MODULATION OF BICHROMATIC WAVE TRAIN IN A WAVE TANK

Motohiko Umeyama[1]

Abstract

The instability of bichromatic wave train in water of intermediate depth is investigated by means of a perturbation method. The analytical solution, treated as the result of an initial value problem, yields the modulation and disintegration of the wave train, caused by the interaction of two incident waves of finite amplitude. A laboratory study was also conducted to measure the surface elevations of bichromatic waves generated by several types of compound sinusoidal signals. The calculation using the third-order finite-amplitude wave equation is carried out, and its prediction is compared with the measured data obtained under several experimental conditions.

Introduction

For surface waves in water of arbitrary depth, small disturbances cause the instability as the waves propagate after a certain distance. Eventually the instability leads to loss of coherence, while the wave energy distributes over a broad spectrum. Benjamin and Feir(1967), using a perturbation approach, studied the instability of nonlinear deep-water waves, and showed that weakly nonlinear free-surface waves are inherently unstable to modulated periodic disturbances. In the following paper, Benjamin(1967) qualitatively explained the experimental evidence, known as side-band instability, using a second-order perturbation solution, and predicted that the side-band frequency components grow exponentially in time. Their results, however, are limited to the initial instability. Since then, much theoretical effort has been expended to search for an uniformly valid equation describing the behavior of the Stokes' wave train; for example, Lighthill(1967) examined the early stage of the nonlinear modulation of a wave packet using Whitham's theory(1965) based on an averaged Lagrangian, and Chu and Mei(1970) extended the Whitham's theory to derive the modulation equation for slowly varying Stokes' waves. On the

[1]Associate Professor, Department of Civil Engineering, Tokyo Metropolitan University, 1-1 Minamiohsawa, Hachiohji, Tokyo 192-0397, Japan

other hand, Zakharov(1968) derived the cubic Schrödinger equation which predicts the spatial evolution of the wave packet, and later, Zakharov and Shabat(1972) obtained an exact solution of the nonlinear Schrödinger equation. Their solutions predict the disintegration of initially uniform wave train into a definite number of wave groups. Yuen and Lake(1975) proved that Whitham's theory yields the same nonlinear Schrödinger equation when applied consistently to the order considered. All these solutions can be only applied to the case of the monochromatic wave train. The instability of the bichromatic wave train, having a set of discrete frequencies, has been regarded as the similar phenomenon which can be described by the evolution equation of the monochromatic wave. The instability by the bichromatic wave motion develops slowly as waves propagate from the wavemaker, and the growth rate of the instability becomes greater as the wave steepness increases. An example is found in a recent literature by Stansberg(1994), who observed a strong variation of wave pattern downstream the wave tank.

Despite the interest in the subject, few theoretically justifiable equation is available, to date, for predicting modulated surface displacements responsible for bichromatic waves. The purpose of this study is to investigate the modulation and disintegration of the bichromatic wave train in water of intermediate depth. A perturbation method is applied to the velocity potential and the vertical displacement of the water surface. The present equation predicts the surface displacements during the period of transition from a stationary state to a steady state in a finite distance from the wavemaker. The solution explains the instability of the periodic wave train mathematically as a result of the interaction of two incident waves propagating in the same direction.

Theory

Now we consider a two-dimensional irrotational wave motion bounded above by a free surface and below by a rigid horizontal bed, and assume the fluid to be inviscid and incompressible. The Eulerian velocity components of the water particles can be expressed in terms of a velocity potential $\Phi(x,y,t)$ such as

$$u(x,y,t) = \frac{\partial \Phi}{\partial x} \tag{1}$$

$$v(x,y,t) = \frac{\partial \Phi}{\partial y} \tag{2}$$

where $u(x,y,t)$ = horizontal velocity; $v(x,y,t)$ = vertical velocity; x = horizontal coordinate; and y = vertical coordinate measured from the still water level, respectively. For incompressible flow, the velocity potential must satisfy Laplace equation such as

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0. \tag{3}$$

The kinematical and dynamical boundary conditions at the free surface are

$$g\eta + \frac{\partial \Phi}{\partial t} + \frac{1}{2}\{(\frac{\partial \Phi}{\partial x})^2 + (\frac{\partial \Phi}{\partial y})^2\} = 0 \quad on\ y = \eta, \tag{4}$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial \Phi}{\partial x}\frac{\partial \eta}{\partial x} - \frac{\partial \Phi}{\partial y} = 0 \qquad on\ y = \eta, \tag{5}$$

where $\eta(x,t)$ = vertical displacement of the water surface measured from $y = 0$; g = acceleration due to gravity; and t = time. No fluid can pass perpendicular to the plane horizontal bottom, and therefore the bottom boundary condition is

$$\frac{\partial \Phi}{\partial y} = 0 \qquad\qquad on\ y = -h, \tag{6}$$

where h = water depth from the still water level.

In the finite-amplitude wave theory, the perturbation method is used to solve eqs.(3), (4), (5) and (6). The dependent variables are defined in terms of a power series, with successively smaller terms defined by a small perturbation parameter raised to a higher power in each succeeding term. Therefore, the velocity potential, vertical displacement and angular frequency for a third-order wave of finite amplitude can be expressed as

$$\Phi = \epsilon\Phi^{(1)} + \epsilon^2\Phi^{(2)} + \frac{1}{2}\epsilon^3\Phi^{(3)} + O(\epsilon^4), \tag{7}$$

$$\eta = \epsilon\eta^{(1)} + \epsilon^2\eta^{(2)} + \frac{1}{2}\epsilon^3\eta^{(3)} + O(\epsilon^4), \tag{8}$$

$$\sigma = \sigma^{(0)} + \epsilon\sigma^{(1)} + \frac{1}{2}\epsilon^2\sigma^{(2)} + O(\epsilon^3), \tag{9}$$

where ϵ = perturbation parameter; σ = angular frequency; and O()= order symbol. The superscripts $^{(1)}$, $^{(2)}$, and $^{(3)}$ denote quantities corresponding to the first-order, second-order and third-order perturbation solutions.

Goda and Abe(1968) developed a progressive wave theory for finite amplitude waves using a power series expansion. The results for a third-order progressive wave are as follows.

