ANALYTICAL MODEL OF SAND SPIT EVOLUTION

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Spits typically consist of sand or gravel and are commonly occurring morphological features at inlets, river mouths, and the down-drift ends of barrier islands. Thus, they may form at the ocean-, lake-, or bay-side of inlets, entrances, and river mouths. Apart from the scientific interest in spits and their evolution, engineers have often studied spits with regard to their penetration into river mouths or inlets, restricting the flow rate and possibly even causing closure of the inlet (river mouth). Governing processes for spit growth under a predominant longshore transport, causing down-drift accumulation of sand, were reviewed. Based on this review, equations for the simulation of spit growth from former studies were improved, and their analytical solutions employed to build a model able to reproduce linear spit elongation. Major modifications were introduced in the equations to account for variation in spit cross-section with time, and to better describe the increase in active profile height and transport at the down-drift end of the spit as it elongates through the inlet channel. The analytical solutions were compared with data from the laboratory and field case studies. The case studies represent situations of unrestricted and restricted growth, including time-varying cross-sectional spit area and increasing active profile height. Results showed that the generalized expression for time-varying spit cross-sectional area enabled the adoption of a more realistic trapezoidal cross-section for the modeled spits. The model also contributed to estimate the net longshore sediment transport rates, facilitating comparison with observations from the different case studies. For unrestricted spit growth, it was possible to give a satisfactory representation of spit elongation over the analyzed periods, although for some study areas a single elongation rate could not accurately predict increasing spit lengths over long periods of analysis, *i.e*., above 50 years. Nevertheless, the model has a high potential to make rapid quantitative predictions, being a useful and valid tool for initial estimates in engineering projects.

Keywords: analytical model; spit; longshore transport

INTRODUCTION

Spits are common morphological features at river mouths, down-drift end of barrier islands and inlets (Hoan *et al*., 2011), typically consisting of sand and gravel. Kraus (1999) defined a spit as “organized surface-piercing accumulations of sediment that grow by transport directed from a landmass or sediment source toward a water body”. For coastal professionals, engineers in particular, it is important to study the evolution of these coastal features because their elongation might result in penetration into river mouths or inlets. This penetration can further obstruct and reduce the flow exchange with the ocean, and in extreme cases cause their closure, *e.g.*, Tanaka *et al*. (1996). Moreover, their evolution can also interfere with navigation as the penetration brings sediment that cause shoaling of the channel.

Analytical models are mathematical solutions for simplified version of the governing equations that may be used for representing shoreline and profile change (Larson and Hanson, 1997). Hanson *et al*. (2003) recognized the role of analytical solutions to describe the basic physics involved, and by means of knowing the interdependence between variables, compute essential features of beach response to incident waves, currents, and changes in water levels. Thus, an important aspect of this paper is to mathematically model linear spit evolution in order to better understand spit development and the physical governing factors. With the use of analytical models, negative interferences to coastal environments and infrastructures caused by spit elongation can quickly be estimated. Thereafter, preventive and mitigation actions can be adopted and implemented.

A general model for linear spit growth under a predominant longshore transport, which causes a down-drift accumulation of sand, is developed after a review of the governing processes. Building on the work of Kraus (1999), improvements were introduced of the governing equations to account for a more varied range of spit cross-sectional areas, to allow the spits cross-section to vary with time, and to better account for the increase in active profile height and transport at the down-drift end of the spit as it elongates into the inlet channel. Considering schematized, yet realistic situations, analytical solutions to a number of spit evolution cases were derived using the derived governing equations.

The analytical solutions are compared with data from laboratory and field case studies. The case studies involve both unrestricted and restricted growth; the case of a more complex setting implying restricted spit growth is typically prevailing in the presence of an inlet. Time-varying cross-sectional spit areas and increasing active profile height during spit growth are also investigated. The model showed a high potential to make quantitative predictions, which will facilitate its use in practical investigations.

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THEORETICAL DEVELOPMENTS

The development of a mathematical model for linear spit evolution in this paper is based on Kraus (1999) (*cf.* also Kraus and Seabergh, 2002). However, the present model will employ different formulations for the influence of spit cross-sectional shape, causing a varying spit width with time, and will also account for changes in active profile height (*i.e.* the sum of berm height and depth of closure).

