# COMPENSATORY REVERSE FLOW OF PROGRESSIVE WAVES WITH FINITE AMPLITUDE

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This paper is devoted to problem of mass transport of fluid for the surface progressive waves. Both Stokes and cnoidal waves are considered. New solutions for the transitional current are obtained. It is discovered that the mass transport of fluid in the direction of wave propagation exists only in the top layer. In the underlying layers a compensatory reverse flow is formed. The existence of a compensatory flow was verified experimentally. It is revealed that theoretical results duly conform to experimental data.

*Keywords: Stokes waves; cnoidal waives; Stokes drift; reverse flow*

## INTRODUCTION

Waves on the water surface can be generated by a large number of causes. These include wind, underwater tectonic activity, astronomical forces, resonance in basins and movement of ships.

Since the propagation of these waves is caused mainly by gravity, they are called gravity waves. These gravity waves have a wide range of properties, and quite often their effect on the shores and structures and their impact on the transfer processes in basins is the main. Hence problems on waves were and still are of great interest to scientists.

Among the reckoned founders of wave theory are the names of Euler, Lagrange, Cauchy, Stokes, Boussinesq, Korteweg, de Vries and others. There are dozens of monographs and hundreds of articles devoted to the theory of gravity waves. Fundamental results in the field of wave theory are due to Levi-Civita (1925), Struik (1926), Kochin (1927), Lamb (1932), Lavrentyev (1946), Keller (1948), Friedrichs and Hyers (1954), Littman (1957), Stoker (1959), Peregrine (1967), Whitham (1974), Lighthill (1978), Ovsyannikov (1983), Fenton (1985), Madsen and Sorensen (1992) and others.

Mostly widespread are the two theoretical trends, one of which assumes a small wave height relative to the wave length, the other – small depth of water relative to the wave length. Moreover, from a mathematical point of view, wave motion can be linear and non-linear, depending on whether convective inertial terms are taken into account or not.

The linear problem concerning waves of small amplitude was already studied in detail in the 19th century. Solutions for kinematic and energy wave characteristics were obtained, the notions of phase and group speed were introduced. It was proved that the group speed is not only a kinematic, but also an energy characteristic as it defines the speed of wave energy propagation.

However there are still many areas where more knowledge is needed, especially in respect of non-linear waves. In particular, the solution of the problem concerning mass transport of fluid by the propagation of progressive waves of finite amplitude needs to be refined.

## STOKES WAVES

The problem concerning stationary progressive waves of finite height on the water surface with a constant depth was first solved by Stokes (1847). It was assumed that the fluid is incompressible and irrotational. The wave profile, axes of coordinates and basic notations are given in Fig. 1.

The wave shape and the speed of its propagation are assumed to be constant. In this case in the system of coordinates $x$, $z$, which moves along the direction of the wave with phase speed $C$, the movement will be steady. Moreover, the flow is two-dimensional and potential. It is known that for such a flow it is possible to find a stream function $\psi$, identically conforming to the equation of continuity and associated with the velocity potential $\varphi$ by the conditions of Cauchy-Riemann

$$\frac{\partial\varphi}{\partial x}=\frac{\partial\psi}{\partial z}\ ,\quad \frac{\partial\varphi}{\partial z}=-\frac{\partial\psi}{\partial x}\ . \tag{1}$$

