Analysis of Evanescent Waves of Linearized Multi-Level Wave Equations

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Introduction

In the viewpoint of ocean development, it is very important to evaluate the wave deformation around the isolated islands fringed by reefs with steep, sometimes step-like bathymetry. In such analysis, applicability for quasi-3 dimensional wave scattering problem is thought to be indispensable for the wave model to be employed. In this paper, the linearized form of the multi-level wave model proposed by Kanayama et.al(1998) is examined analytically on its applicability to the wave scattering problem in step-like bathymetry. The characteristics of the evanescent waves obtained from dispersion relation of model equations are investigated.

Linearized form of Multi-Level Wave Equations(M.L.W.E.) by Kanayama et al.(1998)

【depth-integrated continuity equation】

$$\frac{\partial \eta}{\partial t}+\sum_{i=1}^{N} d_i \frac{\partial u_i}{\partial x}=0$$

【momentum equations for each layer】

$$\frac{\partial u_n}{\partial t}+g\frac{\partial \eta}{\partial x}=\sum_{i=1}^{N}\alpha_{n,i}\frac{\partial^3 u_i}{\partial t \partial x^2}$$

η : surface elevation, d_i : thickness of each layer,
u_i : horizontal velocity of each layer, N: the number of layers

【dispersion coefficients】 $\alpha_{n,i}=\alpha1_{n,i}+\alpha2_{n,i}+\alpha3_{n,i}+\alpha4_{n,i}$

$$\alpha1_{n,i}=\begin{cases}0 & (n\le 1)\\ \sum_{m=1}^{n-1} d_m d_i & (n\ge 2, i\ge n)\\ \sum_{m=1}^{i-1} d_m d_i & (n\ge 2, i\le n-1)\end{cases}
\qquad
\alpha2_{n,i}=\begin{cases}0 & (n\le 1)\\ \frac{1}{2}d_i^2 & (n\ge 2, i\le n-2)\end{cases}$$

$$\alpha3_{n,i}=\begin{cases}\frac{1}{2}d_n d_i & (n\le i-1)\\ 0 & (n\ge i)\end{cases}
\qquad
\alpha4_{n,i}=\begin{cases}0 & (n\ne i)\\ \frac{1}{3}d_i^2 & (n=i)\end{cases}$$

Eigen vectors of linear dispersion relation of M.L.W.E.

By expressing the variables in simple harmonic oscillation style, and eliminating η, the linear dispersion relations of multi-level wave equations are expressed in the style of eigen value problem of $N\times N$ matrix **A**

【Linear dispersion relation of M.L.W.E.】

$$\mathbf{Au}=\frac{1}{(kh)^2}\mathbf{u}\ ,\quad \mathbf{u}=\{u_1,u_2,\cdots,u_N\}^{\mathbf{T}}$$

【 a_{ij} , the elements of Matrix **A**】

$$a_{ij}=\alpha_{ij}-\frac{gd_j}{\omega^2 h^2}$$

g : gravitational acceleration, h : water depth,
k : wave number, ω : wave frequency

One of N eigenvalues is positive and others are negative. Positive eigenvalue corresponds to the progressing mode. On the other hand, negative eigenvalues correspond to the evanescent modes. As shown in **Fig.1**, in the case of uniform layering, the vertical distributions of evanescent waves are almost equivalent to that of the small amplitude wave theory up to 9th mode, the highest mode for 10 layer model. On the other hand, eigenvectors of non-uniform layering show different vertical distributions from that of the small amplitude wave theory.

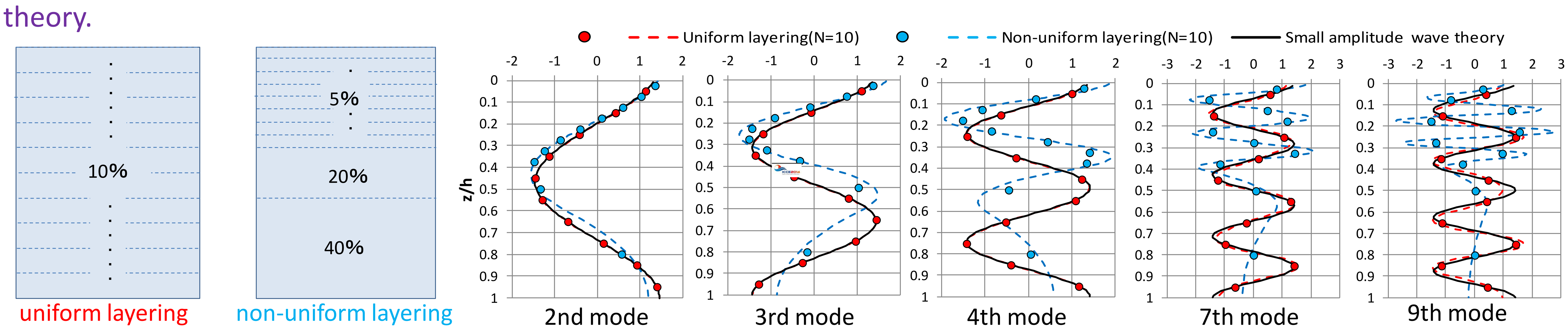


Fig.1 The vertical distribution functions for evanescent waves of multi-level wave equations with 10 layers

