Evolution Equations for Edge Waves and Shear Waves on Longshore Uniform Beaches

James T. Kirby, M.ASCE[1], Uday Putrevu, A.M.ASCE[2] and H. Tuba Özkan-Haller[3]

Abstract

A general formalism for computing the nonlinear interactions between triads of coastally-trapped gravity and vorticity waves is developed. An analysis of the linearized problem reveals that gravity (or edge) waves and vorticity (or shear) waves exist as members of the same non-Sturm-Liouville eigenvalue problem, with unstable shear waves representing the complex eigenvalue portion of the resulting spectrum. Interaction equations derived here cover resonant interactions between three edge waves, three shear waves, or a shear wave and two edge waves. Numerical examples are shown for the case of three edge waves on a planar beach in the absence of a longshore current. It is found that edge waves can exchange significant amounts of energy over time scales on the order of ten wave periods, for realistic choices of edge wave parameters.

Introduction

The low frequency wave climate on an open coastal beach contains a complex mix of trapped gravity wave motions (edge waves) as well as vorticity (or shear) waves associated with the instability of the longshore current. To date, there has been a tendancy in the literature to treat both classes of motion as isolated systems in which the principle effect of nonlinearity is through amplitude dispersion. Formulations of this type typically treat the wave field in terms of a wave envelope modulated by cubic nonlinearity, leading to the cubic Schrödinger equation for conservative edge wave systems (Yeh, 1985) or Ginzburg-Landau equation for marginally unstable shear waves (Feddersen, 1998). However, in field conditions, all of these motions occur in a relatively dense spectral environment, and the existence of combinations of waves satisfying three-wave resonance conditions makes it likely that the dominant nonlinear mechanism affecting edge or shear waves would be through resonant interactions at second order.

Direct numerical simulations (Allen et al, 1997; Özkan-Haller and Kirby, 1998)

[1]Professor, Center for Applied Coastal Research, University of Delaware, Newark, DE 19716 USA.
[2]Research Scientist, NorthWest Research Associates, Inc., PO Box 3027, Bellevue, WA 98009-3027 USA.
[3]Assistant Professor, Department of Naval Architecture and Marine Engineering, University of Michigan, Ann Arbor, MI 48109 USA.

suggest that the growth to finite amplitude of the shear wave climate involves strong nonlinear interaction between the various length scales in the motion. It is likely that there are also opportunities for edge waves to undergo strong interactions, although this pathway has not been heavily investigated to date. All of these interactions contribute to the final evolution of the low frequency climate on a beach, which may or may not have some sort of equilibrium configuration.

The goals of present study are to:

1. Derive evolution equations describing the nonlinearly-coupled evolution of the discrete modes of the low frequency wave climate.
2. Use these equations to investigate the full range of edge wave - edge wave, shear wave - shear wave, and edge wave - shear wave interactions.
3. Couple the resulting system to incident wave conditions.
4. Investigate the equilibrium statistics of the resulting low-frequency wave climate, and compare to field measurements.

The core of our approach to this problem is the development of a spectral model describing nonlinear interactions between the free waves of the system by means of resonant interactions at second order. To date, the literature has identified the possibility of these resonances for the case of three edge waves (Kenyon, 1970; Bowen, 1976) or three shear waves (Shrira et al, 1997). We wish to add to this list the possibility of a triad involving a single shear wave and two edge waves, either of which can be propagating with or against the shear wave. Figure 1 illustrates such a case with all three waves propagating in the same direction as the longshore current. A general framework for computing these interactions is outlined below, and then specialized to the case of edge waves on a planar beach with no current in order to obtain analytical results.

Formulation of the Problem

For simplicity, our attention here is restricted to the case of unforced, undamped nonlinear long wave motions on a longshore uniform beach. The inclusion of forcing would lead to a coupling of the low-frequency motion to the incoming short wave climate (Lippmann et al, 1997). The introduction of longshore variability would extend the present analysis to include both the slow variation of model parameters in the longshore direction as well as the direct scattering of wave modes by wavelength-scale bottom features (Chen and Guza, 1998). These topics will be addressed in extensions of the present work.