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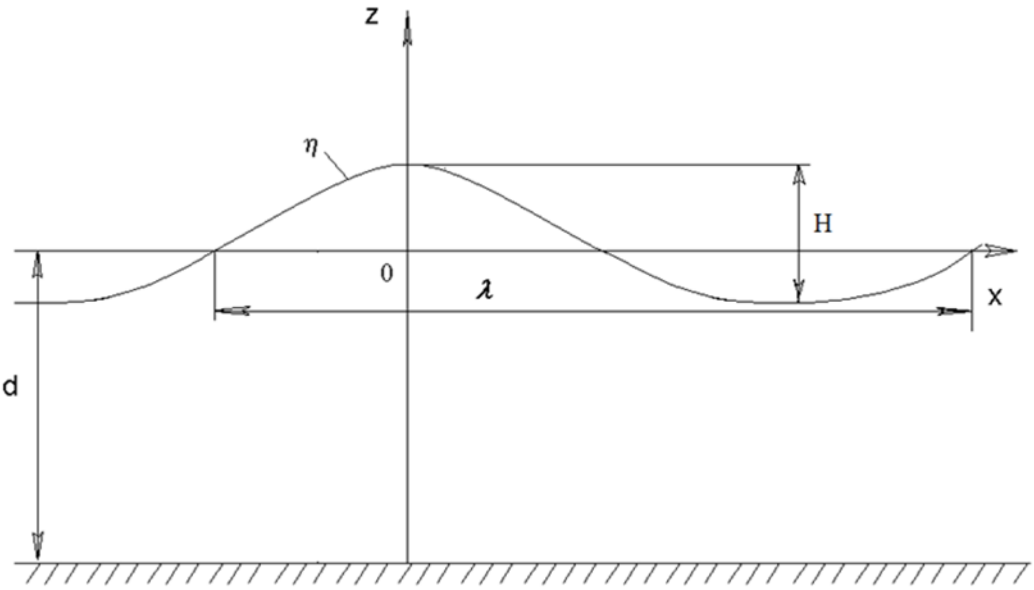


**Figure 1. Progressive wave: *d* = depth; $\eta$ = fluid surface profile; $\lambda$ = wave length; *H* = wave height.**

Generally, it is necessary to find a solution for the free surface profile, as well as the velocity potential and the stream function. However, the majority of published works, investigating stationary progressive waves of finite amplitude with constant depth, give solutions only for the wave profile and for the velocity potential. A refined solution of the progressive wave problem is therefore presented below.

The wave profile $\eta$, moving with the phase speed *C*, the system of coordinates *x z* and the major parameters $d, H, \lambda$ are schematically shown on the Fig.1.

Let us define for the waves of finite amplitude the functions $\eta(x), \varphi(x,z), \psi(x,z)$.

Further let's introduce dimensionless values

$$x^* = kx, \quad z^* = kz, \quad d^* = kd, \quad \varphi^* = \frac{k}{C}\varphi, \quad \psi^* = \frac{k}{C}\psi, \quad \eta^* = k\eta, \quad b^* = \frac{b}{C^2} \;,$$

where $k = 2\pi/\lambda$ =wave number, $b$ – Bernoulli constant.

Below the index (*) is omitted. Let us find the functions $\varphi(x,z)$, $\eta(x)$, which comply with the equation of continuity

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial z^2} = 0 \quad , \tag{2}$$

the boundary condition at the bottom by $z = -d$

$$\frac{\partial\varphi}{\partial z} = 0 \tag{3}$$

and the boundary conditions on the free surface by $z = \eta$

$$\frac{\partial\varphi}{\partial z} = \frac{\partial\eta}{\partial x}\frac{\partial\varphi}{\partial x} , \tag{4}$$

$$\frac{g}{C^2k}\eta + \frac{1}{2}\left[\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial z}\right)^2\right] = b \tag{5}$$

For the stream function and the wave profile the problem is formulated as follows. It is required to find the functions $\psi(x,z)$ and $\eta(x)$, which satisfy the condition of vorticity absence

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial z^2} = 0 , \tag{6}$$

and the boundary condition at the bottom by $z = -d$

$$\frac{\partial \psi}{\partial x} = 0 , \tag{7}$$

and the boundary conditions on the free surface by $z = \eta$

$$\psi = 0 , \tag{8}$$

$$\frac{g}{C^2 k}\eta + \frac{1}{2}\left[\left(\frac{\partial \psi}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial z}\right)^2\right] = b \cdot \tag{9}$$