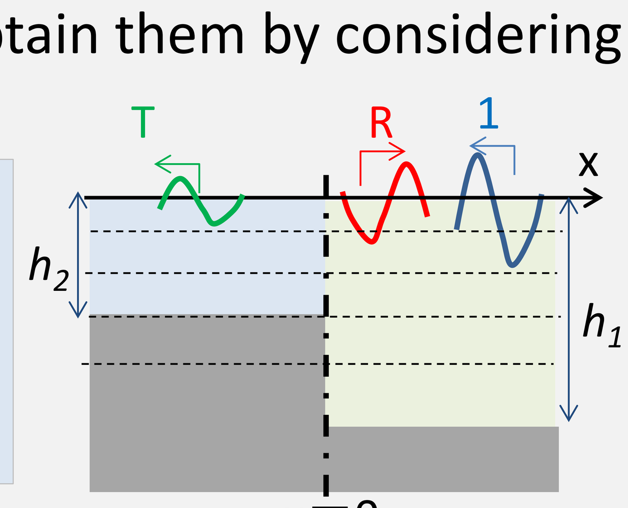
Scattering Analysis for Step Bathymetry with Eigen vectors

To confirm the completeness of these eigenvectors , wave scattering analysis on step-like bathymetry with eigenvectors are carried out as shown in **Fig.2**. Transmission coefficient T, reflection coefficient R, and complex amplitudes of every eigenvector of both regions are unknown variables in this problem. We can obtain them by considering the continuity of velocities and velocity potentials at the boundary of two regions.

REG.2

Radiation mode → $\mathbf{q}_0$, Evanescent mode → $\{\mathbf{q}_n\}$

$$\mathbf{u}_2(x)=\frac{T}{m_T}e^{ik'x}\mathbf{q}_0+\sum_{n=1}^{N_2-1}C_n\mathbf{q}_n e^{k'_n x}\ ,\quad m_T=\sqrt{\frac{C_{G1}C_2(\mathbf{q}_0\bullet\mathbf{q}_0)}{C_{G2}C_1(\mathbf{p}_0\bullet\mathbf{p}_0)}}$$



REG.1

Radiation mode → $\mathbf{p}_0$, Evanescent mode → $\{\mathbf{p}_m\}$

$$\mathbf{u}_1(x)=(e^{ikx}-Re^{-ikx})\mathbf{p}_0+\sum_{m=1}^{N_1-1}S_m\mathbf{p}_m e^{-k_m x}$$

Fig.2 Schematic figure of wave scattering analysis on step-like bathymetry

The calculated scattering velocity field on the step edge shows good agreement to the potential theory(**Fig.3**). Similarly, almost equivalent transmission and reflection coefficients to potential theory are provided (**Fig.4**).