The dependent variables in the present analysis are the surface displacement $\eta(x,y,t)$, cross-shore velocity $u(x,y,t)$ and longshore current $v(x,y,t) + V(x)$, where a distinction is made between the mean current profile $V(x)$ and the wave-induced fluctuations $v(x,y,t)$. The governing equations are given by

$$\frac{d\eta}{dt} + (hu)_x + hv_y \quad = \quad -(\eta u)_x - (\eta v)_y \equiv {}^{\eta}N \qquad (1)$$

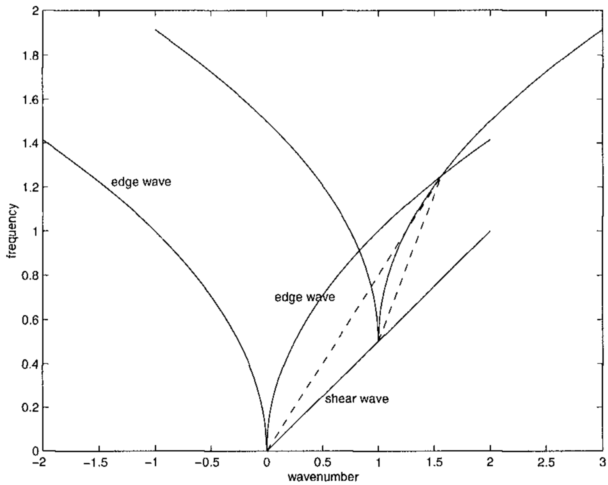

Figure 1: Diagram illustrating hypothetical resonant triad interaction involving a shear wave and two edge waves. Identifying the shear wave as the first wave in the triad, the origin of the edge wave dispersion curve is translated up the shear wave dispersion curve to the locus of shear wave frequency and wavenumber. Resonances involving two edge waves are then indicated by the intersections of the original and the translated edge wave dispersion curves. The two dashed lines here indicate two edge waves with the same mode number and propagating downstream with the longshore current.

$$\frac{d\,(hu)}{dt} + gh\eta_x = -huu_x - hvu_y \equiv {}^{u}N \tag{2}$$

$$\frac{d\,(hv)}{dt} + V'\,(hu) + gh\eta_y = -huv_x - hvv_y \equiv {}^{v}N \tag{3}$$

where a prime denotes differentiation with respect to x, and where

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + V(x)\frac{\partial}{\partial y} \tag{4}$$

is a time derivative following the local mean current. Eliminating u and v from linear terms gives

$$\frac{d}{dt}\left\{\frac{d^2\eta}{dt^2} - g(h\eta_x)_x - gh\eta_{yy}\right\} + 2ghV'\eta_{xy} = (\epsilon)\text{N.L.T.} \tag{5}$$

where

$$\text{N.L.T.} = \frac{d}{dt}\left\{\frac{d\,{}^{\eta}N}{dt} - ({}^{u}N)_x - ({}^{v}N)_y\right\} + 2V'({}^{u}N)_y \tag{6}$$

and where ϵ denotes a small parameter characterizing the weakness of the wave motions.

The Linearized Problem

We first seek solutions to the linearized problem, obtained by taking the limit $\epsilon = 0$ in (5). Solutions will be of the form

$$\eta = F(x)e^{i(\lambda y-\omega t)} \tag{7}$$

$$u = G(x)e^{i(\lambda y-\omega t)}; \quad G(x) = \frac{-ig}{\sigma}F'(x) \tag{8}$$

$$v = H(x)e^{i(\lambda y-\omega t)}; \quad H(x) = \frac{g}{\sigma}\left\{\lambda F - \left(\frac{V'}{\sigma}\right)F'\right\} \tag{9}$$