If $\varepsilon = ka$ is a small parameter ($a$ – typical amplitude of free surface fluctuations) and the solution can be presented in the form of power series of a small parameter $\varepsilon$ , then to the first approximation we obtain (Shakhin and Shakhina, 2009)

$$\begin{aligned}\varphi = -x + \varepsilon \frac{\cosh(d+z)}{\sinh d}\sin x + \frac{3}{8}\varepsilon^2 \frac{\cosh 2(d+z)}{\sinh^4 d}\sin 2x \\ + \varepsilon^3 \frac{(13 - 4\cosh^2 d)\cosh 3(d+z)}{64\sinh^7 d}\cdot \sin 3x - \varepsilon^2 \frac{\cosh d}{\sinh d}\frac{x}{2d}\end{aligned}, \tag{10}$$

$$\begin{aligned}\psi = -z + \varepsilon \frac{\sinh(d+z)}{\sinh d}\cos x + \frac{3}{8}\varepsilon^2 \frac{\sinh 2(d+z)}{\sinh^4 d}\cos 2x \\ + \varepsilon^3 \frac{(13 - 4\cosh^2 d)\sinh 3(d+z)}{64\sinh^7 d}\cdot \cos 3x - \varepsilon^2 \frac{\cosh d}{\sinh d}\frac{(d+z)}{2d}\end{aligned}, \tag{11}$$

$$\begin{aligned}\eta = \varepsilon \cos x + \frac{\varepsilon^2}{4}\frac{\cosh d(1 + 2\cosh^2 d)}{\sinh^3 d}\cos 2x + \varepsilon^3 \frac{1}{64\sinh^6 d} \\ \cdot \left[8\cosh^2 d(5\cosh^4 d - 4\cosh^2 d - 1)\cos x + 3(1 + 8\cosh^6 d)\cos 3x\right] \\ - \varepsilon^3 \frac{1}{2d}\frac{\cosh d}{\sinh d}\cos x\end{aligned} \tag{12}$$

$$b = \frac{1}{2} + \varepsilon^2 \frac{\sinh 2d + d}{4d \cdot \sinh^2 d} . \tag{13}$$

$$C = \sqrt{\frac{g}{k}\frac{\sinh d}{\cosh d}} \cdot \left[1 + \frac{\varepsilon^2}{2}\left(\frac{8\cosh^4 d - 8\cosh^2 d + 9}{8\sinh^4 d} - \frac{1}{d}\frac{\cosh d}{\sinh d}\right)\right]. \tag{14}$$

Wave height $H$, determined as the difference of free surface layers by $x$=0 and $x = \pi$ , will be equal to

$$H = 2\varepsilon + \varepsilon^3 \frac{\left[8\cosh^2 d(5\cosh^4 d - 4\cosh^2 d - 1) + 3(1 + 8\cosh^6 d)\right]}{32\sinh^6 d} - \varepsilon^3 \frac{1}{d}\frac{\cosh d}{\sinh d} . \tag{15}$$

It should be noted that relations Eqns.10, 12, 14, 15 differ from relations obtained by other authors by last members.

In dimensional variables the relations for stream function $\psi(x,z)$ and horizontal velocity component $u(x, z)$ are as follows:

$$\begin{aligned}\psi = -cz + ca\frac{\sinh k(d+z)}{\sinh kd}\cos kx + \frac{3}{8}cka^2 \frac{\sinh 2k(d+z)}{\sinh 4kd}\cos 2kx \\ + ck^2 a^3 \frac{(13 - 4\cosh^2 kd)\sinh 3k(d+z)}{64\sinh^7 kd}\cos^3 kx - cka^2 \frac{\cosh kd}{\sinh kd}\cdot \frac{(d+z)}{2d}\end{aligned} \tag{16}$$

$$u = \frac{\partial \psi}{\partial z} = -c + cka\frac{\cosh k(d+z)}{\sinh kd}\cos kx + \frac{3}{4}ck^2a^2\frac{\cosh 2k(d+z)}{\sinh^4 kd}\cos 2kx + \frac{3}{64}ck^3a^3 \cdot \frac{(13-4\cosh^2 kd)\cosh 3k(d+z)}{\sinh^7 kd}\cos 3kx - \frac{1}{2d}cka^2\frac{\cosh kd}{\sinh kd} \quad (17)$$

