A Nonlinear and Dispersive 3D Model for Coastal Waves using Radial Basis Functions

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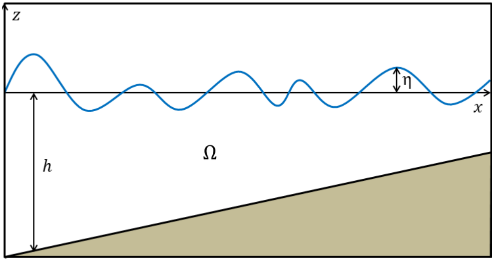
Coastal zone wave modeling

- **Objective:** develop an accurate, nonlinear, phase-resolving nearshore wave propagation model

- **Challenge:** accurate and computationally efficient modeling of the dominant physical processes at a wide range of spatial and temporal scales

Mathematical model
Zakharov Equations



- incompressible flow
- inviscid fluid
- homogeneous atmospheric pressure
- irrotational flow
  $\nabla_{3D}\phi = \underline{u}(\underline{x}, z, t)$, with
  $\underline{x} = (x, y)$
- continuous water column

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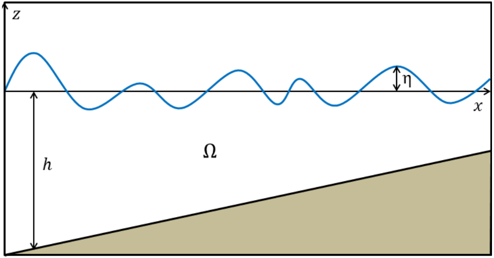
Zakharov Equations



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Temporal evolution of:

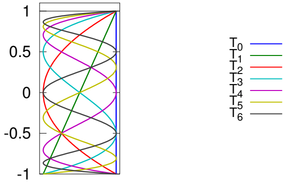
$$\begin{cases} \eta_t = -\nabla\eta\nabla\tilde{\phi} + \tilde{w}\left(1 + (\nabla\eta)^2\right) \\ \tilde{\phi}_t = -g\eta - \frac{1}{2}(\nabla\tilde{\phi})^2 + \frac{1}{2}\tilde{w}^2\left(1 + (\nabla\eta)^2\right) \end{cases} \quad \text{with } \tilde{w} = \left.\frac{\partial\phi}{\partial z}\right|_{z=\eta}$$

(Zakharov, 1968)

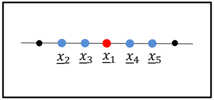
Numerical model

Misthyc code

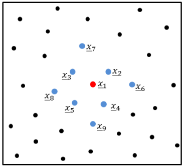
- Spectral method in vertical
  (Tian et al., 2008)
- MUMPS direct linear solver
  (Amestoy et al., 2001, 2006)
- Explicit 4th order Runge-Kutta method
- Horizontal resolution:



- 1DH: high-order finite difference schemes
  (e.g. Bingham et al., 2007)



- 2DH: radial basis functions
  (e.g. Wright and Fornberg, 2006)

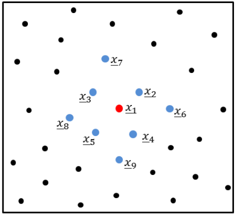


Radial Basis Functions

Finite Difference (RBF-FD) approach

- Scattered node domain discretization
- Evaluation of linear differential operator L applied to f at node $\underline{x}_i$:

$$L(f)(\underline{x}_i) = \sum_{k=1}^{N_{sten}} w_{i,k}^{L} f(\underline{x}_k)$$

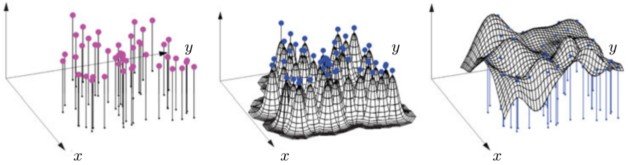


- Choice of radial basis function: $\phi_k(x) = \Phi(||\underline{x} - \underline{x}_k||)$
- Solution of N_{sten} unknown weights $w_{i,k}^{L}$ for each node x_i

Radial Basis Functions
Finite Difference (RBF-FD) approach

Implementation:

- stencil size N_{sten}?
- what type of RBF?
- which RBF function?
- value of the shape parameter C? (for IS RBFs)
- added polynomial of degree n (pn)?