where

$$\sigma \equiv \omega - \lambda V(x) \tag{10}$$

is the local intrinsic frequency of the wave with respect to the local longshore current velocity. Substituting (7)-(9) in (5) gives an eigenvalue problem which may be written in self-adjoint form (Howd et al, 1992) as

$$\left(1 - \frac{g\lambda^2 h}{\sigma^2}F\right) + \left(\frac{ghF'}{\sigma^2}\right)' = 0 \qquad 0 \le x \le \infty \tag{11}$$

$$F \text{ bounded at } x = 0\ , \quad F \downarrow 0 \text{ as } x \to \infty \tag{12}$$

which is not convenient for solution of the eigenvalue problem but which serves as a basis for establishing solvability conditions in the nonlinear problem. The resulting eigenvalue problem is a non-Sturm-Liouville eigenvalue problem for $\{F^r(x), \omega^r\}$ given λ and $h(x)$. There are possible singularities at $\sigma^r = \omega^r - \lambda V_c = 0$, where V_c denotes the critical longshore current velocity. Possible types of solutions include:

1. Gravity motions without a critical level in the current profile → Distorted "regular" edge waves (Howd et al, 1992)
2. Gravity motions in the presence of a double set of critical levels, including:
   (a) Waves trapped against the shore by the faster offshore velocity (Falqués and Iranzo, 1992).
   (b) Waves trapped between the critical levels, propagating upstream relative to the current (Bryan and Bowen, 1998)
   (c) Waves trapped between the offshore critical level and deep water (hypothetical).
3. Vorticity motions representing the unstable growth of meanders in the longshore current (where ω^r is complex; Bowen and Holman, 1989) or the stable propagation of similar meanders (Falqués and Iranzo, 1992; Bowen and Holman, 1989).

For a given λ, the orthogonality condition for two modes with distinct mode numbers n, m and frequencies ω^n, ω^m is easily established,

$$\int_0^\infty \frac{gh(\sigma^n + \sigma^m)}{(\sigma^n)^2(\sigma^m)^2} \left\{ F^{n\prime} F^{m\prime} + \lambda^2 F^n F^m \right\} dx = 0$$

but we do not have a theorem for the completeness of the F^n basis. Since the system is of non-Sturm-Liouville form, we expect to obtain a complex spectrum of eigenvalues, of which the components containing positive imaginary parts will correspond to unstable and growing vorticity modes, or shear waves. We wish to emphasize here that the edge waves and shear waves are members of the same basis of eigenfunctions.

The Nonlinear Problem

Returning to the full problem, we follow the usual approach for obtaining evolution equations for variation of modal amplitudes on slow time and longshore space scales. We introduce multiple scales in order to identify slow changes of modal amplitudes in time and in longshore distance.

$$t \to t + \epsilon t = t + T \qquad (13)$$

$$y \to y + \epsilon y = y + Y \qquad (14)$$

We then introduce an expansion for η,

$$\eta = \eta^{(1)} + \epsilon \eta^{(2)} \qquad (15)$$

The solution for $\eta^{(1)}$ corresponds to a superposition of all eigenmodes of the linearized system,

$$\eta^{(1)} = \sum_n \sum_r \frac{1}{2} A_n^r(Y,T) F_n^r(x) E_n^r + \text{complex conjugate} \qquad (16)$$

where

$$E_n^r = e^{i(\lambda_n y - \omega_n^r t)} \qquad (17)$$

is the oscillatory dependence on fast time and longshore distance, and the F_n^r are the eigenmodes of the linear eigenvalue problem. At $O(\epsilon)$, we get a forced problem for each n, r combination. We require the forcing for each component to be orthogonal to the solution of the adjoint of the original eigenvalue problem. Nonlinear terms in the system may be simplified by imposing resonance conditions, given by

$$\pm\lambda_l \pm \lambda_m - \lambda_n = 0 \qquad (18)$$

$$Re\left\{\pm\omega_\ell^p \pm \omega_m^q - \omega_n^r\right\} = 0 \qquad (19)$$