In stationary system of coordinates the relation for horizontal velocity component is as follows:

$$u_0 = a\omega\frac{\cosh k(d+z)}{\sinh kd}\cos(kx-\omega t) + \frac{3}{4}a^2\omega k\frac{\cosh 2k(d+z)}{\sinh^4 kd}\cos 2(kx-\omega t) + \frac{3}{64}a^3\omega k^2\frac{(13-4\cosh^2 kd)\cosh 3k(d+z)}{\sinh^7 kd}\cdot\cos 3(kx-\omega t) - \frac{a^2\omega}{2d}\frac{\cosh kd}{\sinh kd} \quad (18)$$

where $t$=time, $\omega=2\pi/T$=angular frequency, $T$=wave period.
If the Eq. 18 is integrated by time from 0 to $T$ we derive that in area $-d<z<\eta_{\min}$ ($\eta_{\min}$ = wave troughs mark) the average velocity of the fluid $u_m$ is negative.

$$u_m = \frac{1}{T}\int_0^T u_0 dt = -\frac{a^2\omega}{2d}\coth kd \cdot \quad (19)$$

The Eq. 19 was verified by experiment. The experiments were conducted in a wave flume. The flume bottom was horizontal. The waves were generated by a wave-maker, mounted in a pit near one of the end walls, and were dampened by a wave absorber at the other end wall of the flume. The wave flume is illustrated in Fig.2. The measurements of the average horizontal component of velocity were made by laser Doppler anemometer.


**Figure 2. The wave flume**

The results of measurements of the average velocity at $d$=40 cm, $T$=0,62 s, $H$=5 cm are illustrated in Fig.3

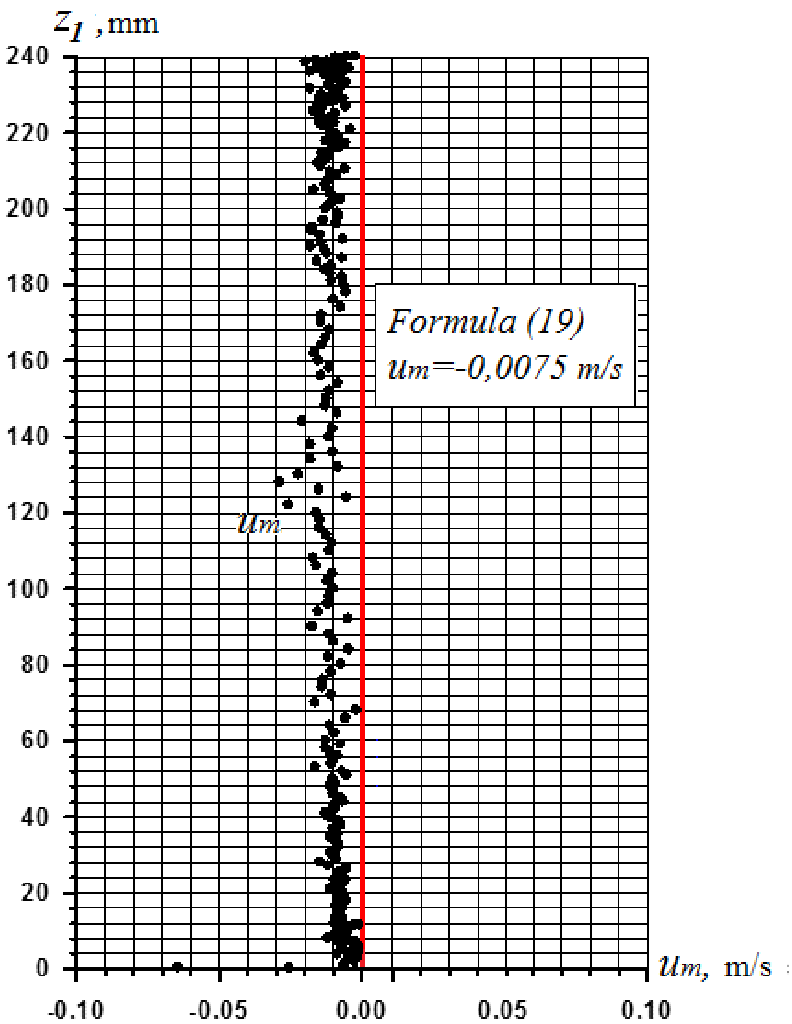


**Figure 3. Profile of the average horizontal component of velocity : d=40 cm, T=0,62 s, H=5 cm**

The calculated value is -0,0075. We can see that the result of calculation is in good agreement with experimental data.