The basic governing equation for spit evolution is obtained by employing the sand volume conservation equation for the entire spit. It is expressed as:

$$\frac{\partial}{\partial t}\left(\int_0^{x_s} A_s dx\right) = Q_{in} - Q_{out} \tag{1}$$

where A_s is the cross-sectional area of the spit at a distance x from the origin, x_s the location of tip of the spit, Q_{in} the longshore sand transport at the updrift end of the spit (x = 0), Q_{out} the transport at the downdrift end of the spit ($x = x_s$), and t time. Eq. 1 is obtained by employing the sand volume conservation equation for the entire spit with Q_{in} and Q_{out} being the boundary fluxes and the time derivative of the integral represents the changes in spit volume.

Existing data for spits often reports their geometric characteristics in connection to mean sea level (MSL). Assumptions regarding the spit shape and the active profile height, that is, the vertical distance over which the sand transport occurs, are required. The assumption considered by Kraus (1999) that the active profile height (D) is the sum of the berm elevation (B) and the depth of closure (D_c) is employed here. Both parameters B and D_c have as reference the MSL. In Kraus' model, A_s is estimated as a constant spit width (W_s) multiplied by D (= $B + D_c$), implying a rectangular shape over the active height. However, the specific cross-sectional shape of the spit may be of interest, although it can be difficult to represent all the details that account for variation in the shape of the spit. Other spit cross-sectional shapes, however, may be more realistic than a rectangular. In the case of a trapezoidal cross section, with a linearly sloping beach at the landward and seaward side, the cross-sectional area is:

$$A_s = DW_s\left(1 + K_\beta \frac{D_c - B}{2W_s}\right) \tag{2}$$

where K_β = 1/tan β_s + 1/tan β_l, in which β_s and β_l are the slope on the seaward and landward side, respectively. In Eq. 2, $A_s = \psi DW_s$, where ψ is a constant that depends on the particular cross-sectional shape of the spit.

Thus, emphasizing an analytical approach, the difference between various assumed shapes may only be a constant form factor. Furthermore, in comparison with data, the shape will not have any major influence on the general trend concerning, for example, spit elongation, but only the magnitude of change and the estimated Q_{in} from the data will be of importance.

For unrestricted spit growth there is no transport at the tip of the spit, that is, Q_{out} = 0, and all sediment supplied to the spit results in growth, either through elongation or widening of the spit. Furthermore, assuming a constant Q_{in} with $A_s = A_e$ (A_s has attained equilibrium), Eq. 1 yields linear spit elongation according to:

$$x_s = \frac{Q_{in}}{A_e} t \tag{3}$$

If the build-up of A_s occurs in proportion to Q/Q_{in}, where Q is the transport along the spit assumed to vary according to:

$$\frac{Q}{Q_{in}} = \left(1 - \frac{x}{x_s}\right)^m \tag{4}$$

in which m is an empirical coefficient, then:

$$A_s = A_e\left(1 - \frac{A_e x}{(m+1)Q_{in}t}\right)^m \quad (5)$$

where $0 < x < (m+1)Q_{in}t / A_e$.

The increase in active profile height D with x from the spit starting at an active height D_o can be described by the following expression:

$$D = D_o + \Delta D\left(1 - \exp(-\alpha x)\right) \quad (6)$$

where ΔD is the increase in D occurring at large distances from the spit, and α a rate coefficient quantifying the approach towards the final depth $D_o + \Delta D$. Substituting Eq. 6 into Eq. 1, and using Eq. 2 for introducing D yields an equation that can be solved analytically. It is also possible to employ a varying spit cross-sectional shape (i.e., W_s), but only for $m = 1$. The solution for this case is given by:

$$\frac{1}{2}x_s\left(1 + \frac{\Delta D}{D_o}\right) + \frac{\Delta D}{\alpha D_o}\left(\frac{1}{\alpha x_s}\left(1 - \exp(-\alpha x_s)\right) - 1\right) = \frac{Q_{in}}{A_e^o}t \quad (7)$$

where A_e^o is the equilibrium cross-sectional area defined based on D_o and x_s is given as an implicit function of t. Again, Eq. 7 is valid for unrestricted spit growth. For small values on x_s, Eq. 7 may be approximated by $x_s = 2Q_{in}\,t / A_e^o$. For the other asymptotic case when x_s becomes large, Eq. 7 reduces to $x_s = 2Q_{in}\,t / A_e^\infty$, where A_e^∞ is the equilibrium area based on $D_\infty = D_o + \Delta D$. Thus, Eq. 7 produces linear growth rates for the spit elongation for small and large values on x_s.