(a) (b) (c)

(Fornberg and Flyer, 2015)

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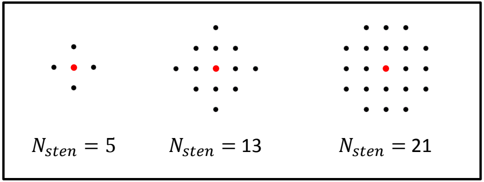

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Regularity	Type	Function $\Phi(r)$	Condition
Infinitely smooth	Multiquadric (MQ)	$\sqrt{r^2+C^2}$	$C \in R$
	Inverse Multiquadric (IMQ)	$1/\sqrt{r^2+C^2}$	$C \in R$
	Inverse Quadratic (IQ)	$1/(r^2+C^2)$	$C \in R$
	Gaussian (GA)	e^{-r^2/C^2}	$C \in R$
Piecewise smooth	Polyharmonic Spline (PHS)	r^m	m odd integer
	Thin Plate Spline (TPS)	$r^m \log r$	m even integer

Radial Basis Functions

Finite Difference (RBF-FD) approach

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Regularity	Type	Function $\Phi(r)$	Condition
Infinitely smooth	Multiquadric (*MQ*)	$\sqrt{r^2 + C^2}$	$C \in \mathbf{R}$
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Radial Basis Functions

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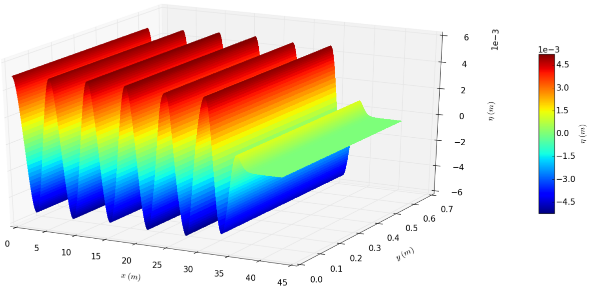
Recommendation: $20 \leq N_{sten} \leq 30$, *PHS* $r^7 + p3$

Regular wave propagation over a flat bottom [1/2]

Analytic test case

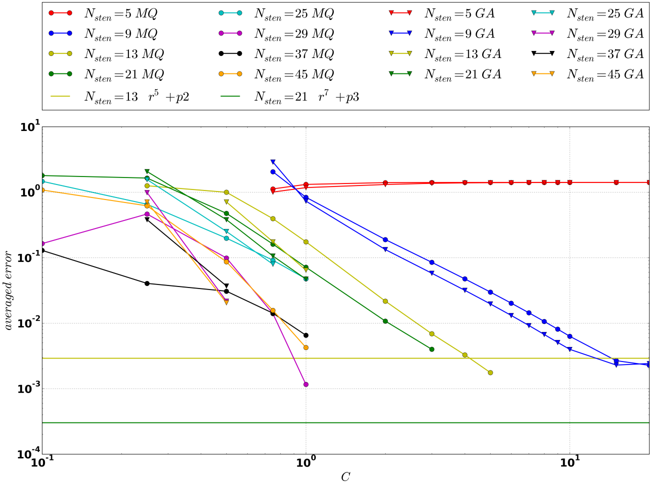


- Regular waves:
  - $A = 0.005$ m
  - $T = 2.26$ s
  - $L = 6.14$ m
  - $h = 1$ m
- Irregularly spaced nodes, $\Delta x = \Delta y \approx L/100$
- Constant time step $\Delta t = T/100$

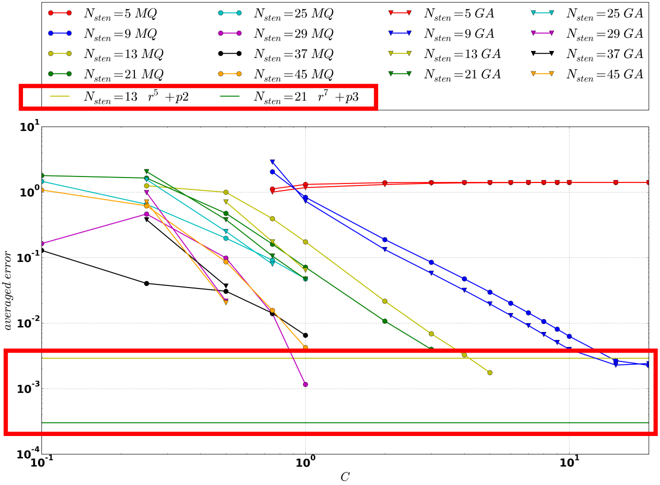
Normalized error:

$$average\ error = \sqrt{\frac{\sum_{i=1}^{N} |\eta(\underline{x}_i) - \eta_{ref_{\mathrm{1DH}}}(\underline{x}_i)|^2}{\sum_{i=1}^{N} |\eta_{ref_{\mathrm{1DH}}}(\underline{x}_i)|^2}}$$