The final evolution equation for each discrete mode in the system has the form

$$\begin{aligned} A_{nT}^r + C_{gn}^r A_{nY}^r = \; & i \sum_l \sum_m \sum_p \sum_q \{ {}_+T_{lmn}^{pqr} A_l^p A_m^q \delta(l+m-n)\delta(\omega_l^p + \omega_m^q - \omega_n^r) \\ & + {}_-T_{lmn}^{pqr} A_l^p A_m^{q*} \delta(l-m-n)\delta(\omega_l^p - \omega_m^q - \omega_n^r) \\ & + {}_-T_{mln}^{qpr} A_l^{p*} A_m^q \delta(m-l-n)\delta(\omega_m^q - \omega_l^p - \omega_n^r)\} \end{aligned} \qquad (20)$$

where ${}_{+}T$ and ${}_{-}T$ are complicated interaction coefficients for sum and difference interactions respectively. The group velocity C_{gn}^r for each mode is given by

$$C_{gn}^r = \frac{\int_0^\infty \left[\frac{2g\lambda_n}{(\sigma_n^r)^2} h (F_n^r)^2 + \frac{2V}{\sigma_n^r} (F_n^r)^2 - \frac{2g\lambda_n V V' h F_n^r F_n^{r'}}{(\sigma_n^r)^4} - \frac{2gV' h F_n^r F_n^{r'}}{(\sigma_n^r)^3} \right] dx}{\int_0^\infty \left[\frac{2}{\sigma_n^r} (F_n^r)^2 - \frac{2g\lambda_n h}{(\sigma_n^r)^4} V' F_n^r F_n^{r'} \right] dx} \tag{21}$$

In the no-current limit, the corresponding group velocity for edge waves on an arbitrary profile reduces to

$$C_{gn}^r = g \left(\frac{\lambda_n}{\omega_n^r} \right) \frac{\int_0^\infty h (F_n^r)^2 dx}{\int_0^\infty (F_n^r)^2 dx} \tag{22}$$

given originally by Pearce & Knobloch (1994).

In order to proceed beyond this point to a numerical determination of a solution, a number of steps need to be carried out. First, a reliable method of determining solutions for the linear eigenvalue problem must be established. Then, given eigenvalue pairs $\{\lambda_n, \omega_n^r\}$, we require an algorithm to reliably search for solutions to resonance conditions. Finally, an accurate means for evaluating integrals in expressions for C_g and the nonlinear coupling coefficients must be developed.

Edge Wave Interactions

In this section, we consider the special case of interaction between triads of edge waves on a planar beach in the absence of currents. In this case, the mode structure and wave dispersion relation is known, and model interaction coefficients may be evaluated analytically.

The possibility of triad interactions between progressive edge waves has been mentioned many times but not often addressed in a direct way. Kenyon (1970) provides a version of the Hasselmann interaction equations for random edge wave interactions, but provided no calculations. Kochergin and Pelinovsky (1989) consider the case of a colinear triad (all waves propagating the same direction) and show results for a single interacting triad. We will establish below that their results are wrong.

For the case of no currents, the interaction coefficients reduce to:

$$\begin{aligned} {}_{\pm}T_{lmn}^{pqr} = {} & \omega_l^p (\pm \omega_m^q) [8\omega_n^r \int_0^\infty (F_n^r)^2 dx]^{-1} \cdot \\ & \int_0^\infty \Big\{ 2(\omega_l^p \pm \omega_m^q) F_l^{p'} F_m^{q'} F_n^r + \omega_l^p F_l^p F_m^{q''} F_n^r \pm \omega_m^q F_l^{p''} F_m^q F_n^r \\ & + [2(\omega_l^p \pm \omega_m^q) \lambda_l^p (\mp \lambda_m^q) - \omega_l^p (\lambda_m^q)^2 \mp \omega_m^q (\lambda_l^p)^2] F_l^p F_m^q F_n^r \Big\} dx \end{aligned} \tag{23}$$