**CNOIDAL WAVES**

The theoretical description of surface waves with a relatively long length is done by the method of asymptotic expansions in power series of a small parameter, which indicates the ratio of the fluid depth to the characteristic length of the wave. The solution for non-linear periodical waves in the shallow water accurate to the 2nd approximation was initially obtained by Korteweg and de Vries (1895).

The equation for the wave profile in dimensional variables is as follows

$$h = h_2 + (h_3 - h_2)cn^2 \frac{2K(\xi)}{\lambda} x \ , \tag{20}$$

where $x$ = horizontal coordinate in the system of coordinates, moving with the phase speed of the wave $C$; $h(x)$ = free surface profile in relation to the bottom level; $h_2$ = wave trough mark; $h_3$ = wave top mark; $H=h_3-h_2$ = wave height; $\lambda$ = wave length; $cn$ = Jacobian elliptic cosine function; $K(\xi)$ = complete elliptical integral of the first kind; $\xi$ = squared modulus of the elliptical function (internal parameter).

Basing on the results, obtained in the work Ovsyannikov (1983), it is possible, knowing $d$, $H$ and $T$, to find the data $h_2$, $h_3$, $\lambda$ and $C$, (Shakhin and Atavin 2004)

$$h_2 = d - H\left(1 - \frac{1}{\xi} + \frac{E(\xi)}{\xi K(\xi)}\right) = d\{1 + \beta K(\xi)[K(\xi) - E(\xi) - \xi K(\xi)]\}, \qquad (21)$$

$$h_3 = h_2 + H = d\{1 + \beta K(\xi)[K(\xi) - E(\xi)]\}, \qquad (22)$$

$$C = \sqrt{gd}\sqrt{(1-\beta KE)[1+\beta K(K-E)][1+\beta K(K-E-\xi K)]}, \qquad (23)$$

$$\lambda = C \cdot T. \qquad (24)$$

Where $\beta = 16d/(3gT^2)$ = parameter of dispersion; $E(\xi)$ = complete elliptical integral of the second kind. Parameters $\xi$, $K$, $E$ are unambiguously defined for certain $d$, $H$ and $T$ from ratio, (Ovsyannikov 1983)

$$\sqrt{\xi}K = \frac{\sqrt{3gH}T}{4d}. \qquad (25)$$

The expressions for the stream function and the horizontal velocity component can be written as follows, (Shakhin and Shakhina 2015)

$$\psi = \frac{dA}{dx} z_1 - \frac{1}{6}\frac{d^3A}{dx^3} z_1^3, \qquad (26)$$

$$u = \frac{\partial \psi}{\partial z_1} = \frac{dA}{dx} - \frac{1}{2}\frac{d^3A}{dx^3} z_1^2, \qquad (27)$$

where $z_1$ = the vertical coordinate relative to the bottom.

The expressions for $\frac{dA}{dx}$ and $\frac{d^3A}{dx^3}$ look as follows, (Ovsyannikov 1983)

$$\frac{dA}{dx} = \frac{Q}{h} + \frac{1}{6}a\left[\frac{2(dh/dx)^2}{h} - \frac{d^2h}{dx^2}\right], \qquad (28)$$

$$\frac{d^3A}{dx^3} = \frac{Q}{h^2}\left[\frac{2(dh/dx)^2}{h} - \frac{d^2h}{dx^2}\right], \qquad (29)$$

where $Q = -Cd$.

