only in terms of improved dispersion but also in terms of a higher accuracy in nonlinear transfer functions. This should generally result in more accurate estimates of higher-order statistics such as skewness and asymmetry.

On the other hand, this expectation of a superior accuracy relative to existing Boussinesq formulations, has not yet been demonstrated in the literature. In fact, Kaihatu & Kirby (1995, 1996) concluded that "the lowest order Boussinesq model, despite its shallow water formalism, yields skewness and asymmetry values closer to those of the experimental data than those of the fully dispersive model". This conclusion was indeed disappointing.

In the present work we shall demonstrate that while the previous formulations by Agnon et al. (1993), Kaihatu & Kirby (1995, 1996), and Agnon & Sheremet (1997) presented consistent nonlinear equations for the velocity potential, they invoked an inconsistent linear relation to obtain the corresponding equations for the surface elevation. We shall show that high accuracy for larger wave numbers is achieved only if a nonlinear transformation is invoked, and we shall derive a new set of fully dispersive evolution equations for the surface elevation. Section 2 contains a brief review of the derivation of deterministic evolution equations in terms of the velocity potential. In Section 3 these equations are converted into equations for the surface elevation using: 1) a linear transformation; 2) a second order transformation. In section 4, a coupled set of stochastic evolution equations for the energy spectrum and the bispectrum is presented. Model validation is given in Section 5, while summary and conclusions can be found in Section 6.

2. Deterministic evolution equations in terms of the velocity potential

In this section we give a brief outline of the derivation of deterministic evolution equations in terms of the velocity potential at the still water level. The equations include leading order nonlinearity and full dispersion. The derivation follows the work by Agnon et al. (1993) and Kaihatu & Kirby (1995).

We adopt a Cartesian co-ordinate system (x, z) with z measured upwards from the still water level. The fluid domain is bounded by the sea bed at $z = -h(x)$ and the free surface $z = \eta(x,t)$. The fluid is assumed incompressible and inviscid, and the flow is assumed to be irrotational. Dimensional quantities are retained with the understanding that leading order nonlinearity is $O(\varepsilon^2)$, where the nonlinearity parameter ε is defined by kA (k being the wave number and A the wave amplitude). After expanding the nonlinear free surface boundary conditions in Taylor series about $z=0$ and retaining terms to $O(\varepsilon^2)$, the truncated boundary value problem in terms of the velocity potential Φ reads,

$$\nabla^2\Phi+\Phi_{zz}=0; \quad -h\le z\le 0, \tag{1}$$

$$\Phi_z+\nabla h\cdot\nabla\Phi=0; \quad z=-h, \tag{2}$$

$$\eta_t-\Phi_z+\nabla\cdot(\eta\nabla\Phi)=O(\varepsilon^3); \quad z=0, \tag{3}$$

$$\Phi_t+g\eta+\eta\Phi_{tz}+\tfrac{1}{2}(\nabla\Phi)^2+\tfrac{1}{2}(\Phi_z)^2=O(\varepsilon^3); \qquad z=0, \tag{4}$$

where ∇ is the horizontal gradient operator. We note that the linearized dynamic free surface boundary condition, i.e.,

$$\Phi_t+g\eta=O(\varepsilon^2); \qquad z=0, \tag{5}$$

can consistently be applied to eliminate η in the nonlinear terms in (3) and (4) by which we get

$$\eta_t-\Phi_z-\frac{1}{g}\nabla\cdot(\Phi_t\nabla\Phi)=O(\varepsilon^3); \quad z=0, \tag{6}$$

$$\Phi_t+g\eta-\frac{1}{g}\Phi_t\Phi_{tz}+\tfrac{1}{2}(\nabla\Phi)^2+\tfrac{1}{2}(\Phi_z)^2=O(\varepsilon^3); \quad z=0. \tag{7}$$

Substituting (7) into (6) yields

$$\Phi_{tt}+g\Phi_z-\left[\frac{1}{g}\Phi_t\Phi_{tz}-\tfrac{1}{2}(\nabla\Phi)^2-\tfrac{1}{2}(\Phi_z)^2\right]_t+\nabla\cdot(\Phi_t\nabla\Phi)=O(\varepsilon^3); \quad z=0 \tag{8}$$

The system of equations (1), (2) and (8) is the starting point for the derivation of evolution equations for weakly nonlinear and fully dispersive water waves.