For a planar beach, the F_n^r are given in terms of Laguerre polynomials by

$$F_n^r(x) = \mathrm{e}^{-|\lambda_n| x} L_r(2|\lambda_n| x) \tag{24}$$

Solutions for isolated triads are obtained in terms of Jacobi elliptic functions. In the cases we have investigated, we have found that cases involving counterpropagating

Wave	Mode	Frequency	Wave number
1	0	ω_1	λ_1
2	0	$\omega_2 = \frac{1}{2}\omega_1$	$\lambda_2 = -\frac{1}{4}\lambda_1$
3	1	$\omega_3 = \frac{3}{2}\omega_1$	$\lambda_3 = \frac{3}{4}\lambda_1$

Table 1: Case 1. Parameters for lowest-order edge wave triad involving counter-propagating zero-mode waves.

waves show strong interactions with energy exchange time scales on the order of 10 wave periods. In contrast, cases involving colinear waves have interaction coefficients of zero, indicating an absence of interaction, contrary to the results of Kochergin and Pelinovsky (1989). Because this result is at odds with the existing literature, we verify it using a direct numerical simulation. The spectral-collocation method of Özkan-Haller and Kirby (1997) is used to obtain direct numerical solutions of the nonlinear shallow water equations with shoreline runup.

Results and Numerical Verification

As a first example, we consider the lowest-order triad involving two counter-propagating zero-mode edge waves, with the relation between frequencies, wavenumbers and mode numbers as indicated in Table 1. The geometry of the triad in wavenumber-frequency space is indicated in Figure 2. The resulting interaction equations are given by

$$A_{1T} = \frac{i\omega_1^3}{8gs^2} A_2^* A_3 \tag{25}$$

$$A_{2T} = \frac{i\omega_1^3}{64gs^2} A_1^* A_3 \tag{26}$$

$$A_{3T} = \frac{9i\omega_1^3}{64gs^2} A_1 A_2 \tag{27}$$

$$|A_1|^2 + |A_2|^2 + |A_3|^2 = \text{constant} \tag{28}$$

In this case, the parameters are chosen such that ω_1 corresponds to a wave with a period of $20s$ on a beach with a slope of $1:10$. In the results illustrated in Figure 3, we have initialized the triad by giving waves 1 and 2 amplitudes of $10cm$, with wave three having no amplitude to start. The resulting solution for the triad interaction is shown in Figure 3 by the smooth curves. The results indicate a complete exchange of energy between one of the Mode 0 waves and the Mode 1 wave propagating the same direction. The exchange occurs in somewhat less than 20 periods of the Mode 0 wave. The counterpropagating Mode 0 wave is crucial to the interaction but exchanges only a small amount of energy with the other modes. This non-reactivity of the counterpropagating wave has been noted for a wide range of initial conditions.

The analytic results shown in Figure 3 have been verified using direct numerical

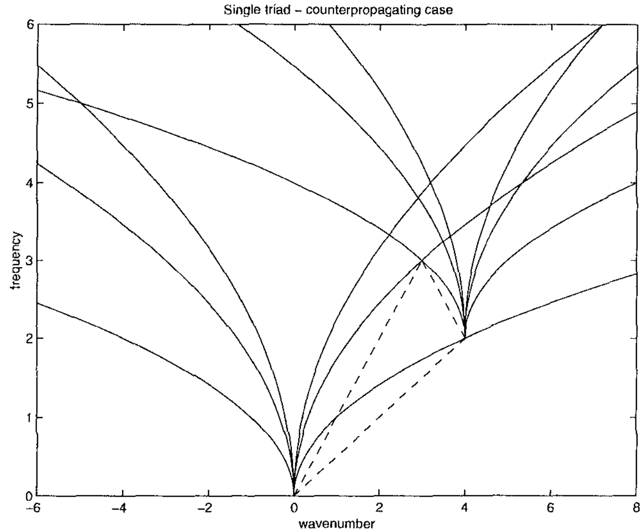

Figure 2: Resonant triad edge-wave interaction with counterpropagating components.