Regular wave propagation over a flat bottom [2/2]

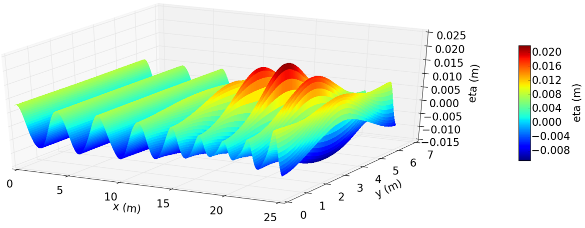


Regular wave propagation over a flat bottom [2/2]

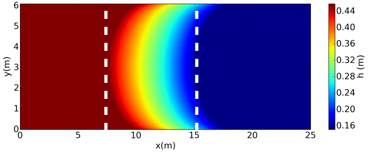


Regular wave propagation over a convergent step [1/2]

Whalin (1971) wave tank experiments



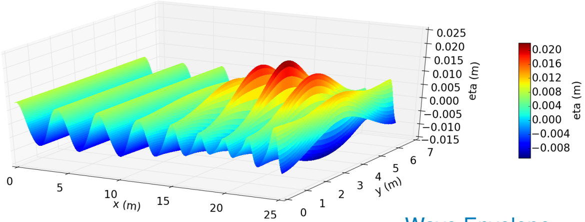
Bathymetry



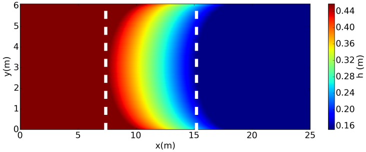
- Regular waves:
  - $A = 0.0075$ m
  - $T = 2$ s
- Regularly spaced nodes, $\Delta x = \Delta y \approx L/98$
- Constant time step, $\Delta t \approx T/112$

Regular wave propagation over a convergent step [1/2]

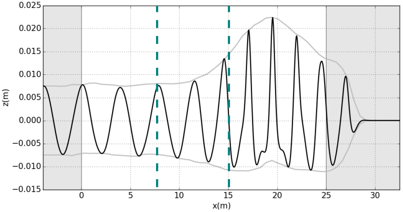
Whalin (1971) wave tank experiments



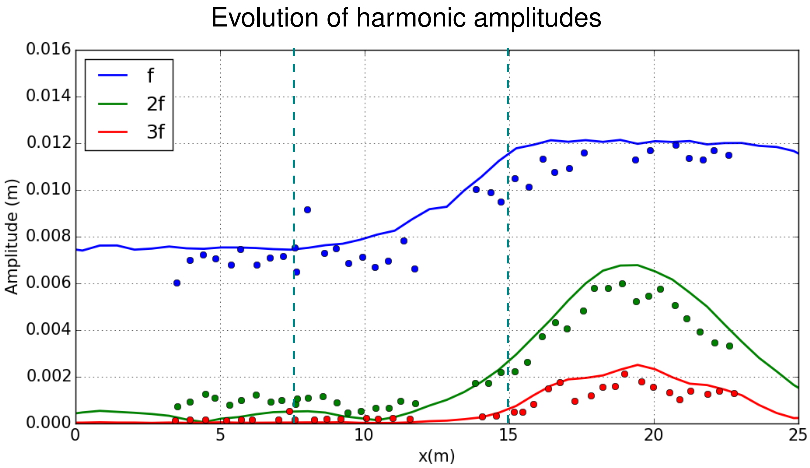
Bathymetry



Wave Envelope



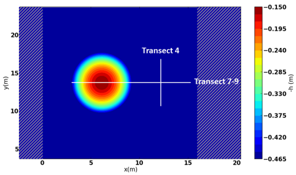
Regular wave propagation over a convergent step [2/2]

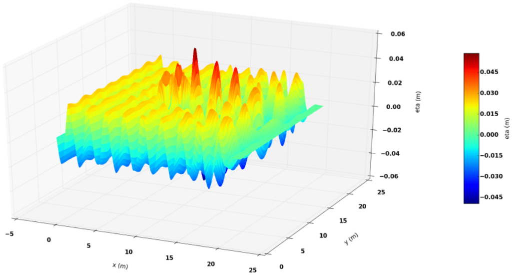


Regular wave propagation over an elliptical shoal [1/4]

