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TEXAS A&M UNIVERSITY

A Model In Frequency Domain For Transformation of Fully Dispersive Nonlinear Waves

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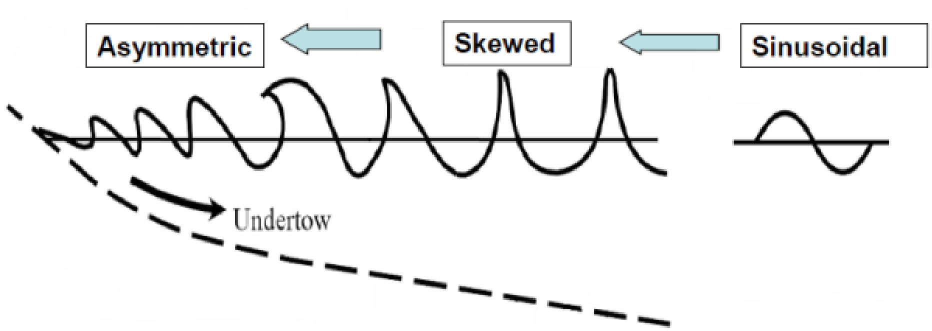
Why Studying Wave Nonlinearity?

Nonlinearity
- Energy exchange among different frequencies during transformation of waves

Shape of the waves
- Capture the higher frequencies
- Accurate modeling of the wave shape parameters

Sediment transport
- Asymmetry in bed shear stress
- Better estimation of instantaneous sediment transport and sandbar migration

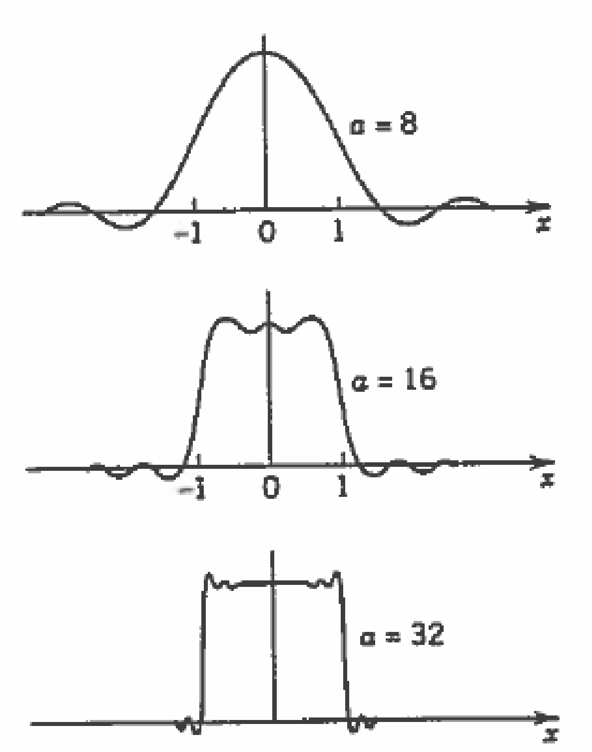
Morphology
- Accurate prediction of surf zone morphology



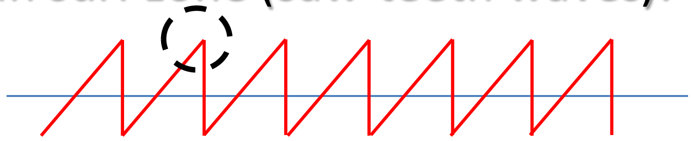
citation: Ole Secher Madsen (2010)

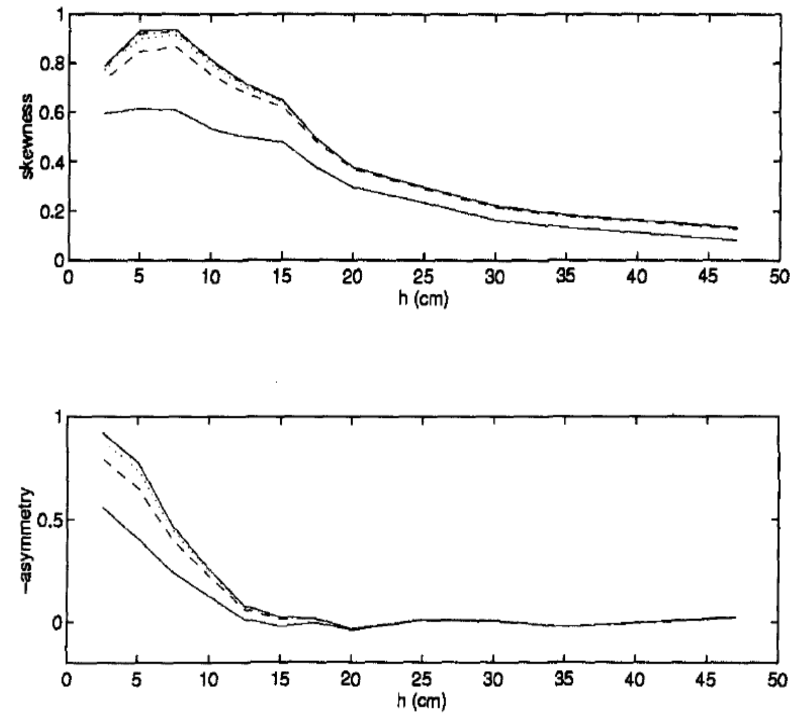
What are the effects of higher frequencies?

Example:



In surf zone (saw-teeth waves):



