CHAPTER 8

THE THEORY OF THE REFRACTION OF A SHORT CRESTED GAUSSIAN SEA SURFACE WITH APPLICATION TO THE NORTHERN NEW JERSEY COAST

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INTRODUCTION

WHAT ARE OCEAN WAVES?

The Thorndike Barnhart Dictionary (1951) defines a wave as a "moving ridge or swell of water." Almost everyone will agree to this definition. Milne-Thompson (1938) in Theoretical Hydrodynamics begins Chapter Fourteen on waves with the two paragraphs quoted in full below:

> "14·10 Wave motion. A wave motion of a liquid acted upon by gravity and having a free surface is a motion in which the elevation of the free surface above some chosen fixed horizontal plane varies with time.
>
> Taking the axis of x to be horizontal and the axis of z to be vertically upwards, a motion in which the vertical section of the free surface at time t is of the form
>
> $$z = a \sin(mx - nt) \tag{1}$$
>
> where a, m, n are constants, is called a simple harmonic progressive wave."

The definition of a wave as a moving ridge or swell of water does not say that all of the waves in a given wave system must have exactly the same amplitude, a, the same direction, toward positive x, the same angular frequency, n, the same wave number, m, and infinitely long crests in the y direction. In fact a wave system need not be a simple harmonic progressive wave at all.

On the open ocean or at a given coast, no man has ever seen a wave system of the form of equation (1). Such a system can only be approximated in a wave tank. Waves in nature, generated by the winds, do not have the properties of equation (1). No man will ever see a wave system on the open ocean like equation (1).

The moving ridges or swells of water on the surface of the ocean do not duplicate each other exactly in height or in the time intervals between successive crests. They do not extend to infinity along the crests. Our contention is that equation (1) is not an adequate re-

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presentation for actual ocean waves. Our contention is also that any physical quantities derived from the assumption that the sea surface is like equation (1) are invalid and inaccurate.

WAVE ANALYSIS

The usual practice in ocean wave analysis is to take a wave record or a pressure record, and count bumps. The maximum swing upward is measured, the following minimum swing is then found and the difference is the "height" of that particular "wave." The time interval between two successive crests is also called the "period" of that particular wave. Upon the completion of the analysis of the record, the result is a set of numbers for the wave "heights" and another set of numbers for the wave "periods."

However, at this point, a difficulty is encountered. We have a whole set of different "heights," and a whole set of different "periods." An inconsistency is evident here in that equation (1) only holds for one height and one period. The dilemma is usually evaded by averaging the height of the one third highest waves and calling the result the "significant" height. The time interval between successive crests of the one third highest waves is also averaged and the result is called the "significant" period.

The result is, lo and behold, two very nice simple numbers, and our troubles are all over. We have just enough numbers to fit equation (1). By the process of brute force, we have thrown away the irregularity of the original record, the short crestedness of the actual sea surface, and the difference between a "sea" wave condition and a "swell" wave condition.

FAULTS OF THE METHOD

One fault of the above method of analysis is that every time an analysis of simultaneous pressure and free surface records has been made, the result is that the "significant" height of the pressure record predicts a "significant" height of the free surface record (based upon the pressure record "significant" period) which is too low and which is in error by any where from 10 to 25 percent. This error has been explained by Pierson (1952) and Pierson and Marks (1952) and the error lies in the complete inadequacy of the "significant" height and period method of analysis.

It is our contention that the method of analysis described above is inadequate and inaccurate in connection with the entire process of ocean wave analysis, ocean wave forecasting, and ocean wave refraction. Pierson (1952) has treated the problem of wave forecasting and wave analysis in a more thorough way which shows that this is the case for wave forecasting and wave analysis.

PURPOSE OF PAPER

In this paper, theories of wave pattern analysis and of wave refraction as developed in electronic theory by Wiener (1949) and Tukey (1949) and as applied by Pierson (1952) will be applied to a model problem of wave refraction of points along the New Jersey coast. The result will be to show that the wave heights not only vary from point to point along the New Jersey coast but that also the "significant" period is not the same from point to point for the same wave system in deep water. Other features of interest will also be pointed out.