Vincent & Briggs (1989) wave tank experiments

Bathymetry



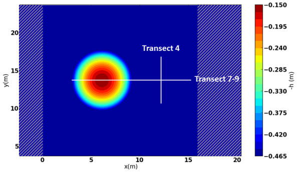


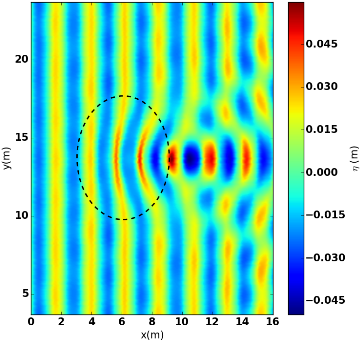
- Regular waves (case M1):
  - $A = 0.02325$ m
  - $T = 1.3$ s
- Regularly spaced nodes, $\Delta x = \Delta y \approx L/30$
- Constant time step, $\Delta t \approx T/36$
- $N_T = 5$

Regular wave propagation over an elliptical shoal [1/4]

Vincent & Briggs (1989) wave tank experiments

Bathymetry





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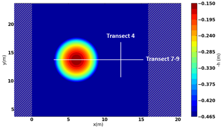
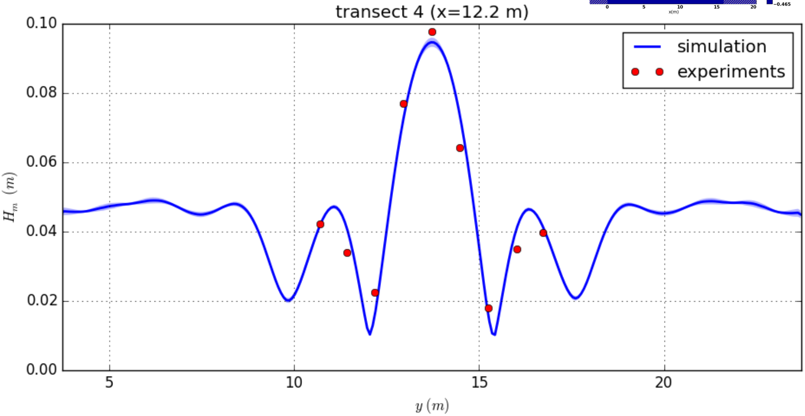
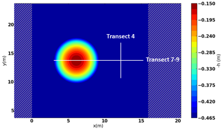
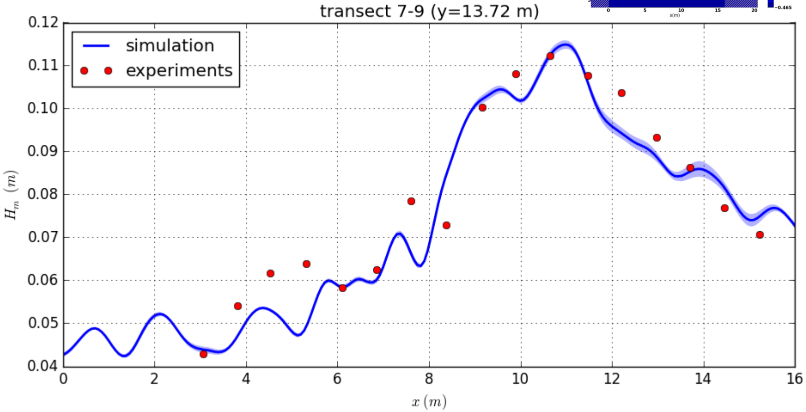
Regular wave propagation over an elliptical shoal [2/4]





Regular wave propagation over an elliptical shoal[3/4]

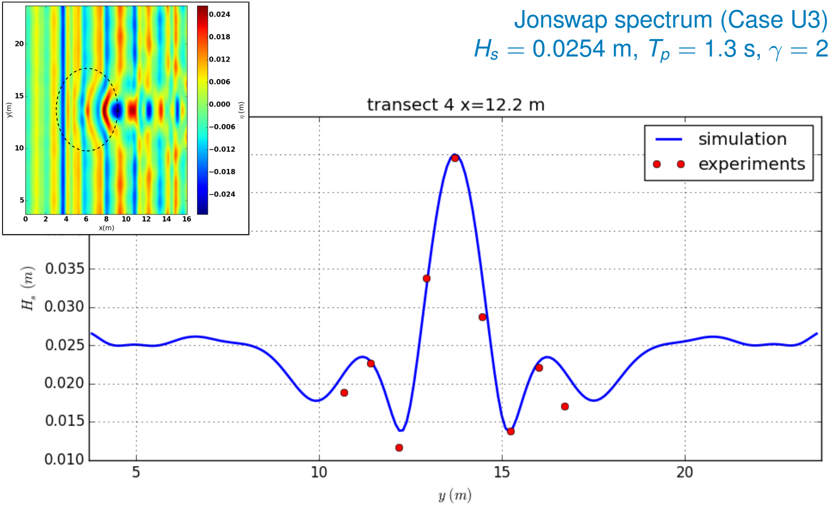




Irregular wave propagation over an elliptical shoal [4/4]