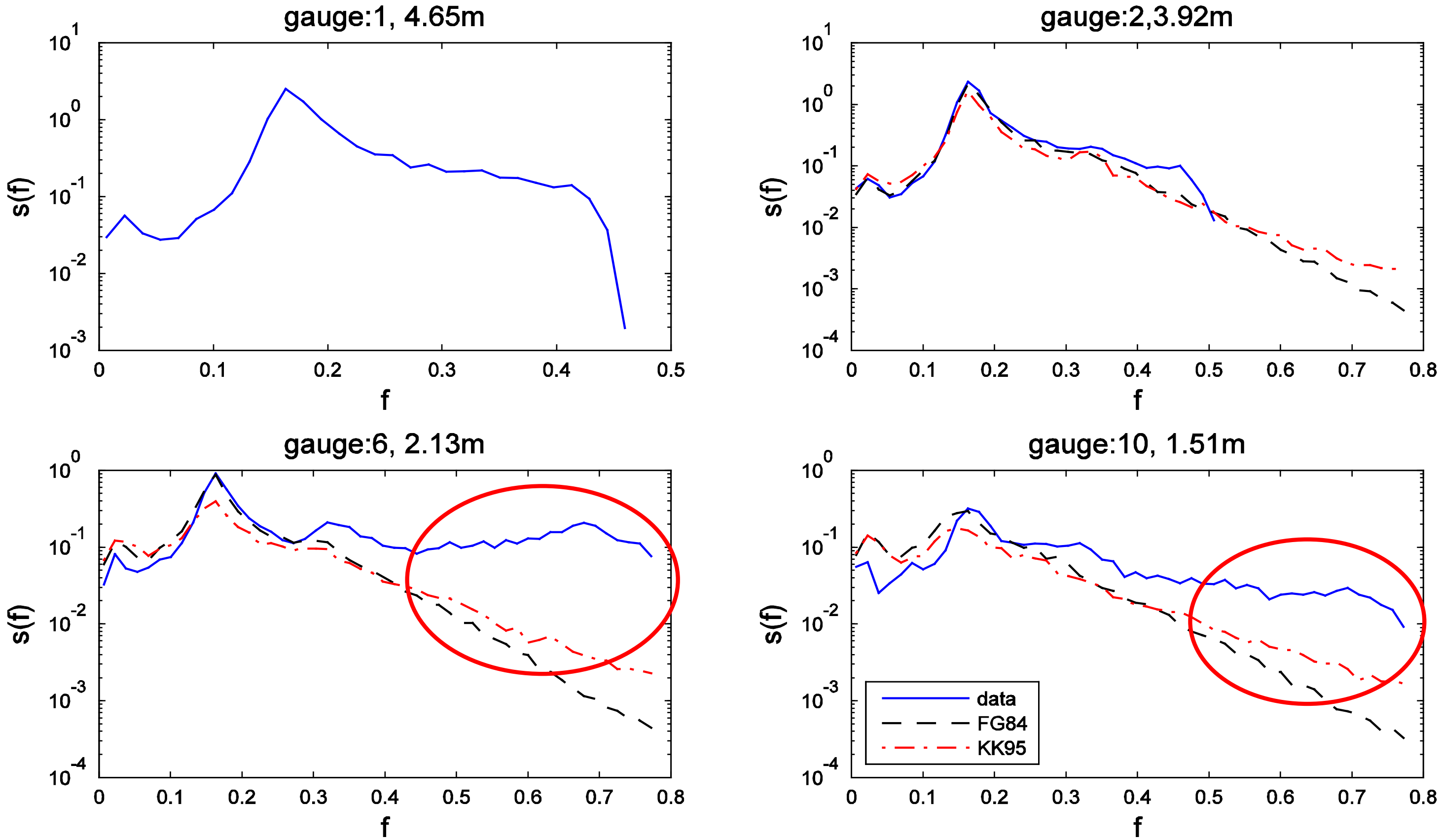


Kreyszig, E.(1988), Advanced engineering mathematics

Kirby and Kaihatu (1996)

Research question:

How to improve the deviation of nearshore models from measured data at high frequencies?



FG84 vs. KK95 in terms of spectral analysis

Wave-wave interaction

Wave-wave interaction arises from:

- Forcing terms in nonlinear boundary conditions.
- Different combinations of free surface elevation (η) and velocity potential ($\emptyset$).
- Products such as : $\eta^2, \phi^2,\ \eta\phi\ ,\ \eta^3, \phi^3,\ \eta^2\phi\ ,\ \emptyset^2\eta$ and the solution contains terms proportional to $1, e^{\mp i\psi}, e^{\mp 2i\psi}, e^{\mp 3i\psi}$

A_n is a **complex or real number**:

$$\eta = \sum_{n=1}^{\infty} \frac{\mathrm{A_n}}{2} e^{i\psi_n}$$

$$\Psi_n = \int k_n dx - \omega_n t$$

ω_n: angular frequency

k_n : Wave number defined by $\omega_n^2 = g\ k_n \tanh k_n h$

Comparison of near-shore nonlinear wave models

1- Extended Boussinesq models:

Consistent model of Freilich and Guza (1984)

- Mild and slow varying bottom.
- Near resonant conditions between triads for the second order expansion.
- Slowly varying amplitudes

$$A_{nx} + \underbrace{\frac{h_x}{4h} A_n}_{\text{Bottom slope term (Green's law)}} - \underbrace{\frac{i n^3 k^3 h^2}{6} A_n}_{\text{Dispersion term}} + \frac{3ink}{8h}\underbrace{\left(\sum_{l=1}^{n-1} A_l A_{n-l} + 2 \sum_{l=1}^{N-n} A_l^* A_{n+l} \right)}_{\text{Nonlinear term}} = 0$$

$$h = h(\delta x)$$

$$\frac{d^n}{dx^n} h = O(\delta^n)$$

Classification of near-shore nonlinear wave models

2- Fully dispersive wave models:

Kaihatu and Kirby (1995)

- Mild slope equation
- Two dimensional fully dispersive model with triad nonlinearity
- Energy dissipation of Thornton and Guza (1983)

$$A_{nx} + \frac{c_{gx}}{2\,c_g} A_n + \alpha_n A_n = -\frac{i}{8kc\,c_g}\sum_{l=1}^{n-1} R\,A_l A_{n-1}\,e^{i\int(k_l + k_{n-l} - k_n)\,dx} + 2\sum_{l=1}^{N-n} S\,A_l^{*} A_{n+l}\;e^{\int(ik_{n+l} - k_l - k_n)dx}$$

Present work

Mathematical derivation

Step 1 - Boundary value problem

$$\nabla_h^2 \phi + \phi_{zz} = 0 \qquad -h < z < \eta$$

$$\phi_z = -\nabla_h h \, . \nabla_h \phi \qquad z = -h$$

$$\eta + \phi_t + \frac{\epsilon}{2} [(\nabla_h \phi)^2 + \phi_z^2] = 0 \qquad z = \eta$$

$$\eta_t - \phi_z + \epsilon \nabla_h \eta . \nabla_h \phi = 0 \qquad z = \eta$$

Using **Taylor series** about $z = 0$, perturbation analysis and substituting $\boldsymbol{\eta = -\phi_t}$:

Second order:

$$\phi_z = -\frac{1}{g}[\phi_{tt} - \frac{1}{2}(\nabla_h \phi)_t^2 - \frac{1}{2}(\phi_z)_t^2 - \frac{1}{2g}(\phi_t)_{zt}^2 + \nabla_h . (\phi_t \nabla_h \phi)]$$

