Interaction of nonlinear interfacial wave and surface gravity wave in two-layer fluids

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INTROCUTION

This paper describes the third-order solution for the nonlinear internal and surface progressive water wave in a two-layer fluid. We use the perturbation method to develop a mathematical derivation. Third-order asymptotic solution which satisfies the governing equation and boundary conditions is obtained. This theoretical solution can be used to describe the mechanism of the interaction between the surface and interfacial water waves. The numerical results demonstrate the influence of the ratio density and thickness of the two fluids on the interfacial and surface profiles, the wave frequency.

FORMULATION OF THE PROBLEM

We consider a two-layer fluid system in which the flow motion is irrotational and the fluids are homogeneous incompressible, inviscid of different density. As sketched in Fig. 1, shows a two-layer fluid system that is stably stratified. The origin of the axes will be located in the undisturbed interface; the subscript *I* and *II* denote the upper layer and the lower layer.

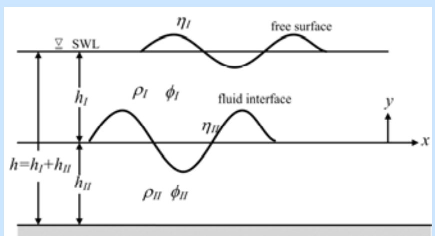

Fig.1 Definition sketch for interfacial wave in a two layer fluid.

The density and depth of the upper layer are ρ_I and h_I, and those of the lower layer are ρ_{II} and h_{II}. Let $\phi_I(x,y,t)$ and $\phi_{II}(x,y,t)$ denote velocity potentials in the upper and lower layers, so that they satisfying the continuity equation lead to the Laplace equations as follows:

$$\nabla^2\phi_I=\frac{\partial^2\phi_I}{\partial x^2}+\frac{\partial^2\phi_I}{\partial y^2}=0,\nabla^2\phi_{II}=\frac{\partial^2\phi_{II}}{\partial x^2}+\frac{\partial^2\phi_{II}}{\partial y^2}=0,$$

x is the horizontal coordinate and y is the vertical coordinate. The wave motion described above has to satisfy the boundary conditions at the bottom, density interface and free surface, respectively. On an immovable and impermeable uniform bottom, the no-flux bottom boundary condition gives $\phi_{IIy}=0,\ \text{at}\ \ y=-h_{II}$.

The kinematic and dynamic boundary conditions at the free surface are

$$\eta_{It}+\eta_{Ix}\phi_{Ix}-\phi_{Iy}=0 \qquad \text{at} \quad y=h_I+\eta_I,$$

$$g(\eta_I+h_I)+\phi_{It}+\frac{1}{2}(\phi_{Ix}^2+\phi_{Iy}^2)=0 \qquad \text{at} \quad y=h_I+\eta_I,$$

where η_I is elevation of water surface measured from $y=h_I$, g is acceleration due to gravity, and t denotes time. The boundary conditions at the density interface are

$$\eta_{IIt}+\eta_{IIx}\phi_{Ix}-\phi_{Iy}=0 \qquad \text{at} \quad y=\eta_{II},$$

$$\eta_{IIt}+\eta_{IIx}\phi_{IIx}-\phi_{IIy}=0 \qquad \text{at} \quad y=\eta_{II},$$

$$\rho_I[g\eta_{II}+\phi_{It}+\frac{1}{2}(\phi_{Ix}^2+\phi_{Iy}^2)]=\rho_{II}[g\eta_{II}+\phi_{IIt}+\frac{1}{2}(\phi_{IIx}^2+\phi_{IIy}^2)] \qquad \text{at} \quad y=\eta_{II},$$

where η_{II} is elevation of density interface.

THE SURFACE WAVE INDUCED BY THE GIVEN INTERFACIAL WAVE (SWIW)

When the internal wave passes, the sea surface looks like boiling. Obviously, when interfacial wave motion at the interfacial surface that will induce waves on the free surface. However, the present theoretical modeling provides a way to discuss this phenomenon. The functions for wave elevation in two layer fluid have been obtained as follow,

$$\eta_I=\varepsilon\eta_{I1}+\varepsilon^2\eta_{I2}+\varepsilon^3\eta_{I3}, \qquad \eta_{II}=\varepsilon\eta_{II1}+\varepsilon^2\eta_{II2}+\varepsilon^3\eta_{II3}.$$

The surface wave induced by interfacial wave (SWIS) can be evaluated and the results are illustrated in the Figs. 2~4. The different indicated wave steepnesses are implemented for comparison. The large wave steepness of surface wave is induced by the large wave steepness of interfacial wave in Fig. 2. Furthermore, there is a wave phase difference π between interfacial wave and surface wave; that is when the interfacial wave is on the crest; the surface wave in on the trough. The Fig. 3 shows that when upper layer depth is thinner than lower layer, , the crest of interfacial wave is larger than the trough of interfacial wave like depression wave.