A SHORT CRESTED GAUSSIAN SEA SURFACE

DEFINITION

A formula which appears to yield all of the known properties of actual ocean waves except those due to non-linearity, is given by equation (2) for waves in infinitely deep water. η (x,y,t) is the free surface. The function, $[A_2(\mu, \theta)]^2$ is the power spectrum of the wave system. The variable, μ, is the spectral frequency, $(2\pi/T)$. The variable, θ, assigns directions to the crests. The function, $\psi(\mu, \theta)$, is a point set function chosen in random phase according to a rectangular probability function from zero to 2π. Equation (2) is not an integral which can be evaluated like those in the back of the calculus book. It is simply a schematic and idealistic way of thinking about a certain type of limiting process.

$$\eta(x,y,t) = \int_{-\pi}^{\pi}\int_{0}^{\infty} \cos\left[\frac{\mu^2}{g}(x\cos\theta + y\sin\theta) - \mu t + \psi(\mu,\theta)\right] \sqrt{[A_2(\mu,\theta)]^2\, d\mu\, d\theta} \qquad (2)$$

The power spectrum is everywhere positive and it is defined over some area in the μ, θ polar coordinate system. The power spectrum has the dimension of cm^2-sec/radian. The form of the power spectrum determines whether the waves are "sea" waves or "swell" waves. If the power spectrum varies over a wide range of μ and θ, say, from $2\pi/15$ to $2\pi/1$ for μ and over a range of 45° for θ, the result is "sea" waves. If the power spectrum varies over a narrow range of μ and θ, say, from $2\pi/14$ to $2\pi/10$ for μ and over a range of 10° for θ, the result is "swell." Evidence for this statement will be cited later.

Equation (2) can be approximated to any desired degree of accuracy by a partial sum. The procedure is to divide the μ, θ polar coordinate system by picking values of μ at the points; μ_0, μ_1, μ_2 .... μ_{2n}, and values of θ at the points; $-\pi$, θ_1, θ_2, θ_3 .... θ_{2p-1}, π. Then the partial sum is given by equation (3) where the values of $\psi(\mu_{2r+1}, \theta_{2q+1})$ are picked at random between the values of zero and 2π.

$$\eta(x,y,t) = \lim_{\substack{r\to\infty \\ p\to\infty \\ \mu_{2r}\to\infty}} \sum_{q=0}^{p-1}\sum_{r=0}^{n-1} \cos\left[\frac{(\mu_{2r+1})^2}{g}(x\cos\theta_{2q+1} + y\sin\theta_{2q+1}) - \mu_{2r+1}t + \psi(\mu_{2r+1},\theta_{2q+1})\right] \cdot \sqrt{[A_2(\mu_{2r+1},\theta_{2q+1})]^2(\mu_{2r+2}-\mu_{2r})(\theta_{2q+2}-\theta_{2q})} \qquad (3)$$

The integral given in equation (2) is the limit of equation (3) as the value of μ_{2r} approaches infinity and as the difference between successive μ's and θ's in the net approaches zero. Since $\psi(\mu, \theta)$ is picked at random, the limit is an infinite number of different forms for the sea surface, all with certain fundamental statistical characteristics for a given power spectrum.

PROPERTIES

Our claim is that equation (2) is a far better representation of actual ocean waves than is equation (1) fitted by the "significant" height and period method. Equation (2) is based upon the linear superposition of an infinite number of infinitesimally high sine waves with different directions and different periods. The result is an irregular pattern of short crested moving swells and ridges which appears to have all of the properties of waves on the ocean surface as they actually are except for non-linear effects.

THE EQUATION OF A WAVE RECORD MADE IN DEEP WATER

Equation (2) is a function of x, y, and t. When waves are observed as a function of time at any fixed point where equation (2) is valid, the result is that a function of the form of equation (4) is observed. Equation (4) can be defined by the limit of a partial sum in a way similar to the way equation (2) was defined above.

$$\eta(t) = \int_0^\infty \cos(\mu t + \psi(\mu)) \sqrt{[A(\mu)]^2 d\mu} \tag{4}$$

It can also be proved that all of the equations given below are properties of the systems defined above when the waves are observed in deep water.

$$\lim_{\substack{\bar{T}\to\infty \\ \bar{y}\to\infty}} \int_{t^*}^{t^*+\bar{T}} \int_{y^*}^{y^*+\bar{y}} \frac{1}{\bar{T}} \frac{1}{\bar{y}} [\eta(x,y,t)]^2 dy\, dt = \frac{1}{2} \int_0^\infty \int_{-\pi}^{\pi} [A_2(\mu,\theta)]^2 d\theta\, d\mu \tag{5}$$

$$\lim_{\bar{T}\to\infty} \frac{1}{\bar{T}} \int_{t^*}^{t^*+\bar{T}} [\eta(t)]^2 dt = \frac{1}{2} \int_0^\infty \left[\int_{-\pi}^{\pi} [A_2(\mu,\theta)]^2 d\theta \right] d\mu = \frac{1}{2} \int_0^\infty [A(\mu)]^2 d\mu \tag{6}$$

$$\int_0^\infty \int_{-\pi}^{\pi} [A_2(\mu,\theta)]^2 d\theta\, d\mu = E \tag{7}$$

The equations state that averages over an infinite distance and infinite time must be taken. Averages in reality over several kilometers or over twenty or thirty minutes are sufficiently long to provide extremely reliable values.