Third order:

$$\phi_z = -\frac{1}{g}\phi_{tt} + \frac{1}{2}\left[\phi_t^2 (\nabla_h^2 \phi)_t\right]_t + \frac{1}{2}\left\{\phi_t \left[(\nabla_{h\phi})^2 + \phi_z^2\right]_z\right\}_t + \frac{1}{2}\phi_t^2 (\nabla_h^2 \phi)_z + \frac{1}{2}\phi_t (\nabla_h \phi_t \nabla_h \phi)_z$$

Mathematical derivation

Step 2 – Separation of depth dependency term (Smith and Sprinks 1975)

$$\phi = \sum_{n=1}^{\infty} f_n\,(k_n, h, z)\, \overline{\Phi}_n(x, y, k_n, \omega_n, t) \qquad \text{where} \qquad f_n = \frac{\cosh k_n(h+z)}{\cosh k_n h}$$

Step 3- Multiple scale analysis

Assuming slowly varying in x :

$$x = x + \epsilon x + \epsilon^2 x$$

$$x = x + X_1 + X_2$$

Second order: $$\overline{\Phi}_{1n} = \frac{-ig}{2\omega_n}\, a(X_1) e^{i\Psi}$$

Third order: $$\overline{\Phi}_{2n} = \frac{-ig}{2\omega_n}\, a(X_2) e^{i\Psi}$$

Step 4- Fourier decomposition

Assuming waves are periodic in time: $$\overline{\Phi}_n = \frac{\emptyset_n}{2} e^{-i\omega_n t} + \frac{\emptyset_n^*}{2} e^{-i\omega_n t}$$

Mathematical formulation of Proposed model

Second order

$$\emptyset_{1n} = \frac{-ig}{\omega_n}\, a(X_1)e^{i\psi}$$

$$\frac{d\psi_n}{dx} = \bar{k} + \epsilon k_1$$

Role of perturbation parameter in the evolution equation (Liu et al. 2016)

$$\frac{d^2\psi}{dx^2} = \epsilon\frac{d\bar{k}}{dX_1}$$

$\bar{k}$ is the linear or reference wave number: $\omega_n^2 = g\bar{k}\tanh\bar{k}\,\mathrm{h}$

$$2F_2 a\bar{k}k_1 + 2iF_2\bar{k}\frac{da}{dx} + iF_2\frac{d\bar{k}}{dx} + iF_{2x}\mathrm{a}\,\bar{k} =$$

$$\sum_{i=1}^{n-1} Ra_l a_{n-l}\, e^{i(\psi_l+\psi_{n-l}-\psi_n)} + \sum_{i=1}^{n-1} 2Sa_l a_{n-l}\, e^{i(\psi_{n+l}-\psi_l-\psi_n)}$$

Mathematical formulation of Proposed model

R and ***S*** are nonlinear coefficients:

$$R = \frac{\omega_n}{\omega_l \omega_{n-l}} \Big[2\omega_n k_l k_{n-l} + \omega_l k_{n-l}^2 + \omega_{n-l} k_l^2 - \left(\frac{k_l \tanh k_l h + k_{n-l} \tanh k_{n-l} h}{g} \right). \left(\omega_l^2 \omega_{n-l} + \omega_l \omega_{n-l}^2 \right) - \omega_n (k_l \tanh k_l h). (k_{n-l} \tanh k_{n-l} h)$$

$$S = \frac{\omega_n}{\omega_l \omega_{n+l}} \Big[2\omega_n k_l k_{n+l} + \omega_l k_{n+l}^2 - \omega_{n+l} k_l^2 + \left(\frac{k_l \tanh k_l h + k_{n+l} \tanh k_{n+l} h}{g} \right). \left(\omega_l^2 \omega_{n+l} - \omega_l \omega_{n+l}^2 \right) + \omega_n (k_l \tanh k_l h). (k_{n+l} \tanh k_{n+l} h)$$

$$F_1 = \frac{1}{\cosh^2 k_n h} \left(\frac{\cosh 2k_n h}{4k_n} - \frac{1}{4k_n} - \frac{h}{2} \right)$$

$$F_2 = \frac{1}{\cosh^2 k_n h} \left(\frac{\cosh 2k_n h}{4k_n} - \frac{1}{4k_n} + \frac{h}{2} \right)$$

Mathematical formulation of Proposed model

Separation of real and imaginary parts:

Real:

$$k_1 = \frac{1}{2F_2 a\bar{k}}\left[\sum_{i=1}^{n-1} R a_l a_{n-l} \cos(\psi_l + \psi_{n-l} - \psi_n) + \sum_{i=1}^{n-1} 2S a_l a_{n-l} \cos(\psi_{n+l} - \psi_l - \psi_n) \right]$$

Imaginary:

$$\frac{da}{dx} + \left(\frac{1}{2\bar{k}}\frac{d\bar{k}}{dx} + \frac{F_{2x}}{2F_2}\right)\mathrm{a} =$$

$$\frac{-i}{2F_2\bar{k}}\sum_{i=1}^{n-1} R a_l a_{n-l} \sin(\psi_l + \psi_{n-l} - \psi_n) + \sum_{i=1}^{n-1} 2S a_l a_{n-l} \sin(\psi_{n+l} - \psi_l - \psi_n))$$

Mathematical formulation of Proposed model

Third order

$$\emptyset_{2n} = \frac{-ig}{\omega_n} a(X_2)e^{i\psi}$$

$$\frac{d\psi_n}{dx} = \bar{k} + \epsilon k_1 + \epsilon^2 k_2$$

$$\frac{d^2\psi}{dx^2} = \epsilon \frac{d\bar{k}}{dX_1} + \epsilon^2 \frac{d\bar{k}}{dX_2} + \epsilon^2 \frac{dk_1}{dX_1}$$