The perturbation functions of velocity potential are

$$\Phi^{(1)} = \frac{\sigma^{(0)}\coth k(y+h)}{k^2\sinh kh}\sin(kx - \sigma t), \tag{10}$$

$$\Phi^{(2)} = \frac{1}{k^2}\sigma^{(0)}\alpha_2^{(2)}\cosh 2k(y+h)\sin 2(kx - \sigma t), \tag{11}$$

$$\Phi^{(3)} = \frac{1}{k^2}\sigma^{(0)}\alpha_1^{(3)}\cosh 3k(y+h)\sin 3(kx - \sigma t), \tag{12}$$

in which

$$\alpha_2^{(2)} = \frac{3}{8}\frac{(c^4-1)}{\cosh 2kh},$$

$$\alpha_1^{(3)} = \frac{(c^2+3)(9c^5 - 22c^3 + 13c)}{32\cosh 3kh}.$$

The perturbation functions of vertical displacement are

$$\eta^{(1)} = \frac{1}{k}\cos(kx - \sigma t), \tag{13}$$

$$\eta^{(2)} = \frac{1}{k}\beta_1^{(2)}\cos 2(kx - \sigma t), \tag{14}$$

$$\eta^{(3)} = \frac{1}{k}\{\beta_1^{(3)}\cos(kx - \sigma t) + \beta_2^{(3)}\cos 3(kx - \sigma t)\}. \tag{15}$$

in which

$$\beta_1^{(2)} = \frac{3c^3 - c}{4},$$

$$\beta_1^{(3)} = \frac{3c^4 + 8c^2 - 9}{8},$$

$$\beta_2^{(3)} = \frac{3(9c^6 - 3c^4 + 3c^2 - 1)}{32}.$$

The perturbation functions of angular frequency are

$$\sigma^{(0)} = \sqrt{gk \tanh kh}, \tag{16}$$

$$\sigma^{(1)} = 0, \tag{17}$$

$$\sigma^{(2)} = k^2 \sigma^{(0)} \frac{9c^4 - 10c^2 + 9}{8}, \tag{18}$$

in which k = wave number; and $c = \coth kh$.

When two incident waves of finite amplitude interact, the free surface conditions in eqs.(4) and (5) cannot be satisfied by a simple superposition of two waves. The two original waves propagating in the same direction deform as a result of the interaction, and the difference due to the deformation results in the formation of nonlinear bichromatic waves. The combination of two (the first and second) incident wave trains with different wave amplitudes, a_I and a_{II} , and a pair of discrete wave periods, T_I and T_{II} is considered here. The solution of bichromatic waves, therefore, must be composed of the first third-order incident wave, second third-order incident wave, and resonance effect. The velocity potential and the surface displacement of bichromatic waves are expressed as

$$\Phi = \epsilon\Phi_I^{(1)} + \epsilon^2\Phi_I^{(2)} + \frac{1}{2}\epsilon^3\Phi_I^{(3)} + \lambda\epsilon\Phi_{II}^{(1)} + (\lambda\epsilon)^2\Phi_{II}^{(2)} + \frac{1}{2}(\lambda\epsilon)^3\Phi_{II}^{(3)} + \epsilon^2\Phi_F^{(2)} + \frac{1}{2}\epsilon^3\Phi_F^{(3)}, \tag{19}$$

$$\eta = \epsilon\eta_I^{(1)} + \epsilon^2\eta_I^{(2)} + \frac{1}{2}\epsilon^3\eta_I^{(3)} + \lambda\epsilon\eta_{II}^{(1)} + (\lambda\epsilon)^2\eta_{II}^{(2)} + \frac{1}{2}(\lambda\epsilon)^3\eta_{II}^{(3)} + \epsilon^2\eta_F^{(2)} + \frac{1}{2}\epsilon^3\eta_F^{(3)}, \tag{20}$$

in which

$$\epsilon = a_I k_I, \quad and \quad \lambda = \frac{a_{II} k_{II}}{a_I k_I}.$$

In these equations, the subscripts I, II, and F denote the first incident wave, second incident wave, and resonance effect, respectively.

The angular frequency of the third-order incident wave increases with increasing wave amplitude if the wave number is given. For bichromatic waves, therefore, the angular frequency of each incident wave is disturbed in order to obtain the correction terms. Taking into account the secondary effect, the perturbations of angular frequencies are expressed as

$$\sigma_I = \sigma_I^{(0)} + \frac{1}{2}\epsilon^2(\sigma_I^{(2)} + \sigma_{IF}), \tag{21}$$

$$\sigma_{II} = \sigma_{II}^{(0)} + \frac{1}{2}\epsilon^2(\lambda\sigma_{II}^{(2)} + \sigma_{IIF}), \tag{22}$$

where σ_{IF} and σ_{IIF} denote the third-order angular frequencies induced by the interaction of the first and second incident waves, respectively. In the above equations, the first and third order perturbation frequencies are already given in eqs.(16) and (18).

Now, a generalized theory for a bichromatic wave system is formulated to the third order. The dynamical and kinematical free-surface boundary conditions are used to obtain linear partial differential equations for each order of the approximation. In formulating the expressions, it is convenient to replace the nonlinear free-surface boundary conditions in eqs.(4) and (5) by conditions to be satisfied about $y=0$ instead of $y=\eta$. Expanding eqs.(4) and (5) into a Taylor series in y, and substituting eqs.(19), (20), (21) and (22) into these equations, the boundary conditions for the second and third powers are as follows.

The second-order equations are :

$$g\eta_F^{(2)}+\frac{\partial\Phi_F^{(2)}}{\partial t}=-\lambda(\frac{\sigma_I^{(0)}}{\sigma_I}\eta_{II}^{(1)}\frac{\partial^2\Phi_I^{(1)}}{\partial t\partial y}+\frac{\sigma_{II}^{(0)}}{\sigma_{II}}\eta_I^{(1)}\frac{\partial^2\Phi_{II}^{(1)}}{\partial t\partial y}+\frac{\partial\Phi_I^{(1)}}{\partial x}\frac{\partial\Phi_{II}^{(1)}}{\partial x}+\frac{\partial\Phi_I^{(1)}}{\partial y}\frac{\partial\Phi_{II}^{(1)}}{\partial y}),\quad(23)$$

$$\frac{\partial\Phi_F^{(2)}}{\partial y}-\frac{\partial\eta_F^{(2)}}{\partial t}=\lambda(\frac{\partial\eta_{II}^{(1)}}{\partial x}\frac{\partial\Phi_I^{(1)}}{\partial x}+\frac{\partial\eta_I^{(1)}}{\partial x}\frac{\partial\Phi_{II}^{(1)}}{\partial x}-\eta_{II}^{(1)}\frac{\partial^2\Phi_I^{(1)}}{\partial y^2}-\eta_I^{(1)}\frac{\partial^2\Phi_{II}^{(1)}}{\partial y^2}).\quad(24)$$