First, we express the surface elevation and the velocity potential as a linear superposition of unidirectional waves,

$$\eta(x,t)=\frac{1}{2}\sum_{p=1}^{\infty}A_p(x)\exp\left[i\left(\omega_p t-\int k_p dx\right)\right]+\quad c.c. \tag{9a}$$

$$\Phi(x,z,t)=\frac{1}{2}\sum_{p=1}^{\infty}f_p(z)\,\tilde{\Phi}_p(x)\exp\left[i\left(\omega_p t-\int k_p dx\right)\right]+\quad c.c. \tag{9b}$$

where $\omega_p=p\Delta\omega$ is the frequency, $\Delta\omega$ is the band-width in the Fourier representation, k_p is the wave-number satisfying the linear dispersion relation, A_p and $\tilde{\Phi}_p$ are the complex spatially varying Fourier amplitudes, f_p represents the vertical structure of the velocity potential and c.c. indicates the complex conjugate.

The specification of the vertical structure f_p is one of the key elements in the derivation of evolution equations for fully dispersive waves. Agnon et al. (1993) initially considered the influence of bound waves as well as free waves on the vertical structure of the potential, but eventually they ignored the effect of the bound wave structure on their equations. Here we simply adopt the procedure of Kaihatu & Kirby (1995) and assume a vertical structure dictated by linear theory, i.e.

$$f_p(z) = \frac{\cosh k_p(h+z)}{\cosh k_p h} \tag{10}$$

where k_p is obtained from

$$\omega_p^{\,2} = g k_p \tanh k_p h \tag{11}$$

The spatial variations of water depth, wave numbers and complex amplitudes are assumed to be weak, and consequently the formulation will include only first derivatives of these quantities, while no derivatives will be included in connection with nonlinear terms. The resulting evolution equations derived by Agnon et al. (1993) and by Kaihatu & Kirby (1995) can be expressed by

$$\frac{d\widetilde{\Phi}_p}{dx} = -\frac{\widetilde{\Phi}_p}{2c_{g,p}}\frac{dc_{g,p}}{dx} + \left(\sum_{m=1}^{p-1} \beta^{+} \widetilde{\Phi}_m \widetilde{\Phi}_{p-m} e^{i\Delta\psi^{+}} + 2\sum_{m=1}^{N-p} \beta^{-} \widetilde{\Phi}_m^{*} \widetilde{\Phi}_{p+m} e^{i\Delta\psi^{-}} \right) \tag{12a}$$

where

$$\beta^{\pm} \equiv \frac{1}{8g^2 c_{g,p}} \left(\frac{g^2}{\omega_p} \left(\omega_{p\mp m} k_m^2 \pm \omega_m k_{p\mp m}^2 \right) \pm 2g^2 k_m k_{p\mp m} + \omega_m^2 \omega_{p\mp m}^2 \mp \omega_p^2 \omega_m \omega_{p\mp m} \right) \tag{12b}$$

$$\Delta\psi^{\pm} = \int \delta^{\pm} dx \quad , \qquad \delta^{\pm} \equiv \left(k_p \mp k_m - k_{p\mp m} \right) \tag{12c}$$

and where c_g is the group velocity. The first term in the right-hand-side of (12a) represents linear shoaling, while the second/third terms represent the nonlinear super/sub-harmonic interactions.

3. Deterministic evolution equations in terms of the surface elevation

In comparison with the evolution equations derived from Boussinesq-type equations (e.g., Freilich and Guza, 1984; Madsen and Sørensen, 1993), the set given by (12a-c) has the potential of being applicable to a wider range of wave numbers as it incorporates full dispersion. However, in order to utilise this potential, it is important to make a consistent transformation from the velocity potential to the surface elevation. In the following we convert (12a-c) into evolution equations for the amplitudes of the surface elevation using two different approaches: a) A linear transformation (Section 3.1); b) A second order transformation (Section 3.2).

3.1 Using the linear relation between the velocity potential and the surface elevation

In this section we follow Kaihatu & Kirby (1995) and apply the linear approximation (5) in combination with the Fourier representation (9), which yields

$$\widetilde{\Phi}_p = \frac{ig}{\omega_p} A_p \tag{13}$$

By substituting (13) into (12a) we obtain

$$\frac{dA_p}{dx} = -\frac{A_p}{2c_{g,p}}\frac{dc_{g,p}}{dx} + i\left(\sum_{m=1}^{p-1}\tilde{\alpha}^{+}A_m A_{p-m}e^{i\Delta\psi^{+}} + 2\sum_{m=1}^{N-p}\tilde{\alpha}^{-}A_m^{*}A_{p+m}e^{i\Delta\psi^{-}}\right) \tag{14a}$$

with

$$\tilde{\alpha}^{\pm} \equiv \pm\frac{g\,\omega_p}{\omega_m\,\omega_{p\mp m}}\beta^{\pm} \tag{14b}$$

where $\beta^{\pm}$ is given by (12b). These are the deterministic evolution equations derived by Agnon et al. (1993), and Kaihatu & Kirby (1995). Also Agnon & Sheremet (1997) used these equations as the basis for their stochastic formulation.