Jonswap spectrum (Case U3)

$H_s = 0.0254$ m, $T_p = 1.3$ s, $\gamma = 2$



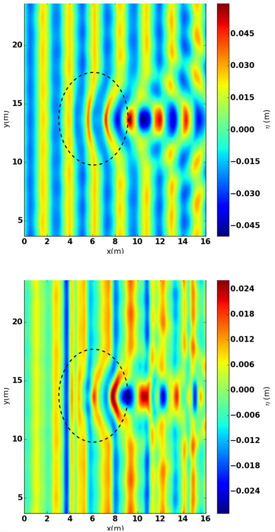
Summary

Conclusions:

- Accurate reproduction of nonlinear and dispersive effects
- Robustness of RBF-FD approach to maintain flexibility
- Recommandation: PHS $r7 + p3$ with $20 - 30$ nodes

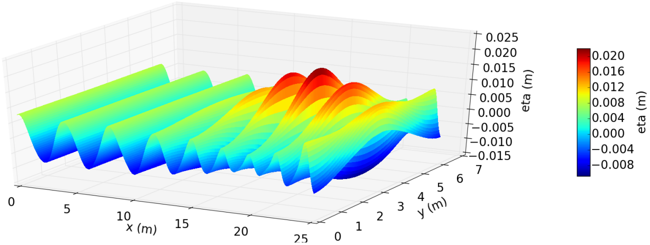
Challenges and ongoing work:

- Instabilities at boundaries
- Optimal node spacing
- Computational efficiency



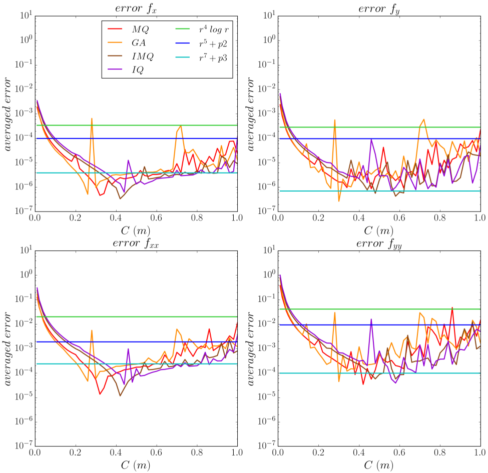
Raoult et al. (2018), *JCP*

Thank you!

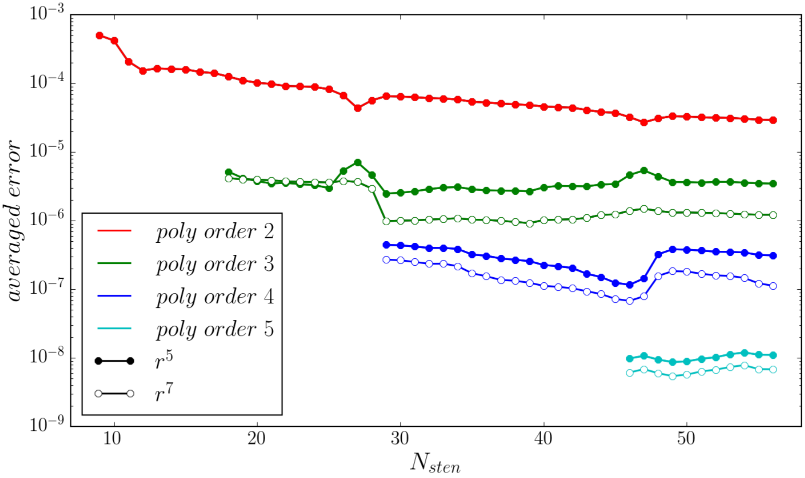


Acknowledgements: EDF R&D and ANRT for the Cifre PhD fellowship for Cécile Raoult