The third-order equations for $[\lambda]$ are :

$$\begin{aligned}\frac{1}{2}(g\eta_F^{(3)} &+ \frac{\partial\Phi_F^{(3)}}{\partial t}+\lambda\frac{\sigma_{IIF}}{\sigma_{II}}\frac{\partial\Phi_{II}^{(1)}}{\partial t})\\
= &- \lambda\frac{\sigma_{II}^{(0)}}{\sigma_{II}}\eta_I^{(2)}\frac{\partial^2\Phi_{II}^{(1)}}{\partial t\partial y}-\lambda\frac{\sigma_I^{(0)}}{\sigma_I}\eta_{II}^{(1)}\frac{\partial^2\Phi_I^{(2)}}{\partial t\partial y}-\frac{\sigma_I^{(0)}}{\sigma_I}\eta_F^{(2)}\frac{\partial^2\Phi_I^{(1)}}{\partial t\partial y}-\eta_I^{(1)}\frac{\partial^2\Phi_F^{(2)}}{\partial t\partial y}\\
&- \lambda\frac{\sigma_I^{(0)}}{\sigma_I}\eta_I^{(1)}\eta_{II}^{(1)}\frac{\partial^3\Phi_I^{(1)}}{\partial t\partial y^2}-\frac{\lambda}{2}\frac{\sigma_{II}^{(0)}}{\sigma_{II}}\eta_I^{(1)2}\frac{\partial^3\Phi_{II}^{(1)}}{\partial t\partial y^2}-\lambda\eta_I^{(1)}\frac{\partial\Phi_I^{(1)}}{\partial x}\frac{\partial^2\Phi_{II}^{(1)}}{\partial x\partial y}-\lambda\eta_{II}^{(1)}\frac{\partial\Phi_I^{(1)}}{\partial x}\frac{\partial^2\Phi_I^{(1)}}{\partial x\partial y}\\
&- \lambda\eta_I^{(1)}\frac{\partial\Phi_{II}^{(1)}}{\partial x}\frac{\partial^2\Phi_I^{(1)}}{\partial x\partial y}-\lambda\eta_I^{(1)}\frac{\partial\Phi_I^{(1)}}{\partial y}\frac{\partial^2\Phi_{II}^{(1)}}{\partial y^2}-\lambda\eta_{II}^{(1)}\frac{\partial\Phi_I^{(1)}}{\partial y}\frac{\partial^2\Phi_I^{(1)}}{\partial y^2}-\lambda\eta_I^{(1)}\frac{\partial\Phi_{II}^{(1)}}{\partial y}\frac{\partial^2\Phi_I^{(1)}}{\partial y^2}\\
&- \lambda\frac{\partial\Phi_{II}^{(1)}}{\partial x}\frac{\partial\Phi_I^{(2)}}{\partial x}-\lambda\frac{\partial\Phi_{II}^{(1)}}{\partial y}\frac{\partial\Phi_I^{(2)}}{\partial y}-\frac{\partial\Phi_I^{(1)}}{\partial x}\frac{\partial\Phi_F^{(2)}}{\partial x}-\frac{\partial\Phi_I^{(1)}}{\partial y}\frac{\partial\Phi_F^{(2)}}{\partial y},\end{aligned}\quad(25)$$

$$\begin{aligned}\frac{1}{2}(\frac{\partial\Phi_F^{(3)}}{\partial y} &- \frac{\partial\eta_F^{(3)}}{\partial t}-\lambda\frac{\sigma_{IIF}}{\sigma_{II}}\frac{\partial\eta_{II}^{(1)}}{\partial t})\\
= &- \lambda\eta_I^{(2)}\frac{\partial^2\Phi_{II}^{(1)}}{\partial y^2}-\lambda\eta_{II}^{(1)}\frac{\partial^2\Phi_I^{(2)}}{\partial y^2}-\eta_F^{(2)}\frac{\partial^2\Phi_I^{(1)}}{\partial y^2}-\eta_I^{(1)}\frac{\partial^2\Phi_F^{(2)}}{\partial y^2}\\
&- \lambda\eta_I^{(1)}\eta_{II}^{(1)}\frac{\partial^3\Phi_{II}^{(1)}}{\partial y^3}-\frac{\lambda}{2}\eta_I^{(1)2}\frac{\partial^3\Phi_{II}^{(1)}}{\partial y^3}+\lambda\frac{\partial\eta_I^{(2)}}{\partial x}\frac{\partial\Phi_{II}^{(1)}}{\partial x}+\lambda\frac{\partial\eta_{II}^{(1)}}{\partial x}\frac{\partial\Phi_I^{(2)}}{\partial x}\\
&+ \lambda\frac{\partial\eta_F^{(2)}}{\partial x}\frac{\partial\Phi_I^{(1)}}{\partial x}+\lambda\frac{\partial\eta_I^{(1)}}{\partial x}\frac{\partial\Phi_F^{(2)}}{\partial x}+\lambda\eta_I^{(1)}\frac{\partial\eta_I^{(1)}}{\partial x}\frac{\partial^2\Phi_{II}^{(1)}}{\partial x\partial y}+\lambda\eta_{II}^{(1)}\frac{\partial\eta_I^{(1)}}{\partial x}\frac{\partial^2\Phi_I^{(1)}}{\partial x\partial y}\\
&+ \lambda\eta_I^{(1)}\frac{\partial\eta_{II}^{(1)}}{\partial x}\frac{\partial^2\Phi_I^{(1)}}{\partial x\partial y}.\end{aligned}\quad(26)$$

The third-order equations for $[\lambda^2]$ are :

$$\frac{1}{2\lambda}(g\eta_F^{(3)} + \frac{\partial \Phi_F^{(3)}}{\partial t} + \frac{\sigma_{IF}}{\sigma_I}\frac{\partial \Phi_I^{(1)}}{\partial t}) = \cdots, \tag{27}$$

$$\frac{1}{2\lambda}(\frac{\partial \Phi_F^{(3)}}{\partial y} - \frac{\partial \eta_F^{(3)}}{\partial t} - \frac{\sigma_{IF}}{\sigma_I}\frac{\partial \eta_I^{(1)}}{\partial t}) = \cdots. \tag{28}$$

The right hand sides of eqs.(27) and (28) are same as those of eqs.(25) and (26) except for the subscripts I and II which are switched.