It turns out that the use of the linear approximation (5) results in inaccuracies in the nonlinear transfer functions. This can easily be demonstrated by the following example: Let us consider a velocity potential (at z=0) determined from Stokes second-order theory for regular waves, i.e.,

$$\tilde{\Phi}(x,t) = \tilde{\Phi}_1 sin(kx-\omega t) + \tilde{\Phi}_2 sin(2kx-2\omega t) \tag{15}$$

If we apply the linear approximation (5) on (15) we obtain as a consequence the second order surface elevation

$$\eta(x,t) = A_1 cos(kx-\omega t) + G_L\frac{A_1^2}{h}cos(2kx-2\omega t) \tag{16}$$

where

$$G_L \equiv \frac{3}{4}kh\frac{cosh2kh}{coshkh\,sinh^3 kh} \tag{17}$$

This obviously deviates from Stokes reference solution, which reads

$$G_{Stokes} \equiv \frac{1}{4}kh\frac{coshkh}{sinh^3 kh}(2+cosh2kh) \tag{18}$$

A Taylor expansion of (17) and (18) yields

$$G_L \rightarrow \frac{3}{4k^2h^2}\left(1+k^2h^2-\frac{11}{15}k^4h^4+O(k^6h^6)\right)$$

$$G_{Stokes} \rightarrow \frac{3}{4k^2h^2}\left(1+\frac{2}{3}k^2h^2+\frac{7}{45}k^4h^4+O(k^6h^6)\right) \tag{19}$$

which shows that the two expressions converge in shallow water. For comparison we may also consider the transfer function corresponding to the Boussinesq formulation of Madsen & Sørensen (1993) i.e.

$$G_{Bous} = \frac{3}{4k^2h^2}\left(1+\frac{8}{45}k^2h^2\right) \tag{20}$$

Fig. 1 shows the variation with kh of G_L and G_{Bous} relative to the target solution G_{Stokes}. In both cases the nonlinearity is significantly underestimated for larger kh values. This lack of accuracy in nonlinear transfer shows up in the numerical calculations in Section 5.

3.2 Using the nonlinear relation between the potential and the surface elevation

In the following we shall use (7) to establish the second-order relation between the surface elevation and the velocity potential. To a first approximation (9b) and (10) yield

$$\nabla\Phi = -ik\Phi, \qquad \Phi_t = i\omega\Phi, \qquad \Phi_z = \frac{\omega^2}{g}\Phi, \qquad z = 0 \tag{21}$$

Hence by the use of (21), the Fourier transformation of (7) now yields

$$A_p = \frac{-i\omega_p}{g}\widetilde{\Phi}_p + 2\left(\sum_{m=1}^{p-1}\widetilde{\gamma}^{+}\widetilde{\Phi}_m\widetilde{\Phi}_{p-m}e^{i\Delta\psi^{+}} + 2\sum_{m=1}^{N-p}\widetilde{\gamma}^{-}\widetilde{\Phi}_m^{*}\widetilde{\Phi}_{p+m}e^{i\Delta\psi^{-}}\right) \tag{22a}$$

where

$$\widetilde{\gamma}^{\pm} \equiv \frac{1}{8g^3}\left[\pm g^2 k_m k_{p\mp m} + \omega_m^2\omega_{p\mp m}^2 \mp \omega_p^2\omega_m\omega_{p\mp m}\right] \tag{22b}$$

One possibility is to solve (22) along with the evolution equation (12) in order to calculate the local variation of the surface elevation (see e.g. Chen et al., 1997). A better option is, however, to invert (22a) by the use of successive approximations and to eliminate the velocity potential from the evolution equations. Thus, we apply the linear approximation (13) in the nonlinear terms of (22a) and obtain

$$\widetilde{\Phi}_p = \frac{ig}{\omega_p}A_p + 2i\left(\sum_{m=1}^{p-1}\gamma^{+}A_m A_{p-m}e^{i\Delta\psi^{+}} + 2\sum_{m=1}^{N-p}\gamma^{-}A_m^{*}A_{p+m}e^{i\Delta\psi^{-}}\right) \tag{23a}$$

where

$$\gamma^{\pm} \equiv \pm\frac{g^3}{\omega_p\omega_m\omega_{p\mp m}}\widetilde{\gamma}^{\pm} \tag{23b}$$