Similarly, when upper layer thickness is thicker than lower layer fluid, the crest of interfacial wave is smaller than the trough of interfacial wave likes elevation wave.

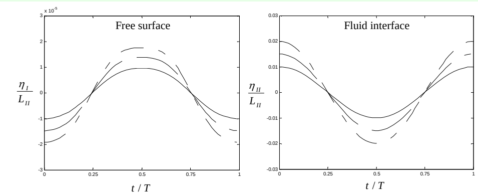

Fig. 2 The wave elevations of SWIW under three different steepnesses of interfacial wave. ($h_I/h_{II}=1$, $(\rho_{II}-\rho_I)/\rho_I=0.005$, $kh=0.5$ Solid line: $kH_{II}=0.04\pi$ Dashed line: $kH_{II}=0.06\pi$ Dash-Dotted line: $kH_{II}=0.08\pi$).

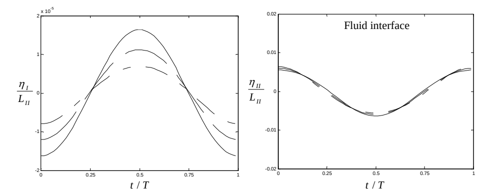

Fig. 3 The wave elevations of SWIW under three different thickness ratio of two-layer fluids. ($\frac{H_{II}}{h}=0.05$, $(\rho_{II}-\rho_I)/\rho_I=0.005$ Solid line: $h_I/h_{II}=0.5$ Dashed line: $h_I/h_{II}=1$ Dash-Dotted line: $h_I/h_{II}=2$).

THE INTERFACIAL WAVE INDUCED BY THE GIVEN SURFACE WAVE(IWSW)

In this section, the behavior of the interfacial wave induced by the surface wave will be discussed and the interfacial mode dispersion relation. The most apparent difference between the SWIS and IWSW is the wave phase shift. The wave phase shift between SWIW and interfacial wave is π. But there is no wave phase shift between IWSW and surface wave that is shown in Fig. 4 and Fig. 2. The effect of h_I/h_{II} on the wave height of SWIW is shown in Fig5. The height of SWIW is directly proportional to h_I/h_{II}. However, the effect of h_I/h_{II} on the wave height of IWSW is opposite. The height of SWIW and h_I/h_{II} are in inverse proportion (see Fig. 6).

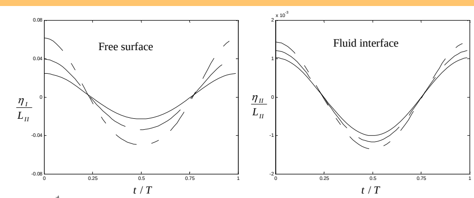

Fig. 4 The wave elevations of IWSW under three different steepnesses of surface wave. ($h_I/h_{II}=1$, $(\rho_{II}-\rho_I)/\rho_I=0.005$, $kh=0.5$ Solid line: $kH_{II}=0.04\pi$ Dashed line: $kH_{II}=0.06\pi$ Dash-Dotted line: $kH_{II}=0.08\pi$).

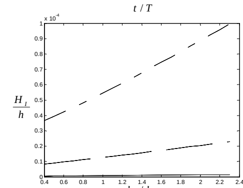

Fig. 5 Dimensionless wave height of SWIW versus thickness ratio of two-layer fluids h_I/h_{II} under three different dimensionless given interfacial wave height. ($kh=1.5$, $(\rho_{II}-\rho_I)/\rho_I=0.005$, Solid line: $H_{II}/h=0.0001$ Dashed line: $H_{II}/h=0.0003$ Dash-Dotted line: $H_{II}/h=0.0005$).

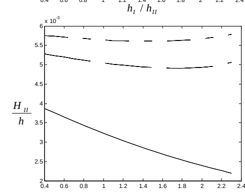

Fig. 6 Dimensionless wave height of IWSW versus thickness ratio of two-layer fluids h_I/h_{II} under three different dimensionless given interfacial wave height. ($kh=1.5$, $(\rho_{II}-\rho_I)/\rho_I=0.005$, Solid line: $H_I/h=0.01$ Dashed line: $H_I/h=0.03$ Dash-Dotted line: $H_I/h=0.05$).

CONCLUSIONS

This paper provides a third-order asymptotic solution for the interaction of interfacial wave and surface gravity wave in two-layer fluids based on the perturbation method. This solution satisfies the governing equation and boundary conditions. The wave elevations at the free surface and the interface are calculated in different wave conditions under the thicknesses and densities of the upper and lower layers in two-layer fluids. We have shown a way to by which the different dynamic mechanism between IWSW and SWIW can be described in general.