By differentiating eq.(23) with respect to t, eliminating η_{F1} from eqs.(23) and (24), and substituting the expressions for $\Phi_I^{(1)}$, $\Phi_{II}^{(1)}$, $\eta_I^{(1)}$, and $\eta_{II}^{(1)}$ into the combined equation, the free-surface boundary condition on $y = 0$ is given by

$$g\frac{\partial \Phi_F^{(2)}}{\partial y} + \frac{\partial^2 \Phi_F^{(2)}}{\partial t^2} = \frac{\lambda}{2k_I k_{II}}\{(\gamma_{11} + \gamma_{12})\sin(\chi_I - \chi_{II}) + (\gamma_{21} - \gamma_{22})\sin(\chi_I + \chi_{II})\}, \tag{29}$$

in which

$$\begin{aligned}
\gamma_{11} &= (\sigma_I - \sigma_{II})\{\sigma_I^{(0)^2} + \sigma_{II}^{(0)^2} - \sigma_I^{(0)}\sigma_{II}^{(0)}(1 + c_I c_{II})\}, \\
\gamma_{12} &= (\frac{k_{II}}{k_I}\sigma_I^{(0)}\sigma_I^2 + \sigma_{II}^{(0)}\sigma_{II}^2 - \frac{k_I}{k_{II}}\sigma_{II}^{(0)}\sigma_{II}^2 - \sigma_I^{(0)}\sigma_I^2), \\
\gamma_{21} &= (\sigma_I + \sigma_{II})\{\sigma_I^{(0)^2} + \sigma_{II}^{(0)^2} + \sigma_I^{(0)}\sigma_{II}^{(0)}(1 - c_I c_{II})\}, \\
\gamma_{22} &= (\frac{k_{II}}{k_I}\sigma_I^{(0)}\sigma_I^2 + \sigma_{II}^{(0)}\sigma_{II}^2 + \frac{k_I}{k_{II}}\sigma_{II}^{(0)}\sigma_{II}^2 + \sigma_I^{(0)}\sigma_I^2),
\end{aligned}$$

where $\chi_I = k_I x - \sigma_I t$; and $\chi_{II} = k_{II} x - \sigma_{II} t$. Eqs.(3), (6) and (29) are satisfied by

$$\Phi_F^{(2)} = \frac{\lambda}{k_I k_{II}}\{\alpha_{F1}^{(2)}\sin(\chi_I - \chi_{II}) + \alpha_{F2}^{(2)}\sin(\chi_I + \chi_{II})\}, \tag{30}$$

in which

$$\begin{aligned}
\alpha_{F1}^{(2)} &= \frac{\gamma_{11} + \gamma_{12}}{2\cosh(k_I - k_{II})h}\frac{\cosh(k_I - k_{II})(y+h)}{\omega'_{I-II} - (\sigma_I - \sigma_{II})^2}, \\
\alpha_{F2}^{(2)} &= \frac{\gamma_{21} + \gamma_{22}}{2\cosh(k_I - k_{II})h}\frac{\cosh(k_I + k_{II})(y+h)}{\omega'_{I+II} - (\sigma_I + \sigma_{II})^2}.
\end{aligned}$$

The corresponding surface displacement is obtained by substituting eq.(30) into eq.(23). Finally the second-order surface displacement by the resonance effect are obtained as

$$\eta_F^{(2)} = \frac{\lambda}{2gk_I k_{II}}\{\beta_{F1}^{(2)}\cos(\chi_I - \chi_{II}) + \beta_{F2}^{(2)}\cos(\chi_I + \chi_{II})\}, \tag{31}$$

in which

$$\begin{aligned}
\beta_{F1}^{(2)} &= \frac{\omega'_{I-II}\gamma_{11} - (\sigma_I - \sigma_{II})^2\gamma_{12}}{(\sigma_I - \sigma_{II})\{\omega'_{I-II} - (\sigma_I - \sigma_{II})^2\}}, \\
\beta_{F2}^{(1)} &= \frac{\omega'_{I+II}\gamma_{21} + (\sigma_I + \sigma_{II})^2\gamma_{12}}{(\sigma_I + \sigma_{II})\{\omega'_{I+II} - (\sigma_I + \sigma_{II})^2\}}, \\
\omega'_{I-II} &= g(k_I - k_{II})\tanh(k_I - k_{II})h, \\
\omega'_{I+II} &= g(k_I + k_{II})\tanh(k_I + k_{II})h.
\end{aligned}$$

The second-order surface displacement by the resonance effect have two components with wave numbers and frequencies consisting of the sums and differences of those of the first and second incident waves. These amplitudes are bounded in time and their magnitude is dependent on the water depth.

To find the third-order velocity potential $\Phi_F^{(3)}$ and surface displacement $\eta_F^{(3)}$, the same procedure is followed in the second-order approximation. Eliminating $\eta_F^{(3)}$ from eqs.(25) and (26), the combined free-surface boundary condition on $y=0$ is obtained for $[\lambda]$ as

$$\begin{aligned} g\frac{\partial \Phi_F^{(3)}}{\partial y} &+ \frac{\partial \Phi_F^{(3)}}{\partial t^2} - \frac{2\lambda\sigma_{IIF}}{k_{II}}\left(\frac{\sigma_{II}\sigma_{II}^{(0)}c_{II}}{k_{II}}+\frac{g}{2}\right)\sin\chi_{II} \\ &= \frac{2\lambda}{k_I k_{II}}[\{(2\sigma_I-\sigma_{II})b_1+gb_4\}\sin(2\chi_I-\chi_{II}) \\ &+ \{(2\sigma_I+\sigma_{II})b_2+gb_5\}\sin(2\chi_I+\chi_{II})+(\sigma_{II}b_3+gb_6)\sin\chi_{II}], \end{aligned} \tag{32}$$

in which

$$\begin{aligned} b_1 &= \frac{\sigma_{II}^{(0)^2}\beta_{I1}^{(2)}}{2}+2\sigma_I^{(0)^2}\alpha_{I2}^{(2)}\sinh 2k_Ih-\sigma_I^{(0)}\sigma_{II}^{(0)}\alpha_{I2}^{(2)}(c_{II}\cosh 2k_Ih+\sinh 2k_Ih) \\ &+ \frac{\sigma_{II}^{(0)^2}k_{II}c_{II}}{8k_I}-\frac{\sigma_I^{(0)}\sigma_{II}^{(0)}k_{II}(c_I+c_{II})}{4k_I}-\frac{\sigma_I^{(0)}\sigma_{II}^{(0)}c_{II}}{2}+\frac{\sigma_I^{(0)^2}c_I}{4} \\ &+ \frac{\sigma_I^{(0)^2}\{\omega'_{I-II}\gamma_{11}+(\sigma_I-\sigma_{II})^2\gamma_{12}\}}{4gk_I(\sigma_I-\sigma_{II})\{\omega'_{I-II}-(\sigma_I-\sigma_{II})^2\}}+\frac{(\gamma_{11}+\gamma_{12})(\sigma_I-\sigma_{II})\omega'_{I-II}}{4gk_I\{\omega'_{I-II}-(\sigma_I-\sigma_{II})^2\}} \\ &- \frac{\sigma_I^{(0)}c_I(\gamma_{11}+\gamma_{12})(k_I-k_{II})}{4\{\omega'_{I-II}-(\sigma_I-\sigma_{II})^2\}}+\frac{\sigma_I^{(0)}(\gamma_{21}-\gamma_{22})\omega'_{I+II}}{4gk_I\{\omega'_{I+II}-(\sigma_I+\sigma_{II})^2\}} \end{aligned}$$