The next step is to differentiate (23a) with respect to x while retaining terms to $O(\varepsilon^2)$. Consistent with the derivation of (12a-c), we ignore, in connection with the nonlinear terms, spatial derivatives of the slowly varying amplitudes and of the group velocity. With this assumption the differentiation of (23a) yields

$$\frac{d\widetilde{\Phi}_p}{dx} = \frac{ig}{\omega_p}\frac{dA_p}{dx} - 2\left(\sum_{m=1}^{p-1}\delta^{+}\gamma^{+}A_m A_{p-m}e^{i\Delta\psi^{+}} + 2\sum_{m=1}^{N-p}\delta^{-}\gamma^{-}A_m^{*}A_{p+m}e^{i\Delta\psi^{-}}\right) \tag{24}$$

where $\delta^{\pm}$ is defined by (12c). Finally after substituting (23a-b) and (24) into (12a), we get

$$\frac{dA_p}{dx} = -\frac{A_p}{2c_{g,p}}\frac{dc_{g,p}}{dx} + i\left(\sum_{m=1}^{p-1}\alpha^{+}A_m A_{p-m}e^{i\Delta\psi^{+}} + 2\sum_{m=1}^{N-p}\alpha^{-}A_m^{*}A_{p+m}e^{i\Delta\psi^{-}}\right) \tag{25a}$$

with

$$\alpha^{\pm} \equiv \pm\frac{g\omega_p}{\omega_m\omega_{p\mp m}}\left(\beta^{\pm} - \frac{g\tilde{\gamma}^{\pm}}{c_{g,p}}\Gamma^{\pm}\right), \qquad \Gamma^{\pm} \equiv \frac{2\delta^{\pm}c_{g,p}}{\omega_p} \tag{25b}$$

which, by inserting $\beta^{\pm}$ and $\tilde{\gamma}^{\pm}$ from (12b) and (22b), can be expressed as

$$\begin{aligned}\alpha^{\pm} = \pm\frac{\omega_p}{8g\,c_{g,p}\omega_m\omega_{p\mp m}}\Big[&g^2\omega_p^{-1}\left(\omega_{p\mp m}k_m^2 \pm \omega_m k_{p\mp m}^2\right)\\ &\pm\left(2-\Gamma^{\pm}\right)g^2k_m k_{p\mp m} + \left(1-\Gamma^{\pm}\right)\left(\omega_m^2\omega_{p\mp m}^2 \mp \omega_p^2\omega_m\omega_{p\mp m}\right)\Big]\end{aligned} \tag{25c}$$

The new deterministic evolution equation (25) is the main result of this work. The dispersion characteristics of the model are dictated by fully-dispersive linear theory. The present evolution equation insures, through the new complex interaction coefficient $\alpha^{\pm}$, a higher accuracy in the nonlinear transfer function. We emphasize that the nonlinear transformation between the velocity potential and the surface elevation is retained in the parameter Γ. The formulation (14a-b), as used by Agnon et al (1993), Kaihatu & Kirby (1995), and Agnon & Sheremet (1997), corresponds to setting $\Gamma = 0$ in (25).

The importance of including the Γ-terms in (25) is illustrated in Fig. 2, which shows the ratio of $\alpha^{\pm}_{\Gamma=0}$ to $\alpha^{\pm}$ as a function of ω_m and $\omega_{p\mp m}$. The upper triangle in Fig. 2 illustrates the super-harmonic interactions while the lower triangle illustrates the sub-harmonic interactions. The second-harmonic interaction is represented by the diagonal line and this result agrees with Fig. 1. It can be concluded that neglecting Γ has a major effect on super-harmonics which are consequently significantly underestimated, while the sub-harmonics are less sensitive.

4. Stochastic evolution equations for the energy spectrum and the bispectrum

Stochastic evolution equations for the energy spectrum and for the complex bispectrum can be derived on the basis of the deterministic evolution equations given by (25). We follow the procedure as outlined e.g. by Agnon and Sheremet (1997), Herbers & Burton (1997), and Kofoed-Hansen & Rasmussen (1998): Firstly, we multiply equation (25a) by the conjugate of A_p; secondly, the conjugate of equation (25a) is multiplied by A_p; thirdly, the former is added to the latter and finally the result is ensemble averaged. This leads to

$$\frac{dE_p}{dx} = -\frac{E_p}{c_{g,p}}\frac{dc_{g,p}}{dx} - 2\left[\sum_{m=1}^{p-1}\alpha^{+}\Im\left(B^{+}_{m,p-m}\right) - 2\sum_{m=1}^{N-p}\alpha^{-}\Im\left(B^{-}_{m,p+m}\right)\right] \tag{26}$$

where

$$E_p \equiv \left\langle A_p A_p^{*}\right\rangle, \qquad B^{+}_{m,p-m} \equiv \left\langle A_p^{*}A_m A_{p-m}e^{i\Delta\psi^{+}}\right\rangle, \qquad B^{-}_{m,p+m} \equiv \left\langle A_p^{*}A_m^{*}A_{p+m}e^{i\Delta\psi^{-}}\right\rangle \tag{27}$$