Mathematical formulation of Proposed model

$$2F_2 a\bar{k}k_1 + 2iF_2\bar{k}\,\frac{da}{dx} + iF_2\,\frac{d\bar{k}}{dx} + iF_{2x}\mathrm{a}\,\bar{k} =$$

$$\frac{1}{2}\sum_{l=1}^{n-1} -2g^2 i\,\frac{\omega_n}{\omega_l\omega_{n-l}}\{2\omega_n\left[\overline{k_l}\,a_l\,\frac{da_{n-l}}{dx} + a_{n-l}\,\overline{k_{n-l}}\,\frac{da_l}{dx} + a_l a_{n-l}\left(\overline{k_l}k_{n-l}^1 + \overline{k_{n-l}}k_l^1\right)i\right] +$$

$$\omega_l\left[2a_l\frac{da_{n-l}}{dx}\,\overline{k_{n-l}} + a_l a_{n-l}\left(\frac{d\overline{k_{n-l}}}{dx} + 2\,\overline{k_{n-l}}\,k_{n-l}^1\right)i\right] +$$

$$\omega_{n-l}\left[2a_{n-l}\frac{da_l}{dx}\,\overline{k_l} + a_l a_{n-l}\left(\frac{d\overline{k_l}}{dx} + 2\,\overline{k_l}\,k_l^1\right)i\right]e^{i(\psi_l+\psi_{n-l}-\psi_n)}\} +$$

$$\frac{1}{2}\sum_{l=1}^{n-1} -2g^2 i\,\frac{\omega_n}{\omega_l\omega_{n+l}}\{2\omega_n\left[\overline{k_l}\,a_l\,\frac{da_{n+l}}{dx} + a_{n+l}\,\overline{k_{n+l}}\,\frac{da_l}{dx} + a_l a_{n+l}\left(\overline{k_l}k_{n+l}^1 + \overline{k_{n+l}}k_l^1\right)i\right] -$$

$$\omega_l\left[2a_l\frac{da_{n+l}}{dx}\,\overline{k_{n+l}} + a_l a_{n-l}\left(\frac{d\overline{k_{n+l}}}{dx} + 2\,\overline{k_{n+l}}\,k_{n+l}^1\right)i\right] +$$

$$\omega_{n+l}\left[2a_{n+l}\frac{da_l}{dx}\,\overline{k_l} + a_l a_{n+l}\left(\frac{d\overline{k_l}}{dx} + 2\,\overline{k_l}\,k_l^1\right)i\right]e^{i(\psi_{n+l}-\psi_l-\psi_n)}\}$$

$$i\sum_{n}\frac{1}{4}\,a_n^3\,\bar{k}_n{}^2\left[\frac{3}{8g}\,\omega_n^2 + \frac{3}{8}\left(\bar{k}_n\tanh\bar{k}_n h\right)\right]e^{2i\psi_n}$$

Real or complex amplitude?

In the proposed model the free surface amplitudes are treated as real numbers whereas in Kaihatu and Kirby (1995) the complex amplitudes are used.

Complex amplitude:

$$\emptyset_n = \frac{-ig}{\omega_n} A_n e^{i\int k_n dx}$$

k_n : Linear wave number ($\bar{k}$)

A_n: contains all nonlinear effects → $A_n = a_n e^{i\theta}$

θ: all nonlinear effects

Real amplitude:

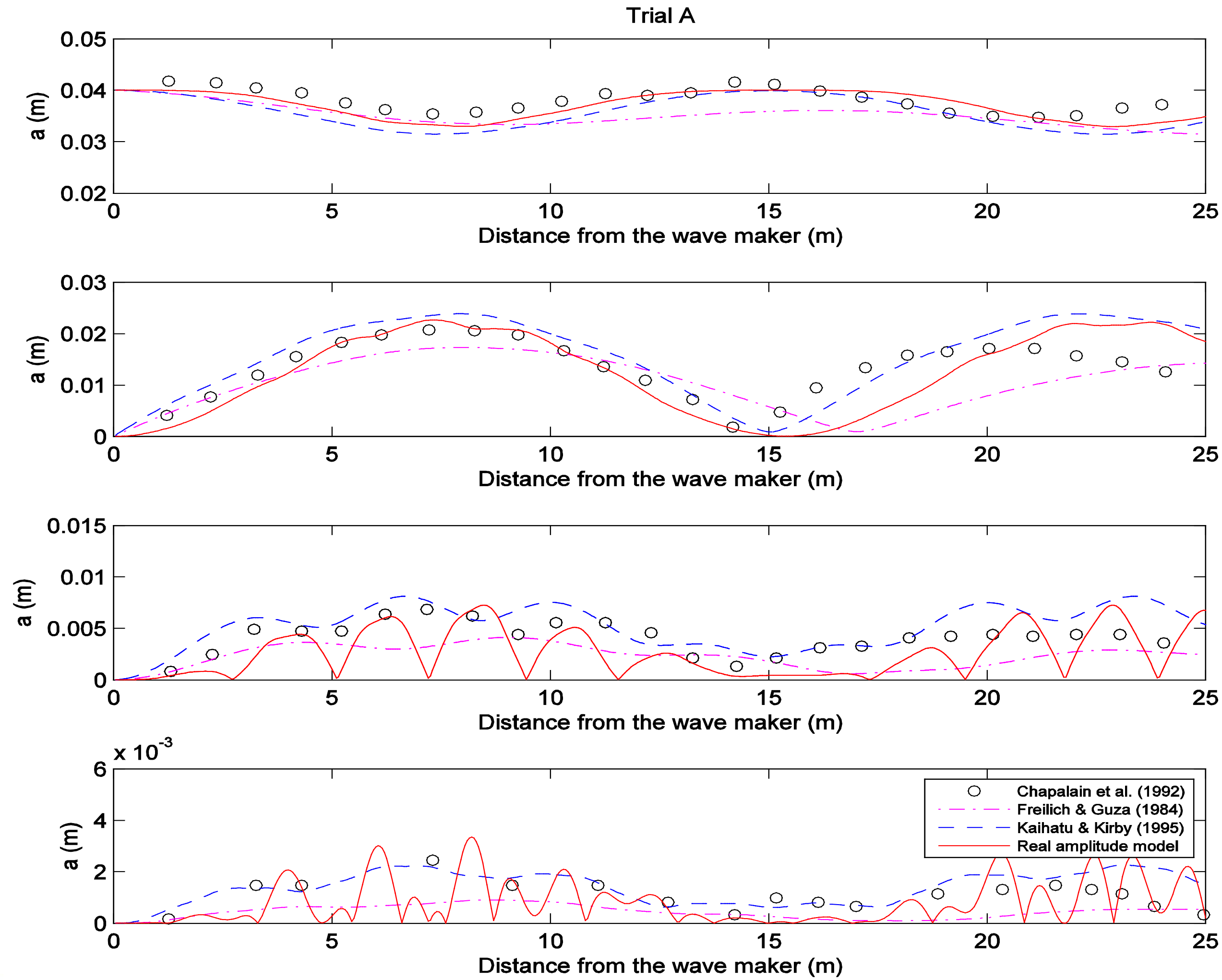
$$\emptyset_n = \frac{-ig}{\omega_n} a_n e^{i\int k_n dx}$$

k_n: **Linear** wave number + modified wave number due to **bottom slope** + modified wave number due to the **effects of bound waves**