$$\begin{aligned} b_2 &= \frac{\sigma_{II}^{(0)^2}\beta_{I1}^{(2)}}{2}+2\sigma_I^{(0)^2}\alpha_{I2}^{(2)}\sinh 2k_Ih-\sigma_I^{(0)}\sigma_{II}^{(0)}\alpha_{I2}^{(2)}(c_{II}\cosh 2k_Ih-\sinh 2k_Ih) \\ &+ \frac{\sigma_{II}^{(0)^2}k_{II}c_{II}}{8k_I}-\frac{\sigma_I^{(0)}\sigma_{II}^{(0)}k_{II}(c_I-c_{II})}{4k_I}+\frac{\sigma_I^{(0)^2}c_I}{4} \\ &+ \frac{\sigma_I^{(0)^2}\{\omega'_{I+II}\gamma_{21}+(\sigma_I+\sigma_{II})^2\gamma_{22}\}}{4gk_I(\sigma_I+\sigma_{II})\{\omega'_{I+II}-(\sigma_I+\sigma_{II})^2\}}+\frac{(\gamma_{21}-\gamma_{22})(\sigma_I+\sigma_{II})\omega'_{I+II}}{4gk_I\{\omega'_{I+II}-(\sigma_I+\sigma_{II})^2\}} \\ &- \frac{\sigma_I^{(0)}c_I(\gamma_{21}-\gamma_{22})(k_I+k_{II})}{4\{\omega'_{I+II}-(\sigma_I+\sigma_{II})^2\}}+\frac{\sigma_I^{(0)}(\gamma_{11}+\gamma_{12})\omega'_{I-II}}{4gk_I\{\omega'_{I-II}-(\sigma_I-\sigma_{II})^2\}} \end{aligned}$$

$$\begin{aligned} b_3 &= \frac{\sigma_{II}^{(0)^2}k_{II}c_{II}}{4k_I}-\frac{\sigma_I^{(0)}\sigma_{II}^{(0)}k_{II}c_I}{2k_I}-\frac{\sigma_I^{(0)}\sigma_{II}^{(0)}c_{II}}{2}-\frac{\sigma_{II}^{(0)^2}c_I}{2} \\ &+ \frac{\sigma_I^{(0)^2}\{\omega'_{I-II}\gamma_{11}+(\sigma_I-\sigma_{II})^2\gamma_{12}\}}{4gk_I(\sigma_I-\sigma_{II})\{\omega'_{I-II}-(\sigma_I-\sigma_{II})^2\}}+\frac{(\gamma_{11}+\gamma_{12})(\sigma_I-\sigma_{II})\omega'_{I-II}}{4gk_I\{\omega'_{I-II}-(\sigma_I-\sigma_{II})^2\}} \\ &- \frac{\sigma_I^{(0)}c_I(\gamma_{11}+\gamma_{12})(k_I-k_{II})}{4\{\omega'_{I-II}-(\sigma_I-\sigma_{II})^2\}}-\frac{\sigma_I^{(0)}(\gamma_{11}+\gamma_{12})\omega'_{I-II}}{4gk_I\{\omega'_{I-II}-(\sigma_I-\sigma_{II})^2\}} \end{aligned}$$

$$
\begin{aligned}
&+ \frac{\sigma_I^{(0)^2}\{\omega'_{I+II}\gamma_{21} - (\sigma_I+\sigma_{II})^2\gamma_{22}\}}{4gk_I(\sigma_I+\sigma_{II})\{\omega'_{I+II} - (\sigma_I+\sigma_{II})^2\}} + \frac{(\gamma_{21}-\gamma_{22})(\sigma_I+\sigma_{II})\omega'_{I+II}}{4gk_I\{\omega'_{I+II} - (\sigma_I+\sigma_{II})^2\}} \\
&- \frac{\sigma_I^{(0)} c_I(\gamma_{21}-\gamma_{22})(k_I+k_{II})}{4\{\omega'_{I+II} - (\sigma_I+\sigma_{II})^2\}} - \frac{\sigma_I^{(0)}(\gamma_{21}-\gamma_{22})\omega'_{I+II}}{4gk_I\{\omega'_{I+II} - (\sigma_I+\sigma_{II})^2\}}
\end{aligned}
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\begin{aligned}
b_4 &= -\frac{\beta_{I1}^{(2)}\sigma_{II}^{(0)} k_{II} c_{II}}{2} - \alpha_{I2}^{(2)} k_{II}\cosh 2k_I h - 2\alpha_{I2}^{(2)}\sigma_I^{(2)} k_I \cosh 2k_I h - \beta_{I1}^{(2)}\sigma_I^{(0)} k_I c_I \\
&- \frac{\sigma_{II}^{(0)} k_{II}}{8k_I} - \frac{\sigma_I^{(0)} k_{II}}{4} - \frac{\sigma_I^{(0)} k_I}{2} - \frac{\sigma_{II}^{(0)} k_{II}}{4} \\
&- \frac{\sigma_I^{(0)^2} c_I\{\omega'_{I+II}\gamma_{21} - (\sigma_I+\sigma_{II})^2\gamma_{22}\}}{4g(\sigma_I+\sigma_{II})\{\omega'_{I+II} - (\sigma_I+\sigma_{II})^2\}} - \frac{\sigma_I^{(0)} c_I(k_I+k_{II})\{\omega'_{I+II}\gamma_{21} - (\sigma_I+\sigma_{II})^2\gamma_{22}\}}{4gk_I(\sigma_I+\sigma_{II})\{\omega'_{I+II} - (\sigma_I+\sigma_{II})^2\}} \\
&- \frac{(\gamma_{21}-\gamma_{22})(k_I+k_{II})}{4\{\omega'_{I+II} - (\sigma_I+\sigma_{II})^2\}} - \frac{(\gamma_{11}+\gamma_{12})(k_I-k_{II})^2}{4k_I\{\omega'_{I+II} - (\sigma_I-\sigma_{II})^2\}}
\end{aligned}
$$