Restricted spit growth may be modelled using suitable expression on Q_{out} derived from physically based transport equations, for example (see Larson *et al*., 2014):

$$Q_{out} = \frac{Q_o}{\left(1 - x_s / x_i\right)^2} \quad (8)$$

where Q_o is the transport through the inlet before the spit starts growing and x_i the initial inlet width. For $A_s = A_e$, the solution to Eq. 1 under these conditions is:

$$\sqrt{\delta}\,\text{arctanh}\left(\frac{\sqrt{\delta}x_s}{(1-\delta)x_i - x_s}\right) + \frac{x_s}{x_i} = \frac{Q_{in}}{A_e x_i}t \quad (9)$$

where $\delta = Q_o/Q_{in}$ and x_s is given as an implicit function of t. At equilibrium Eq. 9 yields $x_{se} = x_i\left(1 - \sqrt{\delta}\right)$.

SELECTED RESULTS

Unrestricted linear growth

Calculated spit elongations using the analytical solution (Eq. 3) together with the measured values for unrestricted spit growth are presented for two locations, Sangomar spit in Senegal, and Badreveln at Falsterbo Peninsula in Sweden (Figure 1). The Sangomar spit is located along the stretch of the Senegalese coast located to the south of Dakar. The length of this sandy beach barrier has been estimated to be around 20 km, and its width varies from several tens to hundred of meters (Barusseau *et al.* 1995). A net sediment transport directed southwards sustained the elongation of the spit during the 60-year period of analysis, between 1927 and 1987. During recent years, the spit has been breached, which resulted in a major change in its morphology, mainly due to sand starvation (Barusseau *et al.* 1996).

The Badreveln spit has been elongating in the northern part of the Skanör-Falsterbo Peninsula since the latter half of the 19th century (Blomgren and Hanson 1998, Larson and Hanson 2013). The formation and later elongation of the spit started after the construction of the Skanör harbor. This infrastructure caused a redirection of the sediment transport from northeast towards the north. The current length of the spit is around 3 km, based on measurements on images from Google Earth. Hoan *et al.* (2011) estimated from aerial photos that the width of the spit was about 70 m. Data on the spit evolution is available from Blomgren and Hanson (2000) from 1916 to 1994, and from Google Earth for the last 20 years.

The best-fit line of the analytical solution to the data gives for Sangomar spit a mean value on the elongation rate of nearly 124 m/yr (Figure 1). The figure also includes lines corresponding to 20% more and 20% less than the spit length elongation (x_s) given by the best-fit rate. These two lines indicate the sensitivity of x_s to a variation in the elongation factor k. It can be seen from the figure that only the spit length for 27 years after the base year (1927) falls out of the area defined by 20%-more and 20%-less lines, suggesting an initially lower longshore sediment transport rate (Q_{in}) than the mean value.

A rectangular cross-sectional shape (*i.e.*, $\psi = 1$; see Eq. 2) renders a Q_{in} around 186,000 m^3/yr, considering an average width of 300 m for the spit and an active profile height of 5 m. For a trapezoidal cross-sectional shape, after a reasonable assumption about the side slopes (yielding $\psi = 2.5$), Q_{in} would be around 465,000 m^3/yr. Although there is limited specific information on the net sediment transport along the Sangomar spit, the estimations for Q_{in} based on the spit growth with rectangular and trapezoidal cross-sections fall between the sediment transport rates given by Barusseau *et al*. (1995) for the northern and southern Senegalese coast. According to this author, the down-drift sedimentary flow rate along the Senegalese coast decreases southwards from 0.5 - 1 million m^3/yr in the north to 50,000 - 100,000 m^3/yr in the south. This great variability is mainly due to the varying characteristics of the swells and availability of sandy material, which is higher in the north (Barusseau *et al.* 1995).

For the Badreveln spit, the best-fit line gave an elongation rate about 28 m/yr. Viewing the area given by the elongation rate lines 20% lower and 20% higher than the best-fit line, it is clear that the spit growth was slower during the first 60 years.