and where $\Im$ denotes the imaginary part and $\langle \ldots. \rangle$ is the ensemble average operator. The right-hand-side contains the average of the third-order moment, the so-called bispectrum, B. In order to obtain a stochastic description of the effect of the nonlinear interactions one needs to go to higher-order moments, and evaluate the bispectrum. An evolution equation for the bispectrum is derived and the terms including the trispectrum, i.e., fourth-order statistical average, appear. In order to close this system of equations, the trispectrum is expressed as products of second-order averages (the socalled Gaussian closure), and we retain only products of terms with opposite-signed phases. The resulting evolution equations of the bispectrum can be written as follows

$$\frac{dB^{+}_{m,p-m}}{dx} = (i\delta^{+} + F^{+})\, B^{+}_{m,p-m} - 2i\left[\alpha^{+}_{p,m,p-m} E_m E_{p-m} + \alpha^{-}_{m,p-m,p} E_p E_{p-m} + \alpha^{-}_{p-m,m,p} E_p E_m\right] \tag{28}$$

$$\frac{dB^{-}_{m,p+m}}{dx} = (i\delta^{-} + F^{-})\, B^{-}_{m,p+m} - 2i\left[\alpha^{-}_{p,m,p+m} E_m E_{p+m} + \alpha^{-}_{m,p,p+m} E_p E_{p+m} + \alpha^{+}_{p+m,m,p} E_p E_m\right] \tag{29}$$

where we emphasise that the $\alpha^{\pm}$ defined by (25c) is actually a short hand notation for $\alpha^{\pm}_{p,m,p\mp m}$, while $F^{\pm}$ denotes the shoaling term defined by

$$F^{\pm} \equiv -\frac{1}{2}\left(\frac{1}{c_{g,p}}\frac{dc_{g,p}}{dx} + \frac{1}{c_{g,m}}\frac{dc_{g,m}}{dx} + \frac{1}{c_{g,p\mp m}}\frac{dc_{g,p\mp m}}{dx}\right) \tag{30}$$

We note that (26), (28) and (29) comprise a coupled set of stochastic evolution equations for the energy spectrum and the bispectrum. The formulation is identical to Agnon and Sheremet (1997) except for the inclusion of the Γ-terms in $\alpha^{\pm}$.

The stochastic model explicitly takes into account, via the bispectrum, the development of the phase correlation between wave triads due to nonlinearity. In this model, the bispectrum is required to calculate the effects of triad wave interactions on the wave spectrum evolution. It can also be used to calculate the overall third-order statistical parameters such as the skewness and asymmetry.

5. Model verification

In order to validate the models from Sections 3 and 4, we concentrate on the experimental data from Cox et al. (1991). This test case considers the shoaling (and breaking) of irregular waves on a plane beach. The water depth at the offshore boundary is 47 cm and the beach slope is 1:20. The surface elevations are measured at eleven locations (denoted WG1 to WG11) in still water depths of 47, 35, 30, 25, 20, 17.5, 15, 12.5, 10, 7.5, and 5 cm. The target spectrum is a Pierson-Moskowitz type with a peak frequency of 1.0 Hz and with a significant wave height of 6.5 cm. At the incoming boundary the peak frequency corresponds to fairly large wave numbers (kh=1.9), and in combination with the broad-banded spectral shape this allows significant energy on frequencies that are well into the deep water range. Hence this test is quite demanding for models incorporating only weak dispersion and it is well suited for checking the applicability of fully dispersive models.

First we apply the stochastic formulation (26)-(30) with $\Gamma = 0$, which is basically the model of Agnon & Sheremet (1997). We note that similar results (not shown here) can be obtained by using the deterministic evolution equations of Agnon et al. (1993) and Kaihatu & Kirby (1995). Fig. 3a shows the computed and measured values of the skewness, while the asymmetry is shown in Fig 4a. Two different simulations are shown corresponding to offshore boundary conditions based on the measured bispectrum and a zero bispectrum, respectively. With the zero bispectrum, the skewness starts off with a zero and stays at that level on most of the shoal. The computed skewness is clearly significantly underestimated in most of the model area. This is in agreement with the analysis shown in Figs. 1 and 2. On the other hand, using the measured bispectrum as input does not really improve the simulation. Although the skewness is now correct right at the boundary, the mismatch between this value and the nonlinearity sustained by the model equations introduces a local recurrence phenomenon with a sudden decrease in the skewness and an increase in the asymmetry. Further inshore the skewness values computed by the two different boundary conditions are not very different. The large discrepancies between the computed and measured values of the asymmetry (Fig 4a) further inshore are due the mechanism of wave breaking, which is absent in the models considered here. It is emphasised that it requires a rather sophisticated breaking formulation in order to capture the variation of the asymmetry in the surf zone. A few examples of frequency domain formulations of wave breaking are discussed by Chen et al. (1997). This topic is, however, outside the scope of the present work.