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\begin{aligned}
b_5 &= \frac{\beta_{I1}^{(2)}\sigma_{II}^{(1)} k_{II} c_{II}}{2} + \alpha_{I2}^{(2)} k_{II}\cosh 2k_I h - 2\alpha_{I2}^{(2)}\sigma_I^{(0)} k_I \cosh 2k_I h - \beta_{I1}^{(2)}\sigma_I^{(0)} k_I c_{II} \\
&+ \frac{\sigma_{II}^{(0)} k_{II}^2}{8k_I} + \frac{\sigma_I^{(0)} k_{II}}{4} - \frac{\sigma_I^{(0)} k_I}{2} - \frac{\dot\sigma_{II}^{(0)} k_{II}}{4} \\
&- \frac{\sigma_I^{(0)} c_I\{\omega'_{I-II}\gamma_{11} + (\sigma_I-\sigma_{II})^2\gamma_{12}\}}{4g(\sigma_I-\sigma_{II})\{\omega'_{I-II} - (\sigma_I-\sigma_{II})^2\}} - \frac{\sigma_I^{(0)} c_I(k_I-k_{II})\{\omega'_{I-II}\gamma_{11} + (\sigma_I-\sigma_{II})^2\gamma_{12}\}}{4gk_I(\sigma_I-\sigma_{II})\{\omega'_{I-II} - (\sigma_I-\sigma_{II})^2\}} \\
&- \frac{(\gamma_{11}+\gamma_{12})(k_I-k_{II})}{4\{\omega'_{I-II} - (\sigma_I-\sigma_{II})^2\}} - \frac{(\gamma_{21}-\gamma_{22})(k_I+k_{II})^2}{4k_I\{\omega'_{I+II} - (\sigma_I+\sigma_{II})^2\}}
\end{aligned}
$$

$$
\begin{aligned}
b_6 &= -\frac{\sigma_{II}^{(0)} k_{II}^2}{4k_I} - \frac{\sigma_I^{(0)} k_{II}}{2} \\
&- \frac{\sigma_I^{(0)} c_I\{\omega'_{I-II}\gamma_{11} + (\sigma_I-\sigma_{II})^2\gamma_{12}\}}{4g(\sigma_I-\sigma_{II})\{\omega'_{I-II} - (\sigma_I-\sigma_{II})^2\}} - \frac{(\gamma_{11}+\gamma_{12})(k_I-k_{II})^2}{4k_I\{\omega'_{I-II} - (\sigma_I-\sigma_{II})^2\}} \\
&+ \frac{\sigma_I^{(0)} c_I(k_I-k_{II})\{\omega'_{I-II}\gamma_{11} + (\sigma_I-\sigma_{II})^2\gamma_{12}\}}{4gk_I(\sigma_I-\sigma_{II})\{\omega'_{I-II} - (\sigma_I-\sigma_{II})^2\}} - \frac{(\gamma_{11}+\gamma_{12})(k_I-k_{II})}{4\{\omega'_{I-II} - (\sigma_I-\sigma_{II})^2\}} \\
&+ \frac{\sigma_I^{(0)} c_I\{\omega'_{I+II}\gamma_{21} - (\sigma_I+\sigma_{II})^2\gamma_{22}\}}{4g(\sigma_I+\sigma_{II})\{\omega'_{I+II} - (\sigma_I+\sigma_{II})^2\}} + \frac{(\gamma_{21}-\gamma_{22})(k_I+k_{II})^2}{4k_I\{\omega'_{I+II} - (\sigma_I+\sigma_{II})^2\}} \\
&- \frac{\sigma_I^{(0)} c_I(k_I+k_{II})\{\omega'_{I+II}\gamma_{21} - (\sigma_I+\sigma_{II})^2\gamma_{22}\}}{4gk_I(\sigma_I+\sigma_{II})\{\omega'_{I+II} - (\sigma_I+\sigma_{II})^2\}} + \frac{(\gamma_{21}-\gamma_{22})(k_I+k_{II})}{4\{\omega'_{I+II} - (\sigma_I+\sigma_{II})^2\}}.
\end{aligned}
$$

The final results of $\Phi_F^{(3)}$, $\eta_F^{(3)}$, σ_{IF}, and σ_{IIF} become

$$
\begin{aligned}
\Phi_F^{(3)} &= \lambda[\frac{\psi_1}{k_I k_{II}\cosh(2k_I-k_{II})h}\{2\sin\frac{2\sigma_I-\sigma_{II}+\sigma_{F1}}{2}t\sin\frac{2\sigma_I-\sigma_{II}-\sigma_{F1}}{2}t\sin(2k_I-k_{II})x \\
&+ \sin(2\sigma_I-\sigma_{II})t\cos(2k_I-k_{II})x\}\cosh(2k_I-k_{II})(y+h)
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \frac{\psi_2}{k_I k_{II} \cosh(2k_I + k_{II})h} \{2 \sin \frac{2\sigma_I + \sigma_{II} + \sigma_{F2}}{2} t \sin \frac{2\sigma_I + \sigma_{II} - \sigma_{F2}}{2} t \sin(2k_I + k_{II})x \\
&+ \quad \sin(2\sigma_I + \sigma_{II})t \cos(2k_I + k_{II})x\} \cosh(2k_I + k_{II})(y+h)] \\
&+ \quad \lambda[\textit{same as the above except for the subscripts I and II being switched}], \qquad (33)
\end{aligned}
$$

$$
\begin{aligned}
\eta_F^{(3)} &= \frac{\lambda}{g k_I k_{II}} [-\psi_1\{(2\sigma_I - \sigma_{II} + \sigma_{F1}) \cos \frac{2\sigma_I - \sigma_{II} + \sigma_{F1}}{2} t \sin \frac{2\sigma_I - \sigma_{II} - \sigma_{F1}}{2} t \\
&+ (2\sigma_I - \sigma_{II} + \sigma_{F1}) \sin \frac{2\sigma_I - \sigma_{II} + \sigma_{F1}}{2} t \cos \frac{2\sigma_I - \sigma_{II} - \sigma_{F1}}{2} t\} \sin(2k_I - k_{II}x) \\
&- \psi_2\{(2\sigma_I + \sigma_{II} + \sigma_{F2}) \cos \frac{2\sigma_I + \sigma_{II} + \sigma_{F2}}{2} t \sin \frac{2\sigma_I + \sigma_{II} - \sigma_{F2}}{2} t \\
&+ (2\sigma_I + \sigma_{II} - \sigma_{F2}) \sin \frac{2\sigma_I + \sigma_{II} + \sigma_{F2}}{2} t \cos \frac{2\sigma_I + \sigma_{II} - \sigma_{F2}}{2} t\} \sin(2k_I + k_{II}x) \\
&+ (2b_1 - \psi_1 \frac{2\sigma_I - \sigma_{II}}{2}) \cos(2\chi_I - \chi_{II}) - \psi_1 \frac{2\sigma_I - \sigma_{II}}{2} \cos\{(2k_I - k_{II})x + (2\sigma_I - \sigma_{II})t\} \\
&+ (2b_2 - \psi_2 \frac{2\sigma_I + \sigma_{II}}{2}) \cos(2\chi_I + \chi_{II}) - \psi_2 \frac{2\sigma_I + \sigma_{II}}{2} \cos\{(2k_I + k_{II})x + (2\sigma_I + \sigma_{II})t\} \\
&+ \frac{2g(b_3 k_{II} - 2b_6 \sigma_{II}^{(0)} c_{II}}{2\sigma_{II}\sigma_{II}^{(0)} c_{II} + g k_{II}} \cos \chi_{II}] \\
&+ \frac{\lambda^2}{g k_I k_{II}} [\textit{same as the above except for the subscripts I and II being switched}], \qquad (34)
\end{aligned}
$$