Verification of the model

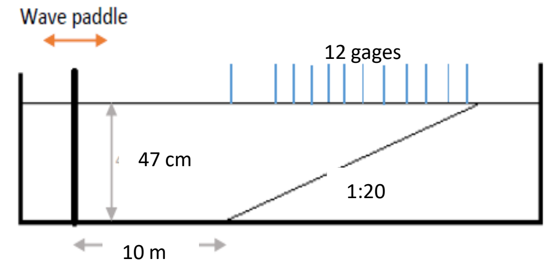
Chapalain et al. (1992) Harmonic experiments

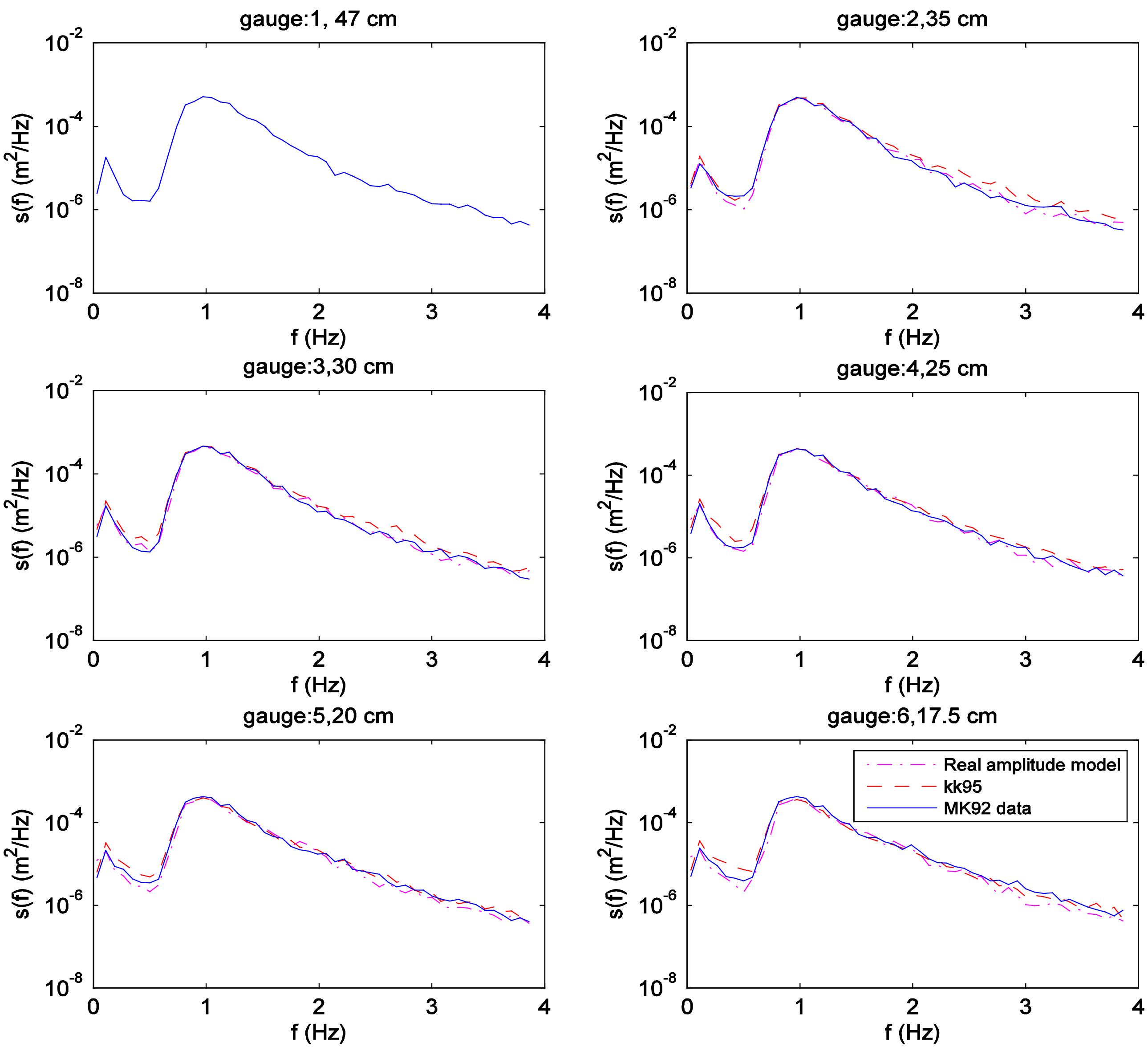
Trial A: h = 0.4 m and T = 2.5 S



Verification of the model

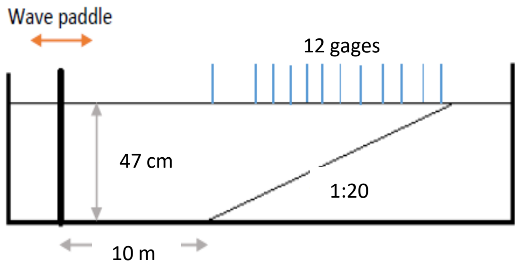
Mase and Kirby (1992) experimental data set, Case 2 *(Kh= 1.9)*