For reference we have included the result of using the stochastic Boussinesq model of Kofoed-Hansen & Rasmussen (1998). As seen in Figs. 3b and 4b the computed skewness and asymmetry are quite similar to the ones obtained in Figs. 3a and 4a, and again the significant underestimate of the skewness is in agreement with the analysis in Fig. 1.

Figs. 3c and 4c show the result of including the new Γ-terms in the stochastic formulation. From Fig. 3c we notice that the computed skewness is significantly improved. The best result is now obtained by using the measured bispectrum as input, but even with a zero initial bispectrum the skewness and asymmetry values quickly fall in line with the measurements after a short distance dominated by recurrence.

Finally, Fig. 5 shows the energy spectra of the surface elevation computed by the stochastic model (26)-(30) including and excluding the new Γ-terms. In both cases the measured bispectrum is applied at the offshore boundary. The spectra are shown at three locations: WG2 (h=0.35m), WG5 (h=0.20m), and WG8 (h=0.125m). With Γ=0 we notice that the lack of nonlinearity in the model results in an artificial release of higher harmonics and an overestimation of the high-frequency tail of the spectrum. On the other hand, the model results obtained with the new Γ-terms are generally in good agreement with the measured spectra.

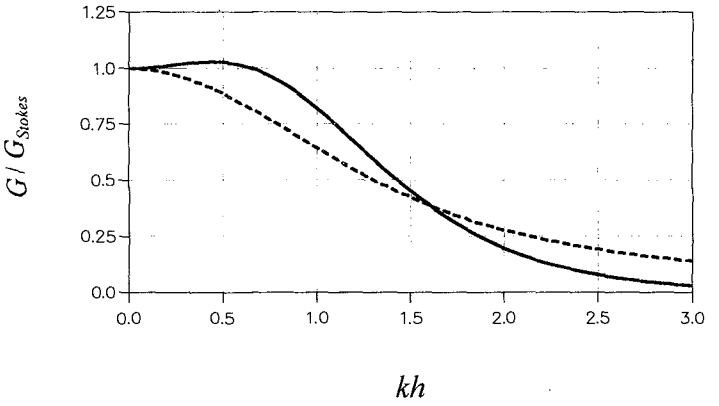

Figure 1. Transfer function for second harmonics. —— G_L / G_{Stokes} ; ----- G_{Bouss} / G_{Stokes}.

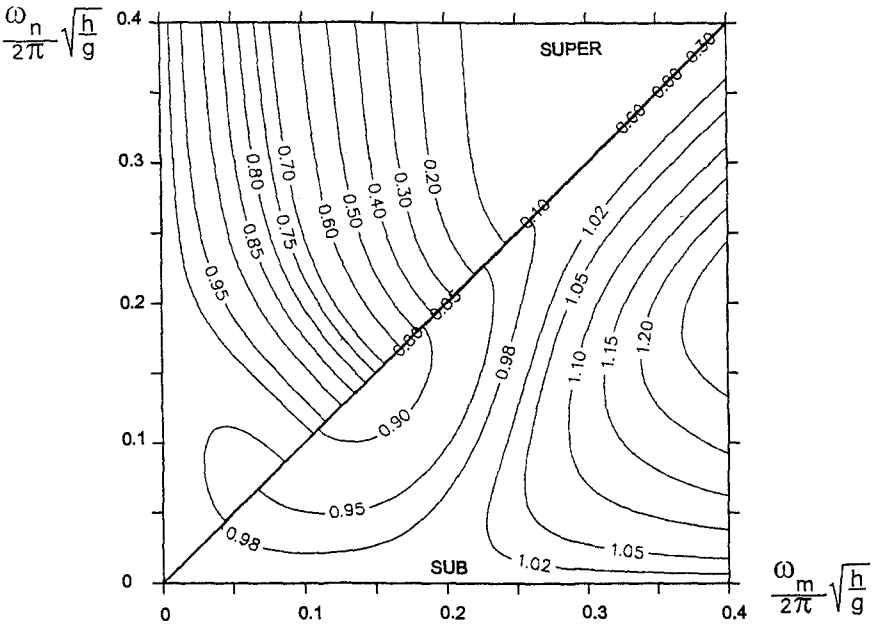

Figure 2. Isoline map of the ratio of $\alpha^{\pm}_{\Gamma=0}$ to $\alpha^{\pm}$ (from eq. 25c) as a function of the interacting frequencies ω_m and $\omega_n \equiv \omega_{p\mp m}$. Super-harmonic results shown above the diagonal; sub-harmonic results shown below it.