THE GAUSSIAN PROPERTY

There is one more important property of this method of representing ocean waves. As was first shown by Rudnick (1951), points picked from a wave record such as equation (4) are distributed according to a normal probability function with a second moment given by E/2 as stated by equation (8).

$$P(-\infty < \eta(t) < K) = \frac{1}{\sqrt{\pi E}} \int_{-\infty}^{K} e^{-\xi^2/E} d\xi \qquad (8)$$

The above property has been verified by a number of different observations. For further details, see the references to Rudnick (1951), Pierson (1952), and Pierson and Marks (1952).

It should be noted that these representations for the wave system change slowly as a function of time and position and that a given power spectrum is only valid for twenty or thirty minutes and over a relatively small area.

WAVE REFRACTION THEORY

INTRODUCTORY REMARKS

Elementary wave refraction theory is developed on the tacit assumption that ocean waves have the form given by equation (1). One direction is taken for the waves, and the "significant" height and "significant" period are assigned to the equation. Then with a refraction diagram, the height and direction of the wave at the point of interest is found. It is assumed (we believe, erroneously) that the "significant" period does not change. If ocean waves were actually like equation (1), then first of all the concept of the "significant" height and period would not be needed at all. All waves would be exactly the same in height, the crests would be infinitely long, and every crest could follow exactly T seconds after its predecessor. Life would be very simple, and theory and observation probably would agree quite well. See, for example, a paper by Marks (1951) where pure sine waves are used in ripple tank studies, and see also all papers reporting on model studies in which pure sine waves were used. However ocean waves are like equation (2), and in current practice, especially for "sea" conditions, one picks out one period and direction from an infinite number of equally important periods and directions, refracts the wave system with just these two values, and then wonders why the process did not work. That it is practically impossible to verify wave refraction theory in actual wave systems for complicated refraction conditions was shown by Pierson (1951b) in a study of wave conditions at Long Branch, New Jersey.

Wave refraction theory as developed in studies of ocean waves is correct for a simple harmonic progressive wave. The papers by Eckart (1951), Johnson, O'Brien, and Isaacs (1948), Peters (1952), Sverdrup and Munk (1944), and Pierson (1951a) are all based upon the assumption that the wave are of the form of equation (1). A forthcoming paper by Arthur, Munk and Isaacs (1952) which will appear in the Transactions of the American Geophysical Union improves on the previous techniques of orthogonal construction as presented by Johnson, O'Brien and Isaacs (1948).

All of the above theoretical results carry over directly to actual ocean waves with all of their fundamental irregularity by virtue of the fact that the wave system is linear. All that we have to do is find out what happens to each term in equation (3), pass to a limit and compute what the new wave system looks like in a form analogous to equation (2) at the new point of observation in the refraction zone. Each term in equation (3) is a simple harmonic progressive wave and theoretically we know everything we need to know about simple harmonic progressive waves.

THE REFRACTION OF A PURE SINE WAVE

The first step, then, in the study of the refraction of a short crested Gaussian sea surface as in equation (2) is to study the refraction of a pure simple harmonic progressive wave as in equation (1). The most general simple harmonic progressive wave in deep water can be represented by an equation of the form of equation (1) where A_1 is the amplitude, θ is the direction toward which the wave is traveling with respect to some x', y' Cartesian coordinate system, μ_1 is a fixed frequency, and δ_1 is an arbitrary phase.

$$\eta_1(x',y',t) = A_1 \cos\left[\frac{\mu_1^2}{g}(x'\cos\theta_1 + y'\sin\theta_1) - \mu_1 t + \delta_1\right] \quad (9)$$

The angle, θ, is most easily associated with the x' axis of a coordinate system drawn with respect to a storm system out over the ocean. As the waves approach the New Jersey coast, it is convenient to define a coordinate system such that positive x points due west and y points to the south. Then the above angle considered with respect to a storm becomes a new angle considered with respect to the coast which will be called θ_F. Equation (9) for the wave still in deep water then becomes equation (10).

$$\eta_1(x,y,t) = A_1 \cos\left[\frac{\mu_1^2}{g}(x\cos\theta_F + y\sin\theta_F) - \mu_1 t + \delta_1)\right] \quad (10)$$

In general, we are interested in the wave system which will be present at some point in the shallower water at some fixed depth, H. We define a third coordinate system at this point with X_R pointing directly on shore, and θ_R measured with respect to the coordinate system.