A longshore sediment transport rate around 10,000 m^3/yr is obtained for a rectangular cross-sectional shape ($\psi = 1$) with the average spit width of 70 m and an active profile height of about 5 m, estimated by Larson and Hanson (2013) from beach profile surveys. The mean transport rate Q_{in} would be around 25,000 m^3/yr for a more realistic trapezoidal cross-sectional shape, after a reasonable assumption about the side slopes ($\psi = 2.5$). The latter sediment transport rate from the analytical solution with a trapezoidal cross-section shape is more in agreement with the estimation by Larson and Hanson (2013). These authors calculated the net sediment transport rate along the stretch of the Skanör-Falsterbo Peninsula south of the Skanör harbor to be around 40,000 m^3/yr using numerical simulations with a shoreline evolution model. They estimated that about half of the sediment volume would pass the harbor and contribute to the elongation of the Badreveln spit, which is in agreement with the results obtained with the analytical spit model.

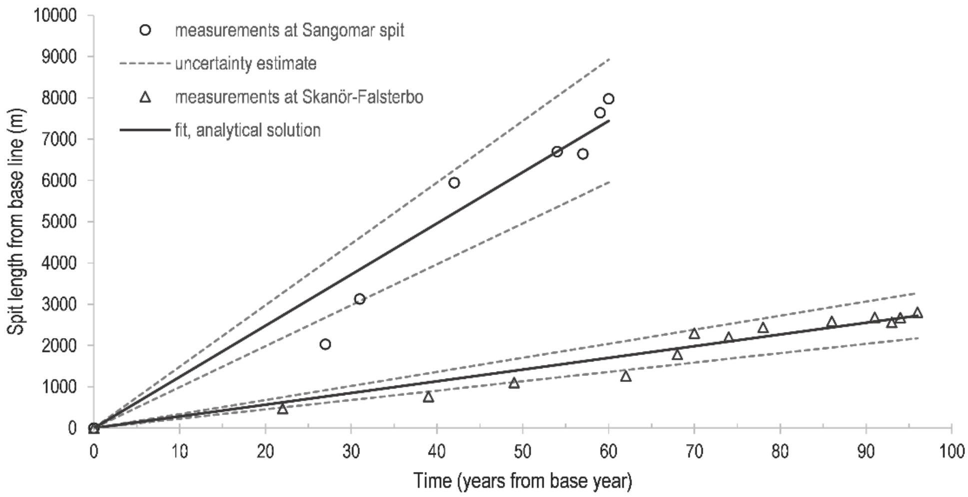

Figure 1 - Unrestricted spit elongation at Sangomar spit and Skanör-Falsterbo Peninsula

Time varying cross-sectional spit shape

For time varying cross-sectional spit shape a data set on spit elongation from a laboratory experiment discussed by Kraus (1999) and Kraus and Seabergh (2002) was used. The physical model was built in the late 1990's by the Coastal Hydraulics Laboratory of the US Army Engineer Research and

Development Center (ERDC). It consisted of a 99 m long, 46 m wide, and 0.6 m deep concrete basin with parallel depth contours in the offshore that followed the equilibrium profile (Dean 1977). The data employed here to assess the performance of the analytical solution for spit elongation with varying cross-sectional shape were from an experiment on spit evolution and its interactions with inlet flow (Kraus and Seabergh 2002). For this experiment, the inlet had an opening of nearly 2.5 m and sand with a median grain size of 0.13 mm was used. Waves were approaching at an angle of 20 degrees, and a maximum water depth in the inlet just over 0.15 m was employed.

The evolution of the spit width towards equilibrium was measured at different locations and this data were used for comparison with predictions by Eq. 5. Using Eq. 2 (assuming a trapezoidal shape for the spit cross-section) computed cross-section areas (A_s) can be converted to spit widths (W_s) and further compared with measurements from the experiment. In addition, to make the comparison possible, parameters such as m, W_e and $Q_{in}/(D\psi)$ which are unknown *a priori* have to be set. Thus, W_e (equilibrium spit width) was specified based on the observations, yielding a value of 0.65 m, and m and $Q_{in}/(D\psi)$ were calibrated through a least-square fit against the data.

Figure 2 illustrates the agreement between computed spit width as a function of time (Eq. 5) with measurements at the locations x = 0.3, 0.6, and 0.9 m downdrift the inlet side from which the spit growth started. Figure 3 illustrates the same type of comparison, but with measurements at the locations x = 1.2, 1.5, and 1.8 m. The best fit for computed spit widths corresponds to m = 1 for both groups of data. The optimum value on the transport parameter $Q_{in}/(D\psi)$ was 0.018 for the first group and 0.010 for the second group of data. As Q_{in} represents the net supply to the spit, it is comprehensible that the transport is smaller for the second group because the reduced inlet cross-section will increase Q_{out}, and consequently reduce the net supply of the sediment to the spit.