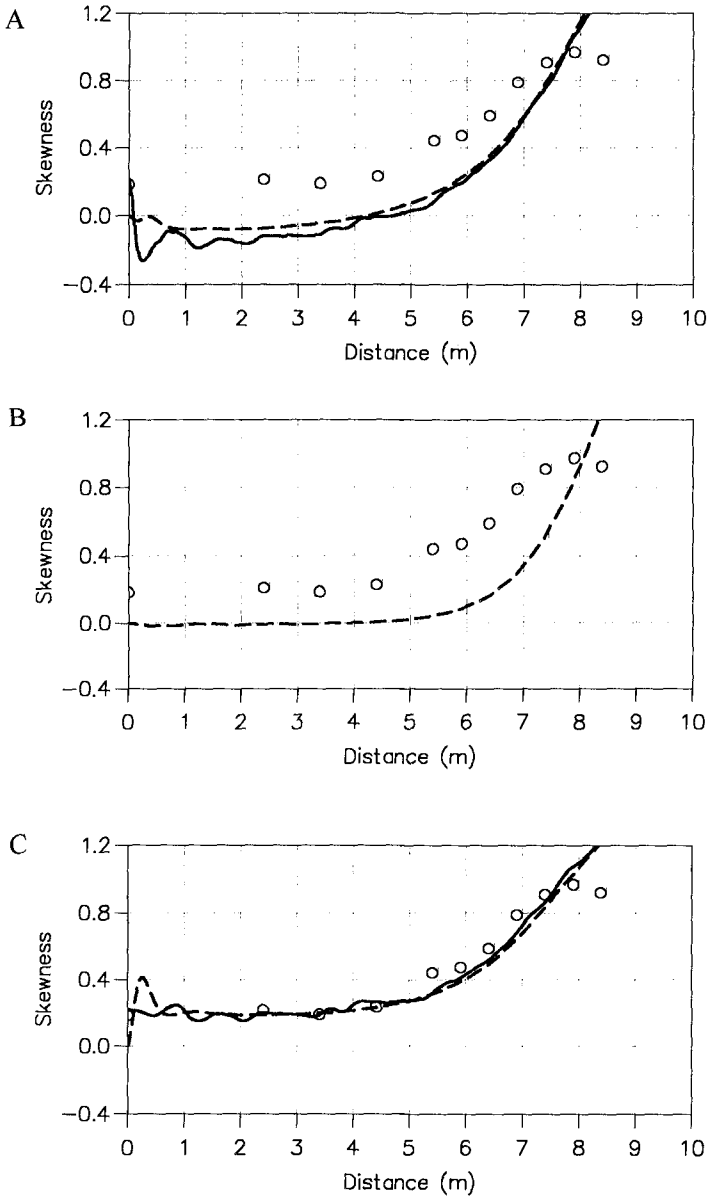

Figure 3. Spatial variations of the skewness during shoaling. Markers are measurements of Cox et al., 1991; ----- Computed with zero bispectrum at the boundary; —— Computed with measured bispectrum at the boundary. A) Stochastic model of Agnon & Sheremet (1997); B) Stochastic Boussinesq model of Kofoed-Hansen & Rasmussen (1998); C) Stochastic model based on the present formulation i.e. eqs (26)-(30) incl. the new Γ-terms.