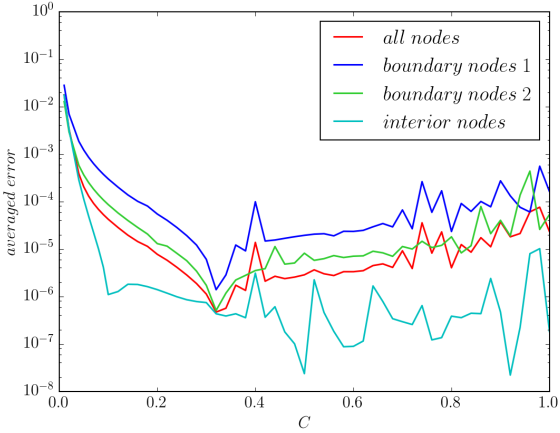
Comparing Types of RBFs
Estimating derivatives



Reducing errors with added polynomials
Estimating derivatives

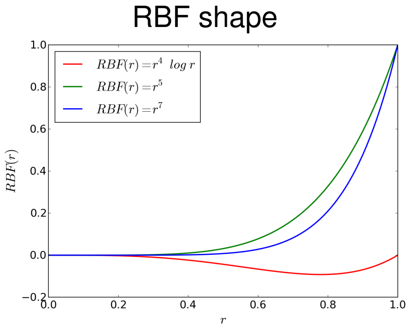


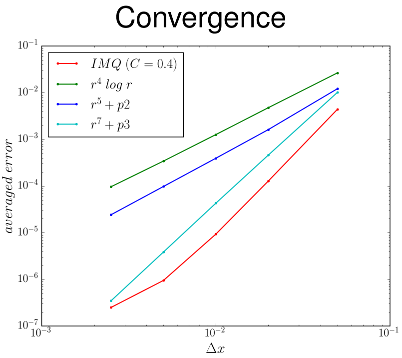
Boundary node errors
Estimating derivatives



PHS functions

Estimating derivatives

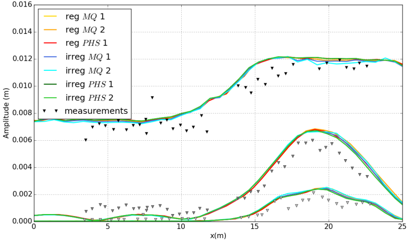




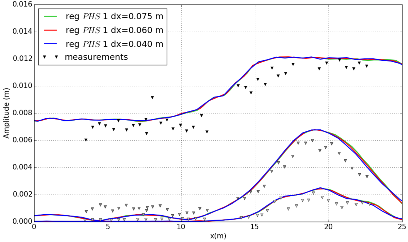
Comparing spatial discretizations

Whalin (1971) experiments

RBF parameters

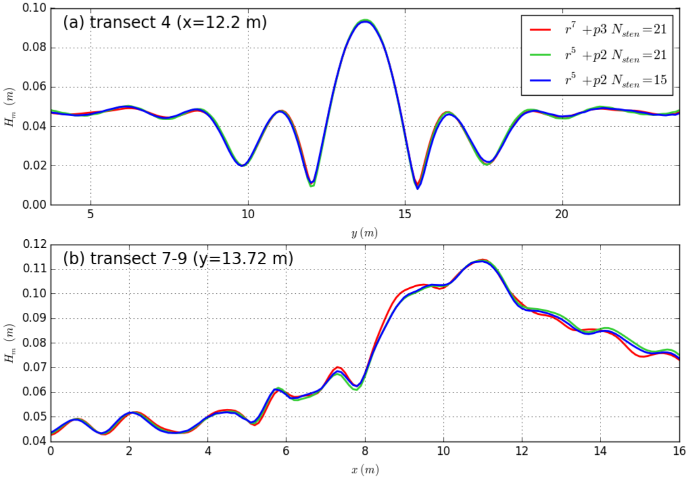


Spatial resolution



RBF PHS functions

Vincent & Briggs (1989) experiments



CPU time

- 1DH cases (processor 3.4 GHz)
  - Solitary wave ($\delta = 0.5$): $NPX = 7000$, $N_T = 7$, cputime/time step $\approx$ 2s
  - Irregular waves propagating over a barred beach (Becq et al. 1999): $NPX = 601$, $N_T = 7$, cputime/time step $\approx$ 0.098s
- 2DH cases (processor 2.7 GHz (cluster))
  - Regular waves over a convergent step (Whalin, 1971): $NPXY = 60716$, $N_T = 7$, $N_{sten} = 13$, cputime/time step$=$ 333 s

Improve computational efficiency with parallelization and pre-conditioned iterative solvers.