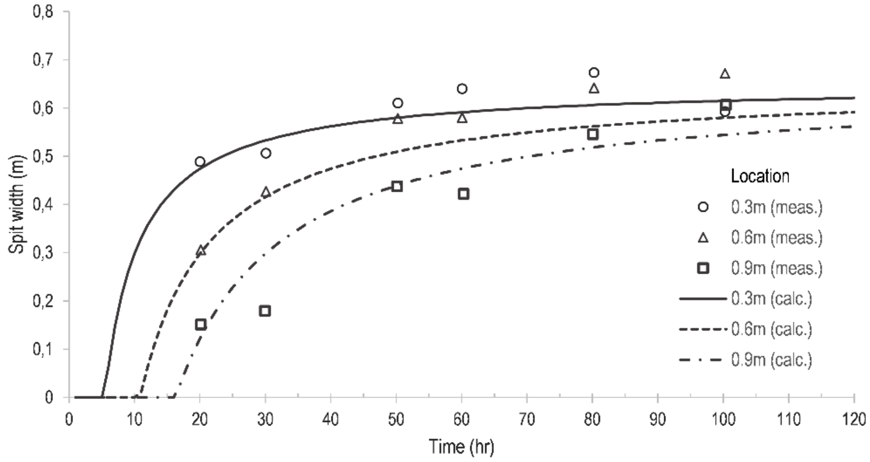

Figure 2 - Comparison between measured and computed time-varying spit widths for unrestricted growth at closest measurement points to the inlet on the downdrift side

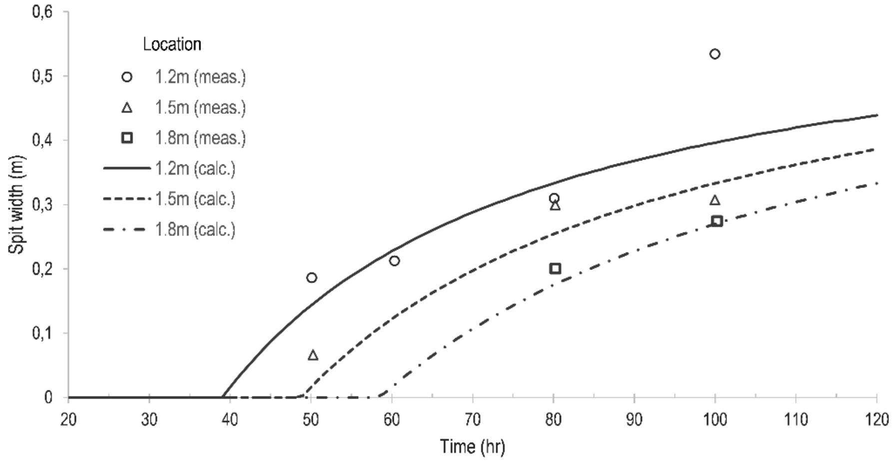

Figure 3 - Comparison between measured and computed time-varying spit widths for unrestricted growth at the more distant points to the inlet on the downdrift side

Increasing active profile height

Data from the same laboratory experiment described by Kraus (1999) was also used to simulate an increasing active profile height of the spit as it penetrates an inlet. In some cases, as there is a variation in the water depth during the spit protrusion into the inlet, Eq. 6 was employed to reproduce the data. This equation assumes an exponential increase in active profile height from an initial height D_o towards a maximum height $D_o + \Delta D$. The solution for spit elongation was computed using Eq. 7, which is valid for $m = 1$.

As previously discussed, reduced forms of Eq. 7 can be derived for small values on x_s, yielding $x_s = 2Q_{in}\, t / A_e^o$, and for large values on x_s, yielding $x_s = 2Q_{in}\, t / A_e^\infty$. From the data presented by Kraus plotted in Figure 4, both earlier and later phases of the spit growth can be approximated by straight lines, with slope $k_o = 4.5$ for small x_s and $k_\infty = 0.45$ for large x_s. For the non-linear stretch, $\Delta D/D_o$, α and Q_{in} / A_e^o should be known to make the computation possible. Thus, $\Delta D/D_o = 9$ was obtained, considering that for Q_{in} constant, the ratio k_o / k_∞ is equal to $1 + \Delta D/D_o$. With $\Delta D/D_o = 9$, optimum values on the two remaining parameters became $\alpha = 0.8$ and $Q_{in} / A_e^o = 0.043$ in the fitting procedure. Figure 4 illustrates the agreement between computed and measured spit lengths x_s. The least-square fitting was performed using a Newton-Raphson technique, solving x_s for specific values on t. It can be seen from Figure 4 that the spit elongation with increasing active profile height is well represented by the analytical solution, although underpredicted a bit during the initial phase, and overpredicted during the final phase.