simulation with the pseudospectral model of Özkan-Haller and Kirby (1997). Results from that model were obtained by Fourier transforming the longshore dependence of the runup tip. Results are shown in Figure 3 as the curves with smaller-scale jitter in time. (This jitter occurs at wave-period or sub-wave-period scales, and is probably associated with the fact that the linear edge waves input as initial conditions differ from fully nonlinear solutions to the problem.) Agreement between analytical triad results and numerical solutions are close, with the numerical solutions indicating a slightly slower energy exchange time and a tendency for energy to leak out of the three components making up the triad.

The fate of the missing energy can be seen in the plot of the frequency-wavenumber spectrum computed from the numerical solution, shown in Figure 4. The spectrum is dominated by the three waves making up the resonant triad, but there are clear contributions at forced, non-resonant peaks representing sum and difference interactions lying off the edge wave dispersion curves. There has also been an excitation of the Mode 0 edge wave at twice the wavenumber of Wave 1, and at a frequency that is not commensurate with any sum or difference combination in the original triad. The mechanism for exciting this free wave is not clear and may be associated with start-up transients in the initial value problem.

Figure 5 shows one longshore period of the numerically computed wave field at two instances in time. The top panel shows the situation at 20 wave periods into the simulation, where the wave field is dominated by the higher-frequency Mode 1

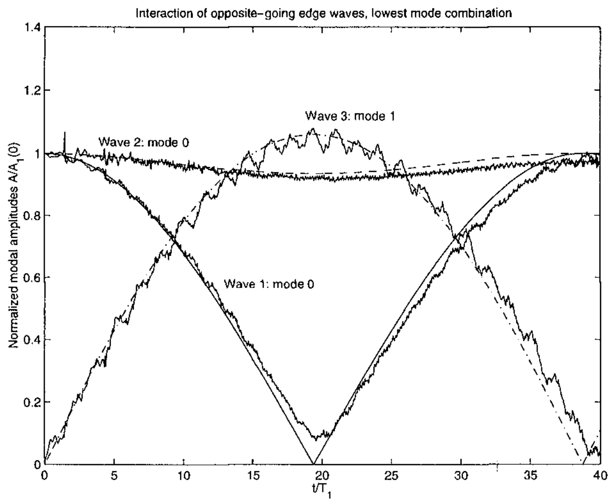

Figure 3: Comparison of time series of modal wave amplitudes: analytic and numerical results.

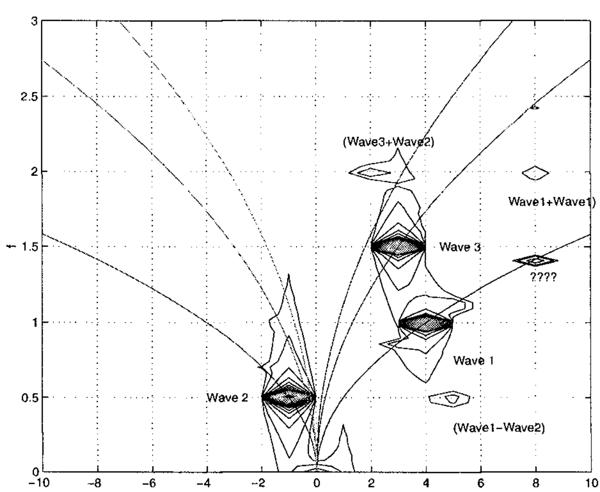

Figure 4: Frequency-wavenumber spectrum for case of counterpropagating waves. Direct numerical simulation.