$$
\sigma_{IF} = -\frac{2\lambda^2 k_I (\sigma_I b_3 + g b_6)}{k_{II}(2\sigma_I \sigma_I^{(0)} c_I + g k_I)}, \qquad (35)
$$

$$
\sigma_{IIF} = -\frac{2k_{II}(\sigma_I b_3 + g b_6)}{k_I (2\sigma_{II}\sigma_{II}^{(0)} c_{II} + g k_{II})}, \qquad (36)
$$

in which

$$
\begin{aligned}
\psi_1 &= \frac{2\{(2\sigma_I - \sigma_{II})b_1 + g b_4\}}{(2\sigma_I - \sigma_{II})^2 - \sigma_{F1}^2}, \\
\psi_2 &= \frac{2\{(2\sigma_I + \sigma_{II})b_2 + g b_5\}}{(2\sigma_I + \sigma_{II})^2 - \sigma_{F2}^2}, \\
\sigma_{F1}^2 &= g(2k_I - k_{II}) \tanh(2k_I - k_{II})h, \\
\sigma_{F2}^2 &= g(2k_I + k_{II}) \tanh(2k_I + k_{II})h.
\end{aligned}
$$

Since $2\sigma_I - \sigma_{II}$ is nearly equal to σ_{F1} in eq.(34), the trigonometric functions $\sin \frac{2\sigma_I - \sigma_{II} - \sigma_{F1}}{2} t$ and $\cos \frac{2\sigma_I - \sigma_{II} - \sigma_{F1}}{2} t$ are slowly varying in comparison with the functions $\sin \frac{2\sigma_I + \sigma_{II} - \sigma_{F1}}{2} t$ and $\cos \frac{2\sigma_I + \sigma_{II} - \sigma_{F1}}{2} t$. The low-frequency factor on $\frac{2\sigma_I - \sigma_{II} - \sigma_{F1}}{2} t$ induces an amplitude modulation on the high-frequency factor on $\frac{2\sigma_I + \sigma_{II} - \sigma_{F1}}{2} t$. The resulting modulated oscillation causes the phenomenon of beats, which are limited to the initial stage.

Experiments

For the purpose of verifying the present wave theory, experiments were conducted in the 26-m-long, 0.5-m-wide, and 0.8-m-deep wave tank at Tokyo Metropolitan University. A piston-type wave generator is placed at one end of the tank, and a wave-absorber is installed at the opposite end to reduce the wave reflection. A series of electronic signals were produced by an arbitrary wave-form synthesizer, which controlled the motion of wave paddle. The surface elevations were measured using resistance wave gauges at four horizontal locations, 10.0, 12.5, 15.0, and 17.5m from the wave paddle. The water depth was varied from 20 to 40cm. In each water depth, several strokes of paddle motion were prescribed. The bichromatic waves composed of a pair of cosine waves have two incident wave amplitudes a_I and a_{II}, and two incident wave periods T_I and T_{II}. The wave conditions used in the present experiment were $T_I = 0.5sec$ and $T_{II} = 0.55sec$, or $T_I = 0.5sec$ and $T_{II} = 0.6sec$ for a pair of bichromatic wave periods and $a_I : a_{II} = 1:1, 1:2$, or $2:1$ for the corresponding ratio of two incident wave amplitudes. In each experimental condition, wave data were collected from four wave gauges continuously at a frequency of 47Hz for about three minutes in order to provide data samples sufficiently long for FFT analyses.

Results

A series of wave records observed at four fixed points and these amplitude spectra are shown in Fig.1. The abscissa and ordinate in figure(a) show the time in seconds, and the surface displacement above the still water level in cm, respectively, while those in figure(b) show the frequency in Hz, and the amplitude in cm, respectively. The periods of the first and second incident waves are $T_I = 0.5sec$ and $T_{II} = 0.55sec$, and the ratio of the first incident wave amplitude to the second incident wave amplitude is $a_I : a_{II} = 1 : 1$. In this case, the water depth is $h = 40cm$. The measured wave profiles are different from the input bichromatic-wave signal, by which the wave paddle motion is prescribed. The wave train at each location exhibits an appreciable modulation, even though the wave form itself is nearly uniform at each point. The modulation becomes smaller as the waves propagate from the wave paddle to the opposite end of the wave tank. Each spectrum contains four major frequency components within relatively narrow range of frequencies. The third-order frequency components by the wave interaction at x=10m are the largest; they decrease their amplitudes rapidly toward the end of the wave tank. Fig.2 shows similar plots of the bichromatic waves for $T_I = 0.5sec$, $T_{II} = 0.6sec$, $a_I : a_{II} = 1 : 1$, and $h = 40cm$. The modulation is not so large in this case. The basic profile of wave train remains the same throughout the entire process. The carrier-frequency and second-harmonic components of incident waves are quite evident but the third-order frequency components by the wave interaction are not seen clearly in the spectrum. The energy transfer within the spectrum is not appreciable, while the dissipation of the total energy is small.

Several free-surface elevations obtained from four different experiments are compared to theoretical results computed by the third-order perturbation equation[(20)]. To describe the natural profile of bichromatic wave train, each calculated

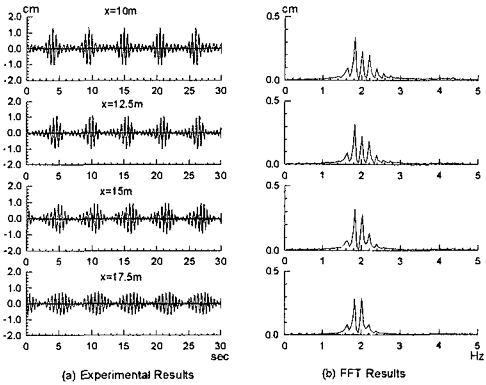

Fig.1 Surface displacements of bichromatic wave train and FFT analyses for $T_I = 0.5sec, T_{II} = 0.55sec, a_I : a_{II} = 1 : 1$, and $h = 40cm$.