Taking into account Eqns. 28, 29 the formulae for the stream function and the horizontal velocity component will become as below

$$\psi = Q\frac{z_1}{h} + \frac{1}{6}Qz_1\left(1 - \frac{z_1^2}{h^2}\right)\left[\frac{2(dh/dx)^2}{h} - \frac{d^2h}{dx^2}\right], \qquad (30)$$

$$u = \frac{Q}{h} + \frac{Q}{2}\left(\frac{z_1^2}{h^2} - \frac{1}{3}\right)\left[\frac{d^2h}{dx^2} - \frac{2(dh/dx)^2}{h}\right]. \qquad (31)$$

In stationary system of coordinates the relation for horizontal velocity component at a certain time is as follows

$$u_0 = C + \frac{Q}{h} + \frac{Q}{2}\left(\frac{z_1^2}{h^2} - \frac{1}{3}\right)\left[\frac{d^2h}{dx^2} - \frac{2(dh/dx)^2}{h}\right]. \qquad (32)$$

It should be noted, that if the Eq. 32 is integrated by length from 0 to $\lambda$ we derive that in area $0<z_1<h_2$ the average velocity of the fluid $u_m$ is nearly constant and negative. The result satisfactorily corresponds with result for stokes waves.

## THE TRANSFER VELOCITY OF FLUID PARTICLES

Since in the stationary two-dimensional potential flow the trajectories of the moving fluid particles coincide with the stream lines, where $\psi$ =const, thus, knowing relations for the stream function $\psi$ ,for the horizontal velocity component $u$ and for the phase speed $C$, it is possible to determine the time $t_z$, during which the virtual fluid particles, being at different levels, pass the distance along the $x$ axis, equal to the length of the wave. Further, knowing the length $\lambda$ and the phase speed $C$ of the wave, it is possible to determine the average transfer velocity, with which the real fluid particles move

$$u_t = C - \lambda / t_z \,. \tag{33}$$

For stokes waves in dimensional variables $\psi$ , $u$, $C$ are represented by the Eqns. 16, 17, 14 and for cnoidal waves $\psi$ , $u$, $C$ are determined by Eqns. 30, 31, 23.

As an example, the respective calculations of the velocity $u_t$, based on Eqns. 16, 17, 14 and 30, 31, 23 are carried out using the following external parameters: water depth $d$ = 10 m; wave height $H$ = 3 m; wave period $T$ = 8 s. The results of the calculations are shown in the Fig. 4, where $z_0$ = vertical coordinate of the fluid particle in the unperturbed state.