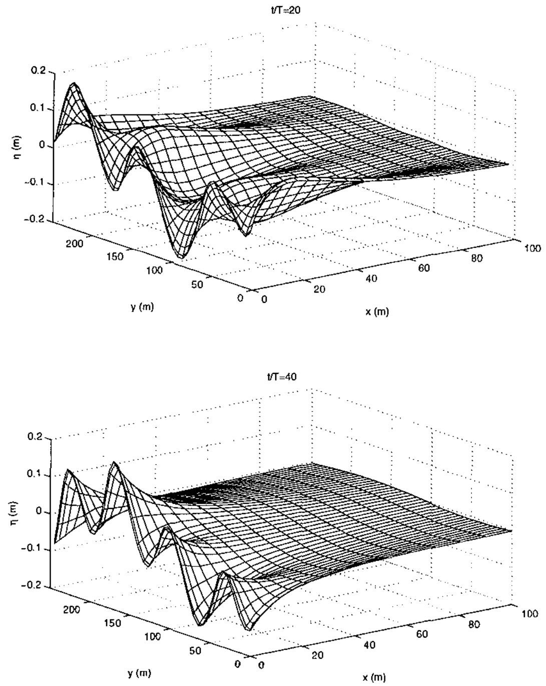

Figure 5: Snapshot of numerically computed instantaneous wave field showing conditions dominated by Mode 1 wave (top panel) and Mode 0 wave (lower panel).

wave riding on the longer, counterpropagating Mode 0 wave. The lower panel shows the situation at 40 periods (close to the end of the recurrence cycle), where the two counterpropagating Mode 0 waves dominate the wavefield.

As a second example, we consider the case elaborated by Kochergin and Pelinovsky (1989) with all waves travelling the same direction, illustrated in Figure 6. The parameters for the lowest-order case are indicated in Table 2. The present theory indicates that nonlinear interaction coefficients reduce to zero, giving solutions $A_1, A_2, A_3 =$ constant. Figure 7 shows time histories for the first twelve Fourier modes of the longshore runup in a direct numerical simulation, with modes $k = 1$ and $k = 3$ corresponding to the initialized low-frequency modes in the triad. The numerical results indicate no interaction between the initialized modes and an absence of growth of the third member of the possible triad. This result is also clear in the resulting frequency-wavenumber spectrum shown in Figure 7, which shows an almost complete lack of energy appearing at the third component, which would appear at scaled wavenumber $k = 4$ and frequency $f = 2$.

Wave	Mode	Frequency	Wave number
1	0	ω_1	λ_1
2	1	$\omega_2 = \omega_1$	$\lambda_2 = \frac{1}{3}\lambda_1$
3	1	$\omega_3 = 2\omega_1$	$\lambda_3 = \frac{4}{3}\lambda_1$

Table 2: Parameters for lowest order triad with waves propagating in the same direction, as in Kochergin and Pelinovsky (1989).

Conclusions

In this paper, we have described a framework for deriving coupled-mode equations for a sea of edge waves and shear waves. Interaction coefficients have been obtained for the special case of edge waves on a plane beach in the absence of currents. For this system, interactions have been shown to exist and to be fairly rapid for triads involving counterpropagating waves. Triads involving unidirectional propagation have been found to not lead to interaction, in contradiction to the existing literature. We do not yet have a conclusive proof that this result holds for all colinear edge wave triads on a planar beach, but it has been found to hold for all combinations tested so far. Results for both cases have been verified by direct numerical simulation. The close agreement between numerical and analytic results also indicates that a weakly nonlinear formulation is appropriate for examining edge wave interactions. This result is to be expected due to the strongly dispersive nature of the edge wave motions.

The work on edge wave interactions is presently being extended to look at more complicated systems involving multiple coupled triads, leading up to an evaluation of equilibrium distribution of energy in a random sea of edge waves. In order to further this goal, we need to:

1. Automate the process of identifying resonances.
2. Extend calculations to a large number of components, in order to investigate the assumptions to be made in going over to a stochastic version of the equations.
3. Implement the stochastic version and couple it to the incident wave climate.