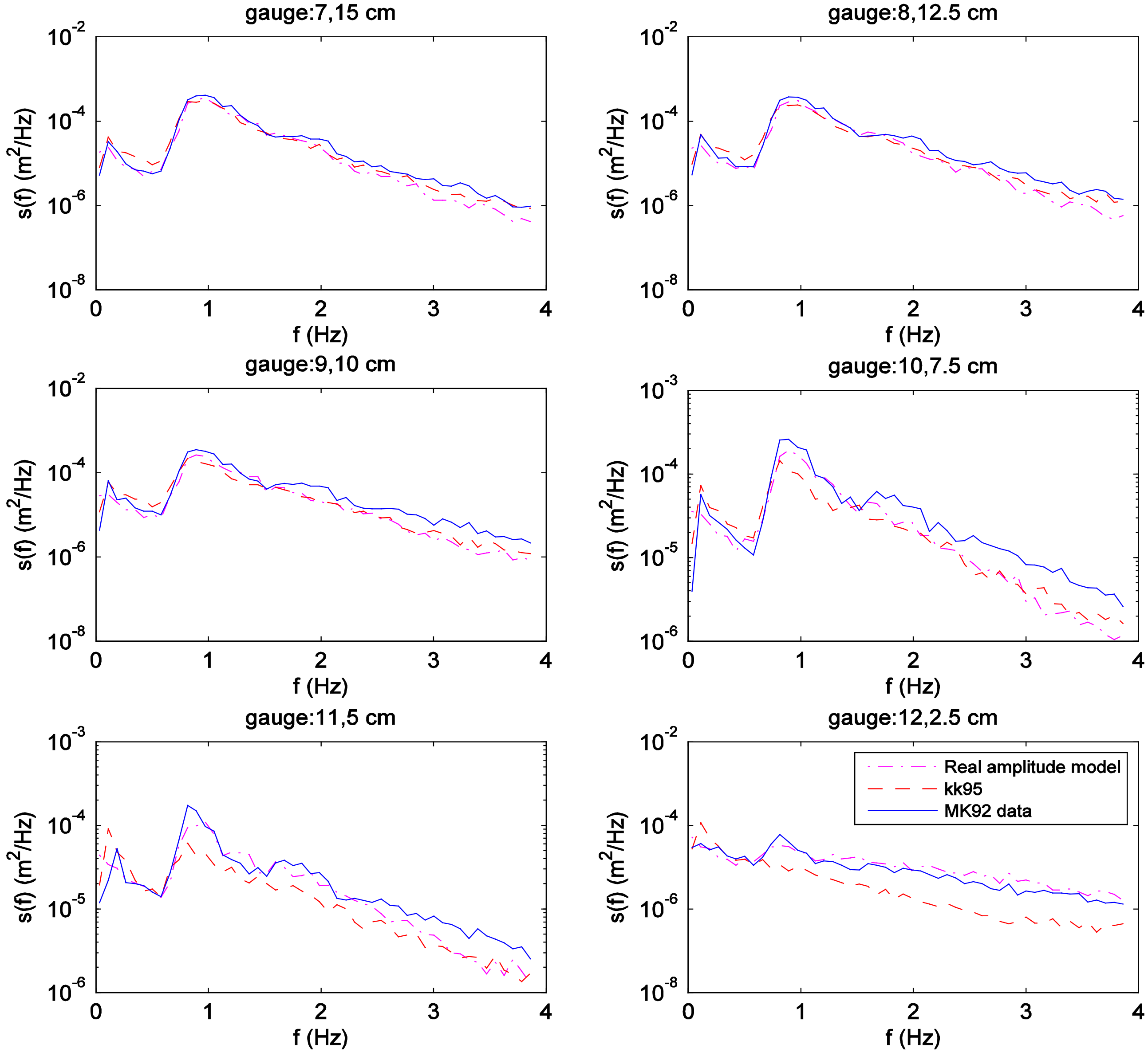




Verification of the model

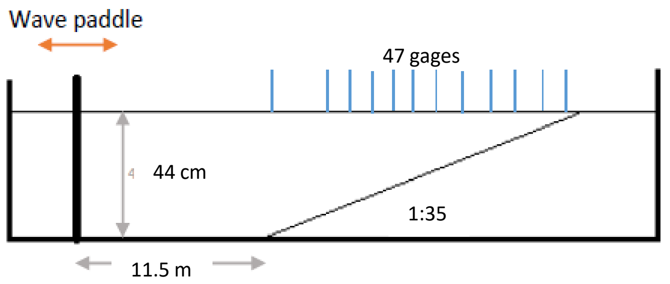
Mase and Kirby (1992) experimental data set, Case 2 (*Kh= 1.9)*