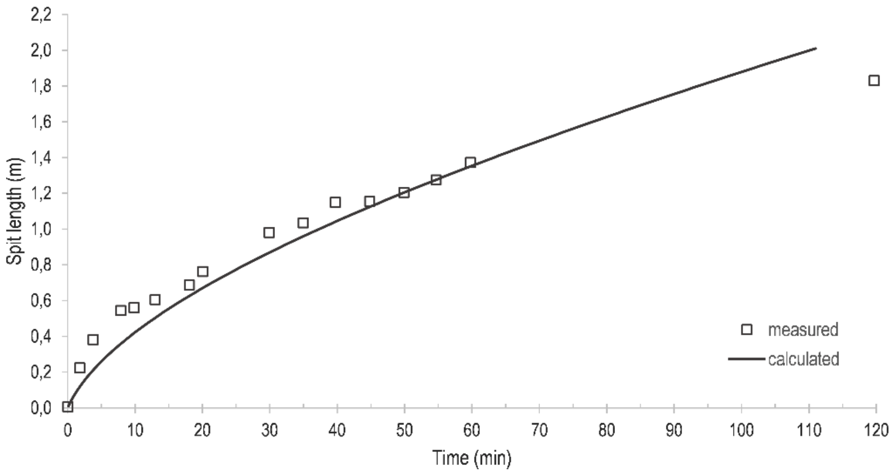

Figure 4 - Spit elongation with increasing active profile height as the spit penetrates into an inlet

Restricted spit growth

The ERDC inlet experimental facility was able simulate tidal flows as tides could be induced through a pumping system. Thus, a set of data presented by Kraus (1999) comprised elongation under the influence of an inlet current that corresponds to the case of restricted spit growth. The analytical solution given by Eq. 9 was used to compute the restricted spit elongation. The solution given by Eq. 9 corresponds to $m = 1$. To make it possible to compare the solution with the measured data, values have to be assigned to δ and Q_{in}/A_e. From the simplified equation at equilibrium $x_{se} = x_i\left(1-\sqrt{\delta}\right)$, and δ can be calculated if x_{se} and x_i are known. Thus, x_{se} was estimated to be about 2.1 m and x_i close to 2.5 m, yielding the value $\delta = 0.04$. Using the latter value, Eq. 9 was then least-square fitted to the data giving an optimum value for the second unknown parameter $Q_{in}/A_e = 0.019$. Because Eq. 9 is implicit, a Newton-Raphson technique was used again to obtain x_s for a specific values on t. The agreement between calculated and measured spit elongation is illustrated in Figure 5. From this figure it can be seen that the spit length is initially underestimated, but just before x_s attains equilibrium it is overestimated.

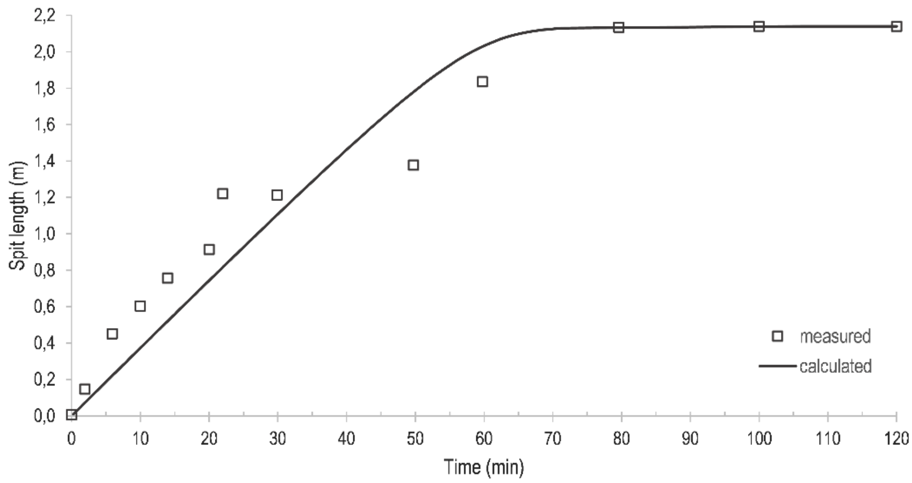

Figure 5 - Comparison between measured and computed spit lengths under restricted growth