In addition, the limitation of the present analytical theory to the case of waves on planar beach topographies is restrictive, and needs to be extended to the case of non-planar topographies such as the exponential profile of Ball (1967). It is also possible that the non-interaction of edge wave triads involving waves propagating the same direction, found here for waves on a planar beach, is an anomalous result that will not hold for arbitrary topographies.

For the case with a net longshore current added to the system, we need to elaborate the process for numerically determining the eigenmodes for an arbitrary topography and longshore current distribution, and then repeat the steps outlined above.

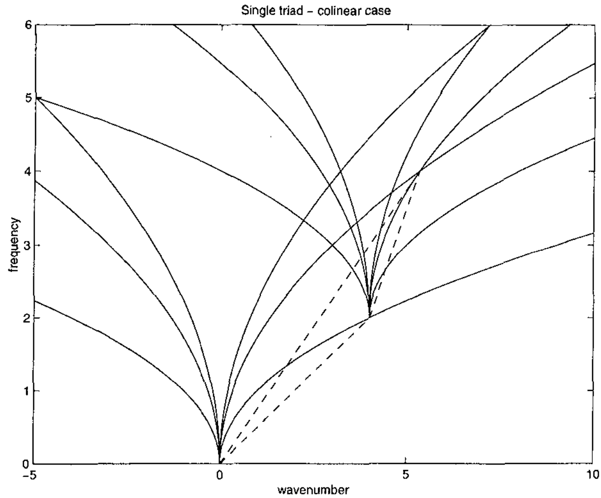

Figure 6: Single triad with colinear components. No resulting interaction.

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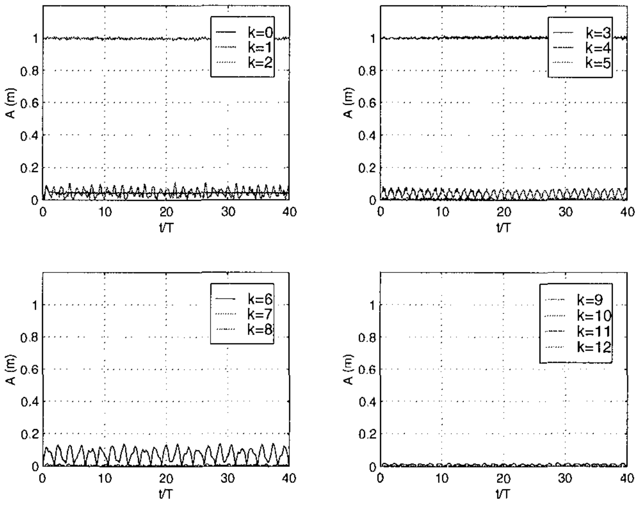

Figure 7: Time series of modal wave amplitudes for colinear case: Direct numerical simulation.

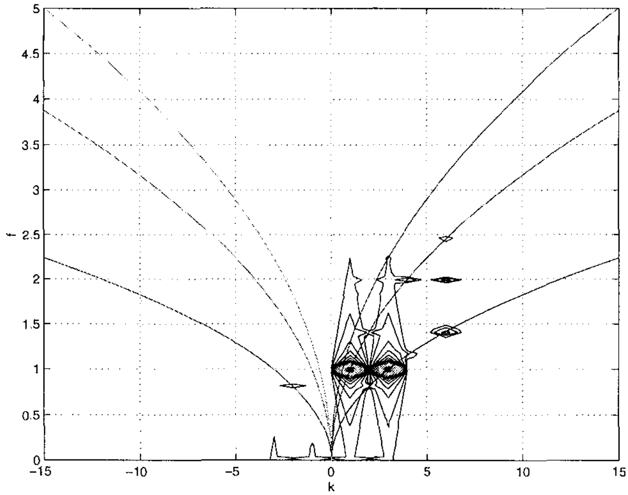

Figure 8: Wavenumber-frequency spectra for colinear case: Direct numerical simulation.

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