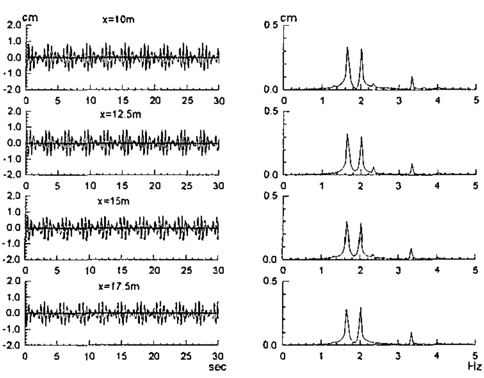

Fig.2 Surface displacements of bichromatic wave train and FFT analyses for $T_I = 0.5sec, T_{II} = 0.6sec, a_I : a_{II} = 1 : 1$, and $h = 40cm$.

value is plotted for thirty seconds at intervals of 0.02 sec. Comparisons of the measured free-surface displacements with the theoretical ones are presented in Figs.3-6. In these figures, (a) shows the experimental data plot, and (b) shows the theoretical profile. A set of time series for $T_I = 0.5sec$, $T_{II} = 0.55sec$, $a_I : a_{II} = 1 : 1$, and $h = 40cm$ are presented in Fig.3(a). The modulation caused by the interaction of two incident waves occurs in all locations, and the resonant nonlinear interaction leads to the disintegration of the wave envelope. The original periods of incident waves, however, remain constant throughout the process of the modulation. Fig.3(b) shows an example of calculated wave profiles for a pair of initial amplitudes, $a_I = a_{II} = 0.37cm$. These initial amplitudes were determined from actual free-surface measurements by the wave gauge at $x = 10m$. Fig.4 shows a comparison of the measured data for $T_I = 0.5sec$, $T_{II} = 0.55sec$, $a_I : a_{II} = 1 : 1$ and $h = 20cm$, and the theoretical calculation for $a_I = a_{II} = 0.34cm$. Although the pe-

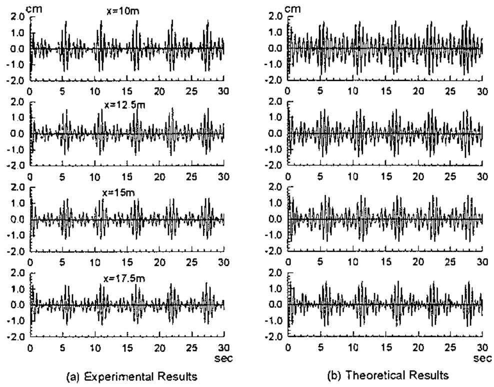

Fig.3 Measured and calculated surface displacements
($T_I = 0.5sec, T_{II} = 0.55sec, a_I : a_{II} = 1:1$, and $h = 40cm$).

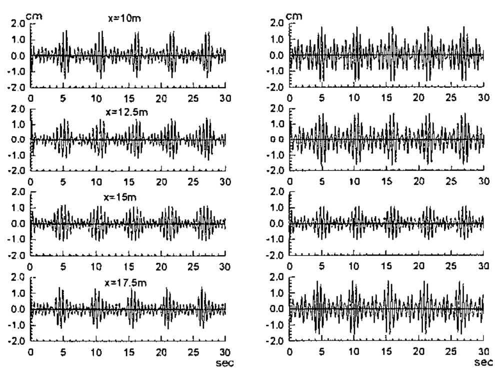

Fig.4 Measured and calculated surface displacements
($T_I = 0.5sec, T_{II} = 0.55sec, a_I : a_{II} = 1:1$, and $h = 20cm$).

riods of the first and second incident waves and the ratio of amplitudes of them are the same as those in the previous case, the water depth is shallower. In this case, the modulation is also found at all locations. A basic difference to the previous experiment for $h = 40cm$ is observed under the nodes of the envelope. Fig.5 presents a set of time series for $T_I = 0.5sec$, $T_{II} = 0.55sec$ and $a_I : a_{II} = 1:2$, in the water depth of $h = 20cm$, and the calculated surface displacements for $a_I = 0.22cm$ and $a_{II} = 0.44cm$. The agreement between the measured data and the theoretical result is reasonable on the tendency of a change of wave train. Fig.6 shows a set of temporal records of surface displacements for $T_I = 0.5sec$, $T_{II} = 0.6sec$, $a_I : a_{II} = 1:2$ and $h = 20cm$. The theoretical results were calculated by the present third-order bichromatic wave equation for $a_I = 0.38cm$ and $a_{II} = 0.76cm$. The measured data are well described by the theoretical curves in spite of such difference in the two wave periods and amplitudes.

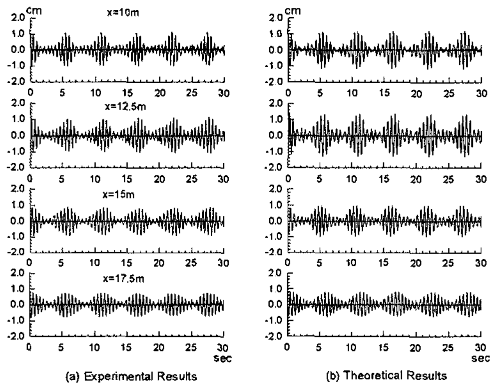

Fig.5 Measured and calculated surface displacements ($T_I = 0.5ecs, T_{II} = 0.55sec, a_I : a_{II} = 1:2$, and $h = 20cm$).

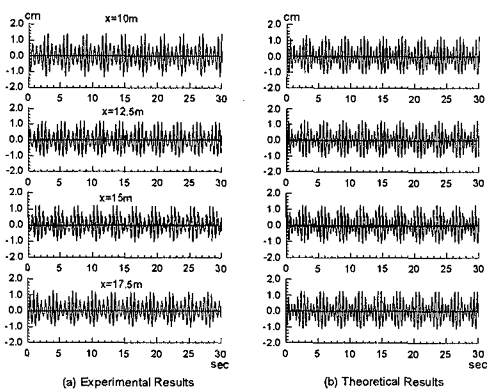

Fig.6 Measured and calculated surface displacements ($T_I = 0.5sec, T_{II} = 0.6sec, a_I : a_{II} = 1:2$, and $h = 20cm$).

Conclusions

The equation based on a finite-amplitude approximation was derived to calculate the surface displacements of bichromatic waves in water of intermediate depth. Experiments were also performed to investigate the modulation of bichromatic wave train in our wave tank. From the results of this study, the following conclusions are drawn.

When a wave train is generated by a pure bichromatic-wave paddle motion, the wave train usually modulates in the wave tank. This tendency is more prominent when the difference of the first and second incident wave periods become small; accordingly, measurable modulation did not develop in our tank length, when the bichromatic waves with the periods $T_I = 0.5sec$ and $T_{II} = 0.6sec$ were generated. The water depth affects to the disintegration of wave train; indeed, the modulation tends to disintegrate into smaller wave envelopes when the water depth increases. The third-order amplitudes by the wave interaction are larger than the first-order amplitudes when the difference between the first and second incident wave periods is smaller. The results obtained in this study are physically valid over the range of our interest.

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