CONCLUSIONS

The mathematical model and resulting analytical solutions derived to describe linear spit elongation were employed to describe observed spit growth. With the mathematical model, unrestricted and restricted growth cases were simulated and the results compared with data from different field case studies and laboratory experiments. Additional analysis was made for cases with the spit cross-section varying in time and with increasing active profile height as the spit elongated.

The generalized expression to describe the spit cross-sectional area enabled the adoption of a more realistic trapezoidal cross-section for modeled spits. It also contributed to approximate the net longshore sediment transport rates computed by the model to estimates from field and laboratory studies. For time-varying spit widths and increasing active profile height, the value of different parameters and coefficients had to be estimated through calibration against data to arrive at optimum solutions.

For unrestricted spit growth, it was possible to give a satisfactory representation of spits elongation over the analyzed periods. However, it is important to mention that a single elongation rate could not predict accurately increasing spit lengths over long periods of analysis, *i.e*., above 50 years. This fact suggests a possible split of the spit growth in several different phases with more homogenous forcing conditions, which can better represent the elongation rates, or employing a numerical approach. Despite this shortcoming, the model has high potential for rapid quantitative predictions, and thus function as a useful and valid tool for first estimates in engineering projects.

The model concentrates on describing linear spit growth in the direction of the net longshore sediment transport and the main shoreline trend. Thus, recurving spits are not addressed. Further efforts should be directed to improve the model so that it can simulate the recurving behavior of spits.

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REFERENCES

Battjes, J.A., and J.P.F.M. Janssen. 1978. Energy loss and set-up due to breaking of random waves, *Proceedings of 14th International Conference on Coastal Engineering*, ASCE, 466-480.

Barusseau, J.P., Bâ, M., Descamps, C., Diop, E.H.S., Giresse, P., and Saos, J.L. 1995. Coastal evolution in Seneal and Mauritania at 10-x year scales. Natural and human records. *Journal of Quaternary International*, 29/30, 61-73.

Blomgren, S, Hanson, H. 2000. Coastal geomorphology at the Falsterbo Peninsula, southern Sweden. *Journal of Coastal Research*, 16(1), 15-25.

Dean, R.G. 1977. Equilibrium beach profiles: U.S. Atlantic and Gulf Coasts. Ocean Engineering Technical Report No. 12, Department of Civil Engineering and College of Marine Studies, University of Delaware, Newark.

Hanson, H., and Larson, M. 2003. Sand transport and coastal development at Skanör-Falsterbo. Report No. 3166, Department of water Resources engineering, Lund Instute of Technology, Lund University, Lund, Sweden (in Swedish).

Hoan, L.X., Hanson, H., Larson, M., and Kato, S. 2011. A mathematical model of spit growth and barrier elongation: Application to FireIsland Inlet (USA) and Badreveln Spit (Sweden). *Journal of Estuarine, Coastal and Shelf Science*, 93, 468-477.

Kraus, N.C. 1999. Analytical model of spit evolution at inlets. *Proceedings Coastal Sediments 99*, ASCE, 1739-1754

Kraus, NC, Seabergh, WC (2002) Inlet spits and maintenance of navigation channels. CHETN-IV-44, Coastal and Hydraulics Laboratory, US Army Engineer Research and Development Center, Vicksburg, MS.

Larson, M., and Hanson, H. 1997. Analytical solutions of one-line model for shoreline change near coastal structures. *Journal of Waterway, Port, Coastal & Ocean Engineering*, 123, 180.

Larson, M., and Hanson, H. 2013. Coastal Erosion and Protection in Sweden. In: *Coastal Erosion and Protection in Europe*, Eds. Pranzini, E and Williams, AT, Earthscan Ltd.

Tanaka, H., Takahashi, F., and Takahashi, A. 1996. Complete closure of The Nanakita River mouth in 1994. *Proceedings 25th Coastal Engineering Conference*, ASCE, 4545-4556.