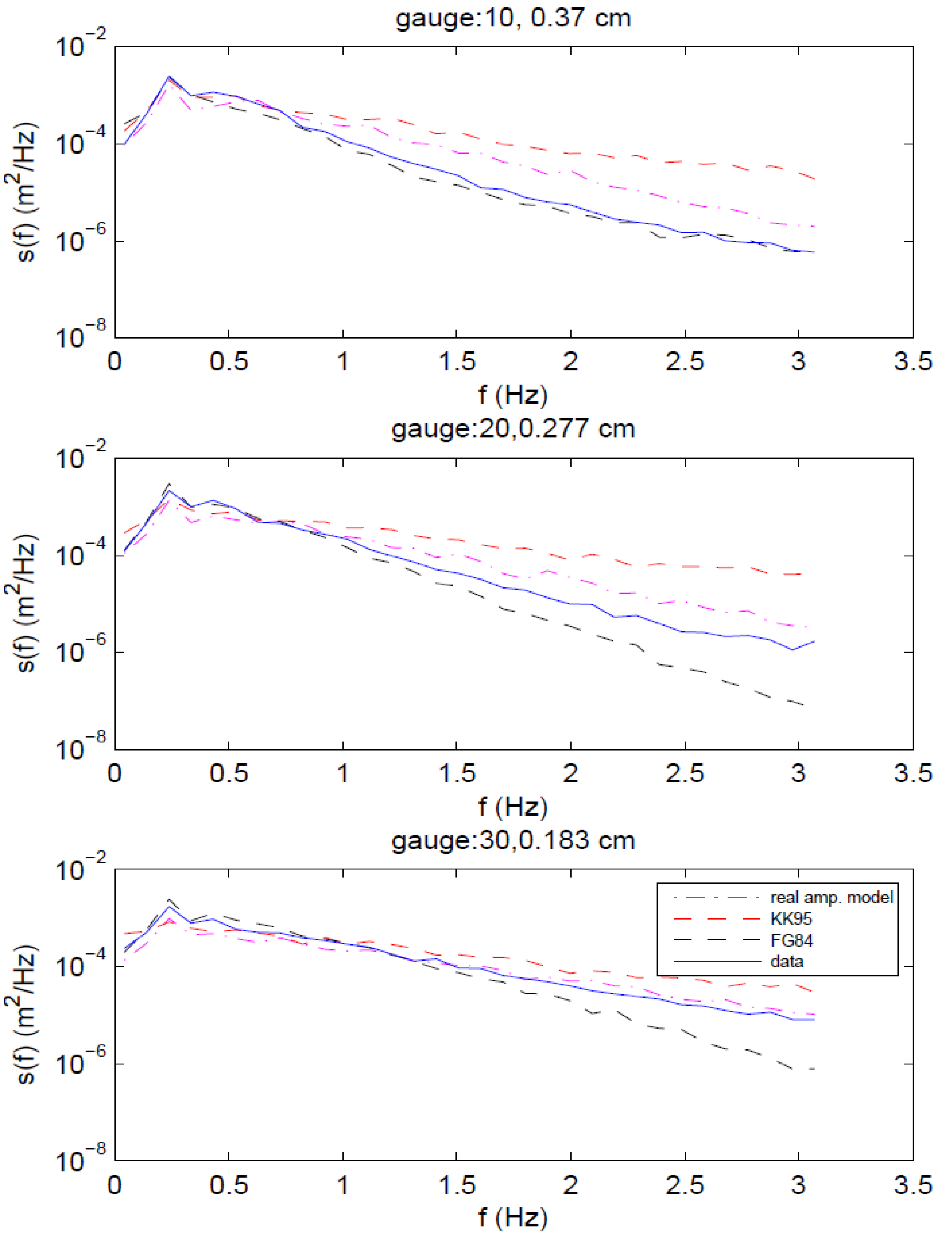




Verification of the model

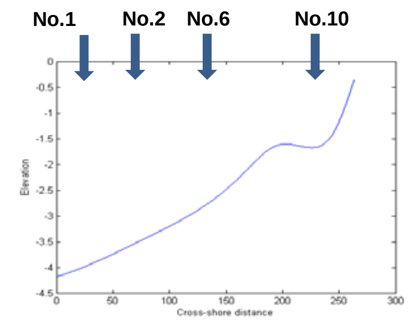
Bowen and Kirby (1994) case B data set



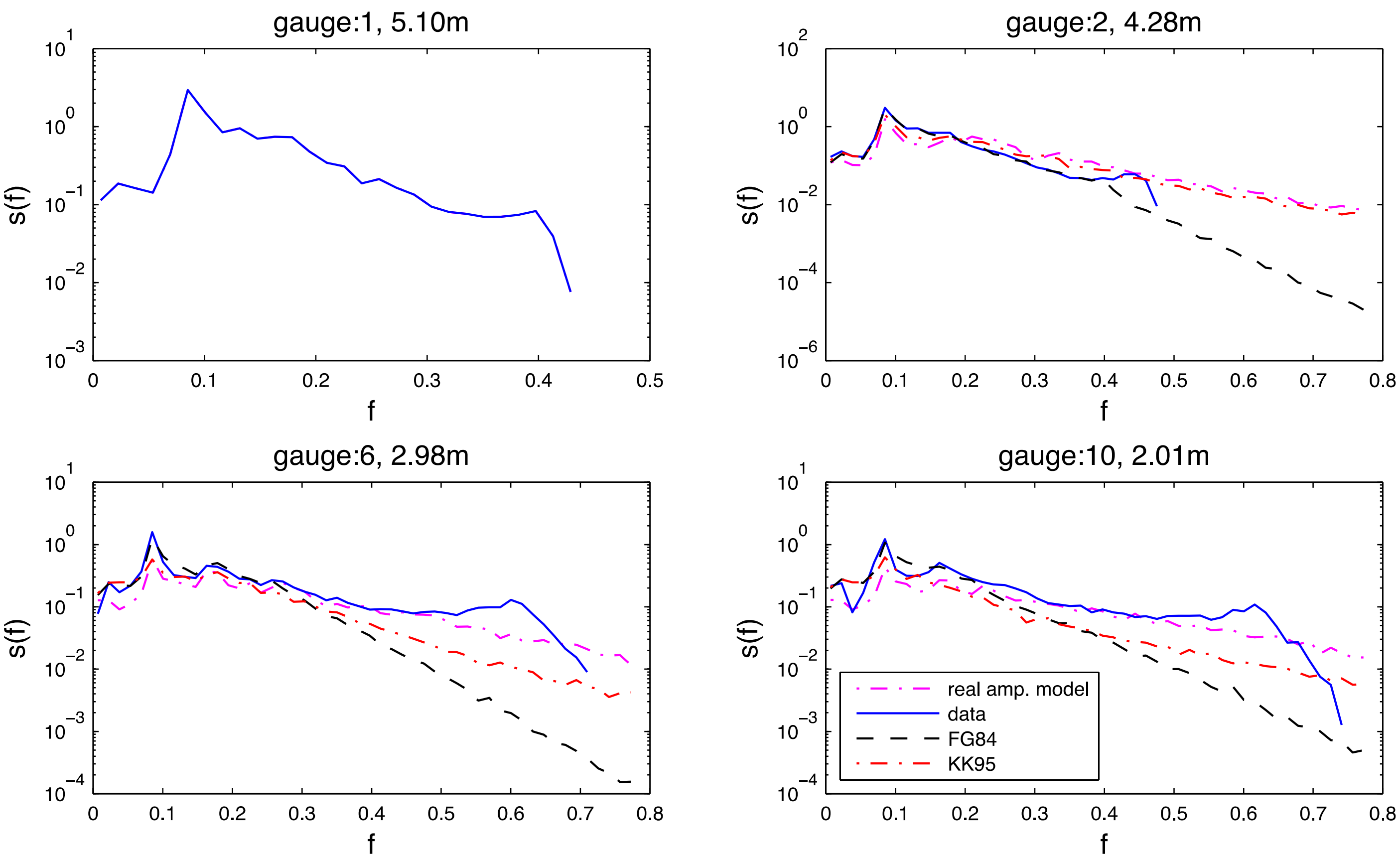


Verification of the model

Bottom profile and the selected gauges:



Duck94 field experiment data set

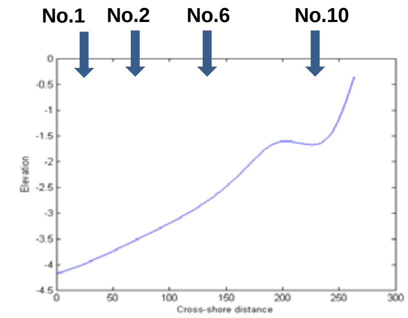


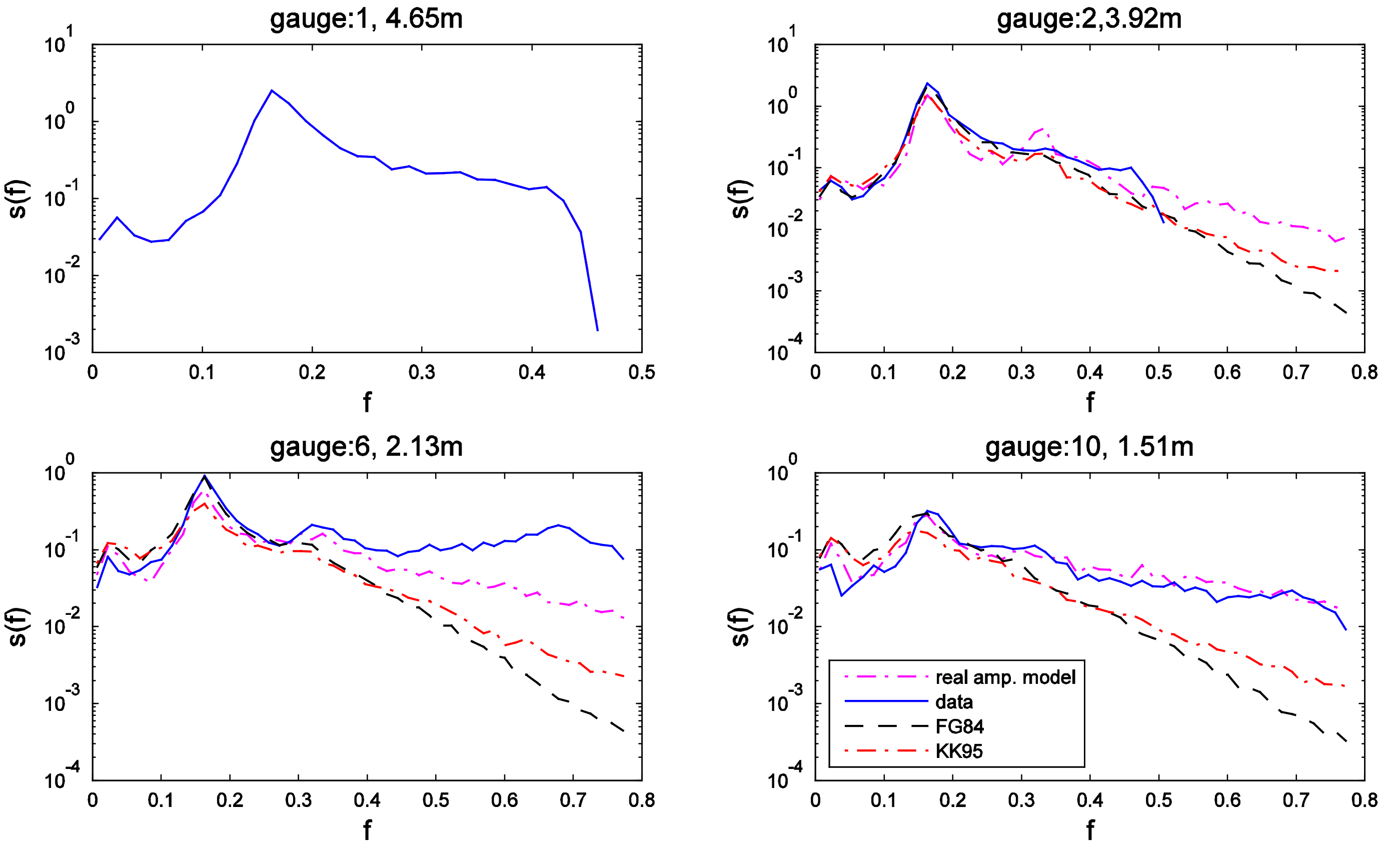
Duck94-09051600, 400 frequency components

Verification of the model

Duck94 field experiment data set

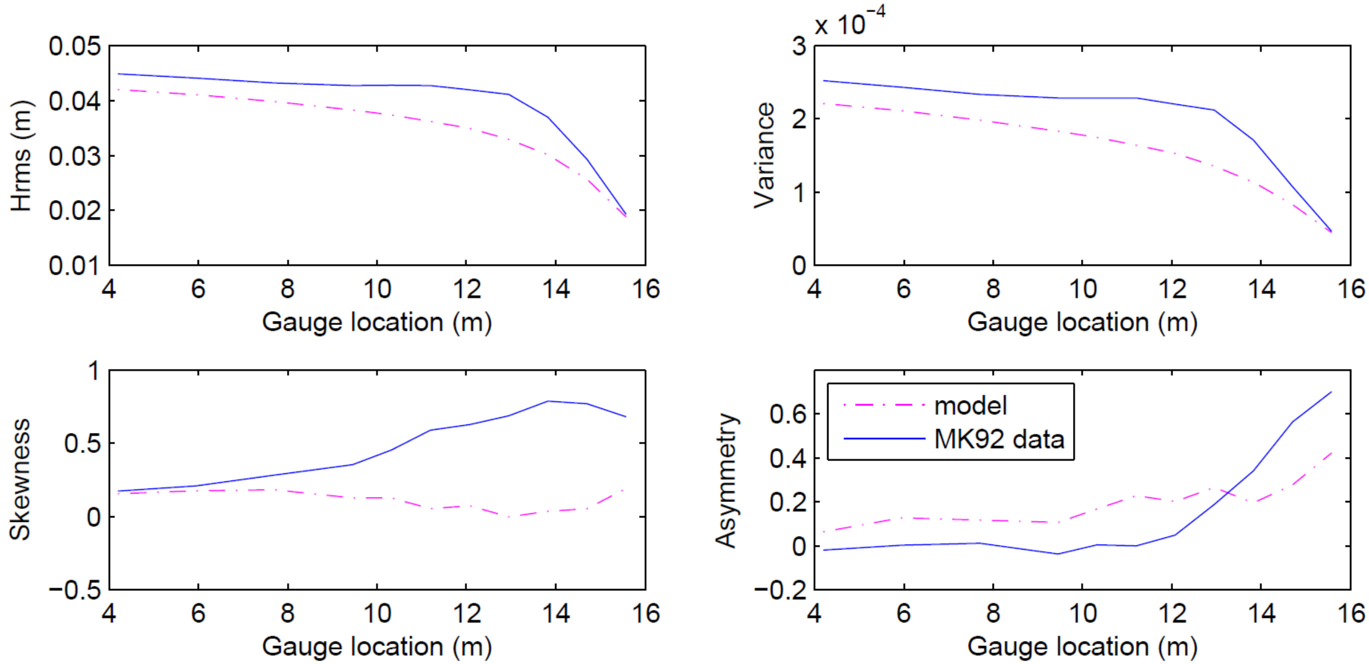
Bottom profile and the selected gauges:





Duck94-09030100, 400 frequency components

Statistical parameters



Findings:

- Higher order Statistical parameters are very sensitive to perturbation parameter, ϵ
- ϵ acts like a free parameter in the model for estimating these quantities

Summary:

- Testing the model using laboratory data sets with a wide range of relative water depths (kh), from a higher value for Mase and Kirby (1992) dataset to the smaller value of kh in the experimental data set of Bowen and Kirby (1994) strengthen the validity of the model for different wave conditions.

- Validity of the model with wide range of relative water depth shows the improvement of the model for higher frequencies.

- The developed model generally agrees well with field data compare to the two other models.

Thank You