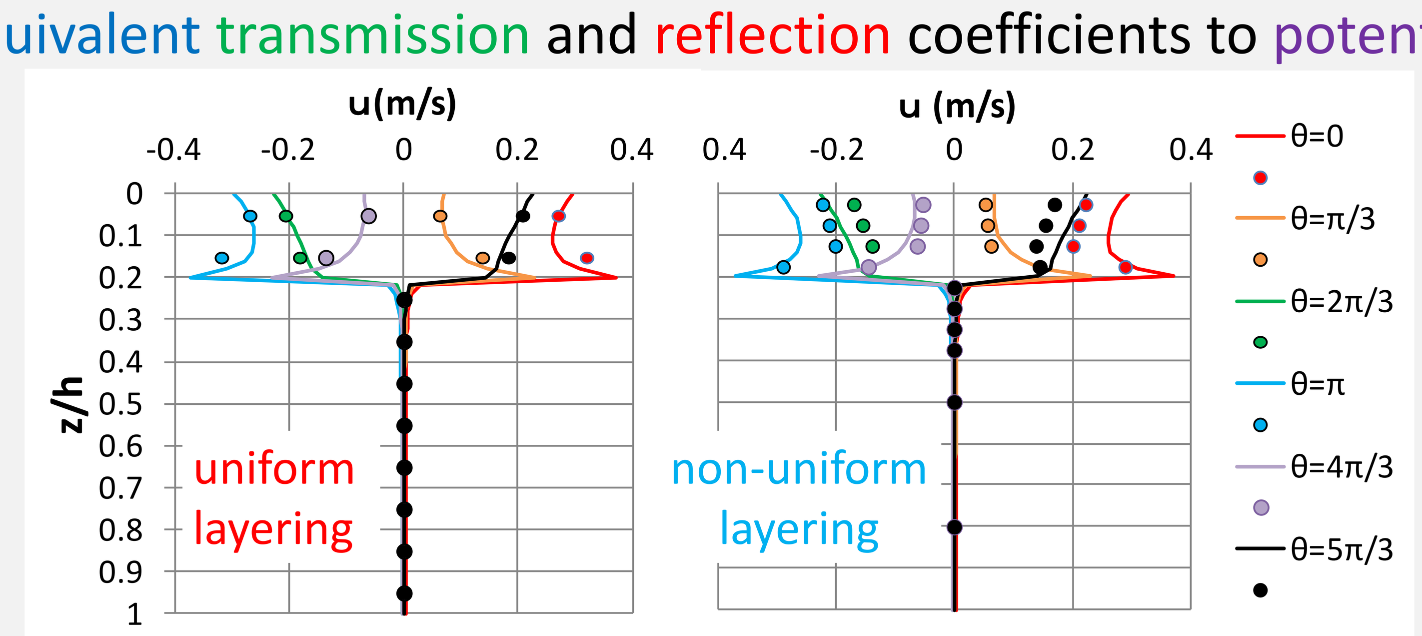


Fig.3 Scattering velocities on step edge

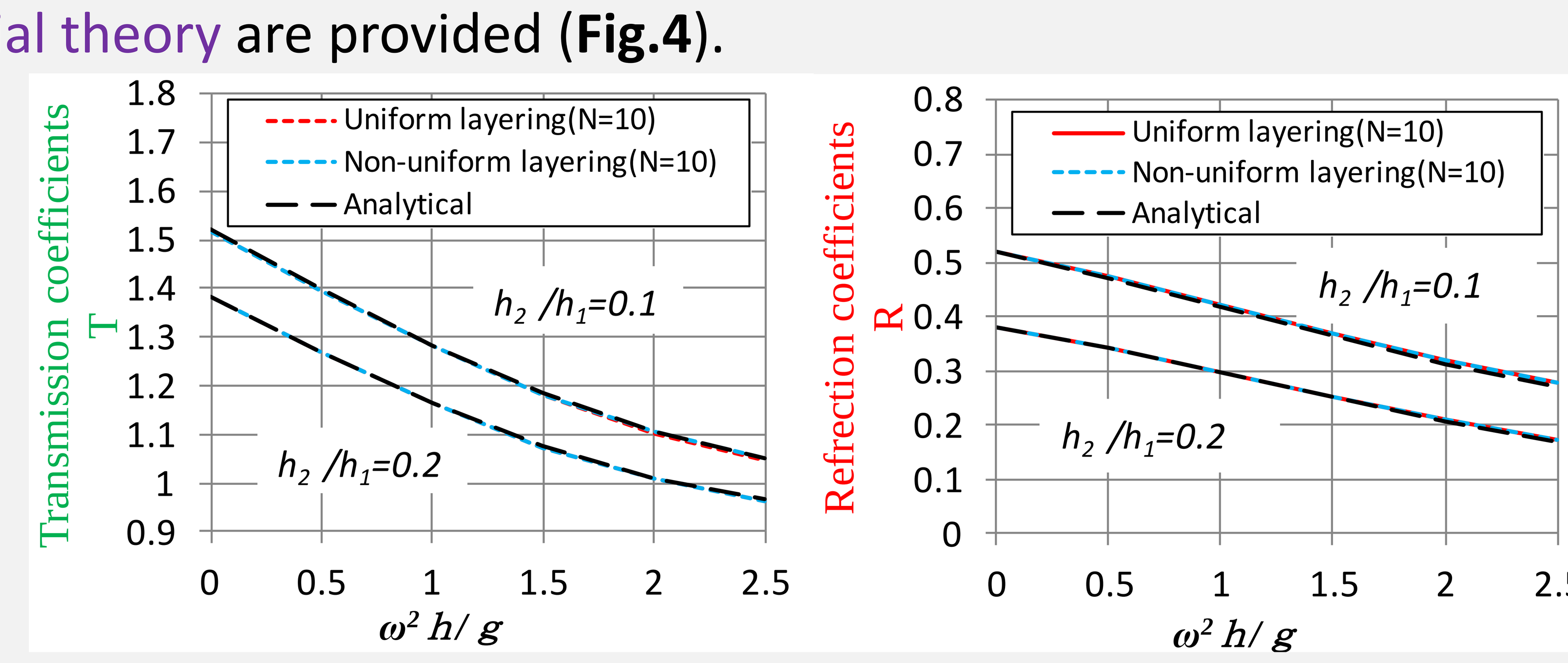


Fig.4 Transmission coefficients and reflection coefficients

Conclusion

1. The vertical distributions of evanescent waves derived from linear dispersion relation of M.L.W.E. depend on layering manner. In the case of uniform layering, they are almost equivalent to that of the small amplitude wave theory
2. The eigenvectors turned out to compose the complete orthonormal system which can reproduce the scattering wave field on step-like bathymetry .

[Reference] Kanayama, S., Tanaka, H. and Shuto, N.: Multi-Level Model for Nonlinear Dispersion Water Waves, Proceedings of 26th International Conference on Coastal Engineering, pp.576-588, 1998.