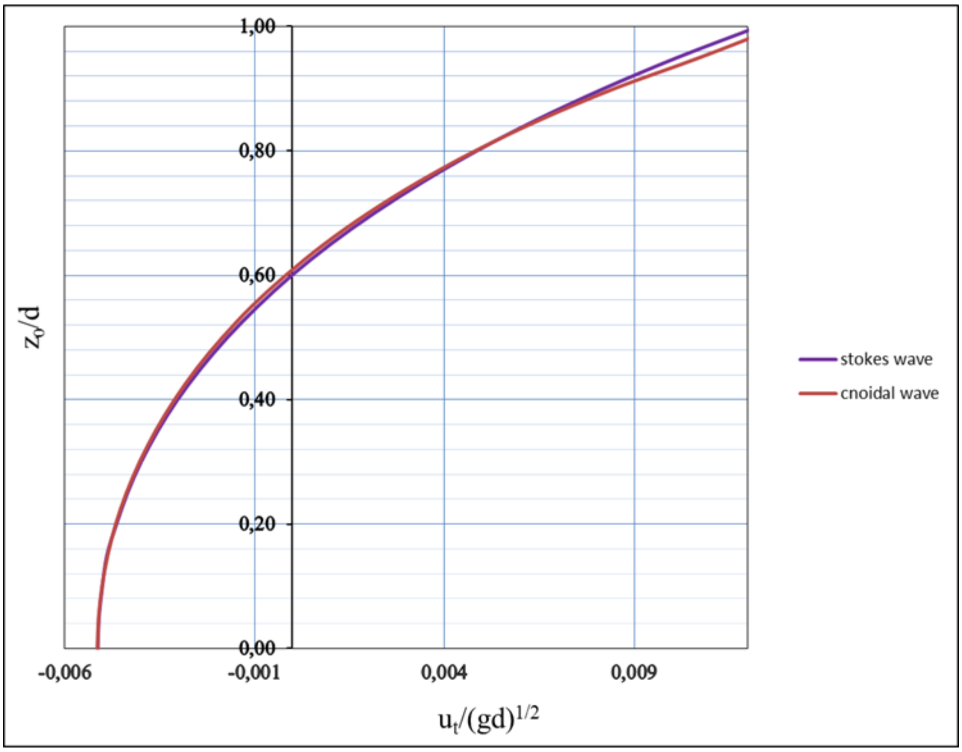


**Figure 4. The profiles of the transfer velocity $u_t$**

It can be seen that the mass transfer, described by the mathematical models, takes place in two directions: in the surface layer the average current is directed along the movement of the waves, while in the bottom layer a compensatory counter-flow is formed.

## EXPERIMENTAL DATA

A series of laboratory experiments to measure the trajectories of particles for progressive waves were performed by Chen et al. (2010). But the trajectories of the particles were measured only in the upper layer in the lower layers has not been studied.

The existence of a compensatory flow was verified by experiment, (Shakhin 2001). The experiments were conducted in a wave flume. The flume bottom was horizontal. The waves were generated by a wave-maker, mounted in a pit near one of the end walls, and were dampened by a wave absorber at the other end wall of the flume. The measurements were taken in the middle part of the flume, approximately 10 meters away both from the wave-maker and the wave absorber. Two series of experiments were conducted.

In the first series it is assumed: depth of water $d$=70 cm; height of wave $H$=7 cm; wave period $T$=1s.

In the second series of experiments these data had the following values: $d$=37 cm; $H$=7 cm; $T$=2 s.

For measuring the velocity in the surface layer, indicator particles were used. Indicator particles had the form of a negative buoyancy triangle plate with the area of 5-7mm$^2$, held by a thin thread at the distance of about 2 cm from the surface with the help of small balls with positive buoyancy. In the bottom layer an ink mark was inducted as an indicator. The transfer velocity was determined by the movement of the indicator particle or the ink mark during 10 wave periods. The record of the indicators movement in relation to the coordinate grid, plotted on the transparent side wall of the flume, was done by a video camera. In course of information processing the positions of the indicators at certain time moments were registered from the screen picture.

The measured and the calculated values of the velocity are presented in the Table 1. The table also highlights the results of calculation by the Stokes Eq. 34

$$u_t = \frac{k^2 a^2 C \cosh 2k(z_0 + d)}{2 \sinh^2 kd} . \tag{34}$$

**Table 1 Comparison of theoretical and experimental values of the transfer velocity**

| Initial parameters | $z_0$,cm | Experimet $u_t$, cm/s | Theory Eqns.16,17,14 $u_t$, cm/s | The Stokes Eq. 34 $u_t$, cm/s |
|---|---|---|---|---|
| $d$=70 cm<br>$H$=7 cm<br>$T$=1 s | -2<br>-67 | 2.2<br>-0.5 | 2.1<br>-0.51 | 2,6<br>0,022 |
| $d$=37 cm<br>$H$=7 cm<br>$T$=2 s | -2<br>-34 | 0.45<br>-0.2 | 0.44<br>-0.23 | 1,17<br>0,63 |

We can see that the Stokes formula gives a positive value of transfer velocity near the bottom. But the experiment gives a negative value.

## CONCLUSION

New results based on Stokes and cnoidal waves theories are obtained. In particular, it is established that, by the propagation of the progressive waves with finite amplitude, in the bottom layers of the fluid a compensatory counter-current is formed. The compensatory reverse flow is an integral part of the wave process, and it is a "reaction" of the fluid to the potential alteration of the average level as a result of mass transfer in the upper layer.

It can be noted, that the impact of the surface waves is not limited by the depth, which is approximately equal to one half of the wave length. A substantial compensatory reverse flow can be formed at the depths, exceeding considerably the wave length. This factor should be taken into account when estimating the water exchange in bottom zone and forecasting the evolution of the shelf terrace as a result of the sediment accumulation.

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