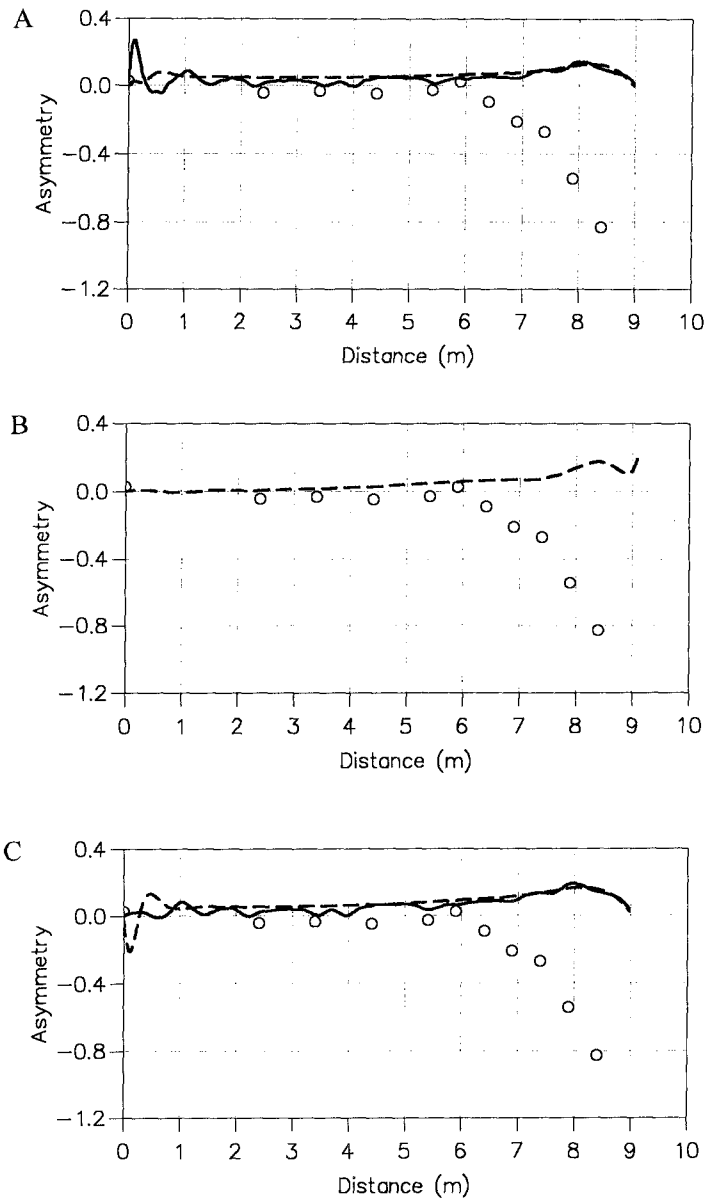

Figure 4. As Fig. 3, but for asymmetry instead of skewness.

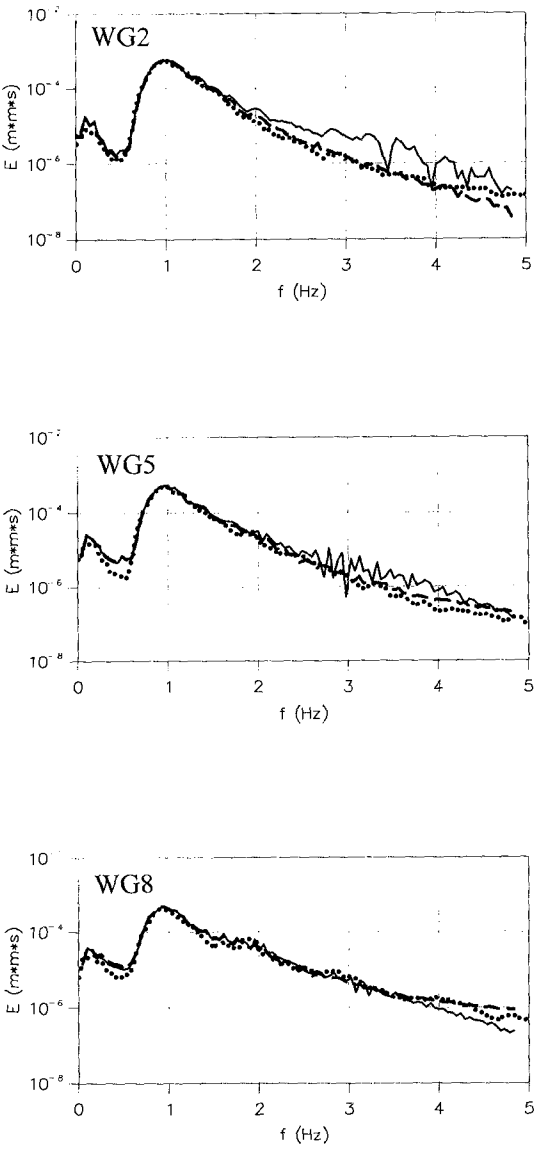

Figure 5. Energy spectra of the sea surface elevation at three locations WG2, WG5 and WG8. Computations made with the measured bispectrum at the boundary. —— Stochastic model of Agnon & Sheremet (1997); ----- Stochastic model based on the present formulation i.e. eqs (26)-(30) incl. the new Γ-terms; Dotted line indicates measurements of Cox et al. (1991).

6. Summary and conclusions

This paper presents a new and more accurate set of deterministic evolution equations for triad interactions involving fully dispersive, weakly nonlinear, irregular waves. Previous formulations from the literature have included consistent nonlinear evolution equations for the velocity potential, while an inconsistent linear relation has been invoked to obtain the corresponding surface elevation. This approximation is only valid in shallow water and as a result, the previous formulations significantly underestimate nonlinear energy transfer for larger wave numbers. As a consequence bound higher harmonics and nonlinear statistical measures such as the skewness are typically underestimated in these formulations.

In the present work we have corrected this inconsistency. Furthermore, in addition to the improved fully dispersive deterministic equations, we present a set of coupled stochastic evolution equations in terms of the energy spectrum and the bispectrum.

The influence of the new terms is demonstrated on a test case involving irregular waves in intermediate water depths. This case requires a combination of dispersion and nonlinearity and it is shown that existing formulations from the literature fail to predict the evolution of the energy spectrum and of the third-order statistics. A considerable improvement is found by including the new terms presented in this work. Further details and results can be found in Eldeberky & Madsen (1998).

Acknowledgements
The present work is financed by the Danish National Research Foundation. Their support is greatly appreciated.

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