A number of things happen to the wave system represented by equation (10) as the waves are refracted by the shallower water. These effects can be computed theoretically by constructing orthogonals by Snell's law and by considering the effect of the shoaling water. The net effect is that five things happen to the wave system. The wavelength of the wave shortens solely as a function of μ_1 and H. The direction toward which the crest is traveling changes due to the change in direction of the orthogonals. The crest becomes higher or lower due to the convergence or divergence of the orthogonals and the effect of shoaling. The crests have some phase difference with respect to the phase in deep water. Finally the wave crests become curves instead of straight lines.

All but this last effect can be represented by equation (11). The curvature of the crests is extremely difficult to represent analytically, and we limit this derivation by saying that equation (11) represents the crests in the vicinity of the point x_R, y_R equal to zero but that at large distances from the point, the derivation will not be satisfactory.

$$\eta_R(x_R, y_R, t) = A_{IR} \cos\left[\frac{I(\mu, H)\mu_I^2}{g}(x_R \cos\theta_R + y_R \sin\theta_R) - \mu_I t + \delta_I + \delta_{IR}\right] \quad (11)$$

The coefficient of the space variables, namely $I(\mu_I, H)\mu_I^2/g$ is equal to $2\pi/L_1$ where L_1 is the wave length of a wave with a period equal to $2\pi/\mu_1$ in water of depth H. It can be shown that a function, $I(\mu, H)$, can be found which easily yields the needed number by which μ_I^2/g (equal to $2\pi/L_{o1}$, where L_{o1} is the deep water wavelength) must be multiplied in order to obtain the value of $2\pi/L_1$.

The change in amplitude is a function of the deep water direction, θ_F, and of the deep water period (or frequency). A function of these variables can be found such that when A_1 is multiplied by it the result is A_{IR}, the amplitude after refraction. This function has been found as a function of period and direction for many places along the coasts of the United States. Examples are given of the forms it can take by Munk and Traylor (1947) and by Pierson (1951a) along with many others. This function can just as easily be plotted as a function of μ and θ_F, and the result will be a function defined as $K_H D(\mu, \theta_F)$ where the effects of refraction and shoaling are both included.

The angle, θ_R, is also a function of θ_F and μ, and it can be found by the same techniques that the above function was found. We define θ_R by equation (12)

$$\theta_R = \Theta(\mu, \theta_F) \quad (12)$$

In the refraction of a system like equation (2), a result will be that the phase change is unimportant although for precise treatment of any partial sum it should be theoretically known. We shall neglect the added refinement of considering δ_{IR} as a function of μ and θ_F.

THE REFRACTION OF A PARTIAL SUM

Under the above assumptions, the refraction of the sum of purely sinusoidal progressive waves as given by equation (3), is a straight forward procedure. The system is first referred to the μ, θ_F coordinate system. Then each term in the partial sum is treated by multiplying the amplitude of the term by the value of $K_H D(\mu, \theta_F)$ and by changing θ_F to θ_R with the aid of equation (12) for the appropriate direction and frequency. The wave length is changed to its new value for the shallower water.

The justification of such a procedure is that the system is linear and the total disturbance is the sum of all of the individual terms in the partial sum. For several hundred terms in the partial sum, the system would already have the appearance of actual ocean waves. The preponderance of the theoretical evidence and of the observational evidence is that the spectrum of ocean waves is continuous, and that an infinite number of terms must be considered, at least theoretically, in order to describe the sea surface properly.

THE REFRACTION OF A SHORT CRESTED GAUSSIAN SEA SURFACE

The final part of the theoretical derivation is to consider what happens to equation (2) when the wave system represented by it is refracted. One condition which must be preserved in the limit is that the average square of the refracted wave system both as represented by the partial sum and by the power integral is the same.

<u>Definition of terms</u> - Let the function, $[K_H D(\mu, \theta_F)]^2$ which is the square of the function mentioned before be defined to be the <u>spectrum amplification function</u>. Also let the function, $\theta_R = \Theta(\mu, \theta_F)$, be defined to be the <u>direction function</u>.

Now, if the direction function is a function of μ and θ_F, it can be inverted and θ_F can be expressed as a function of μ and θ_R. Theoretically, the inversion would involve a mathematical representation for the function and solving for θ_F in terms of θ_R and μ. Practically, it involves reading off the values of θ_F along a line on which θ_R is a constant in the direction function, plotting those values in a μ, θ_R polar coordinate system, and isoplething the lines for θ_F equal to a constant. The <u>inverse direction function</u> can then be defined by equation (13).

$$\theta_F = \Theta^*(\mu, \theta_R) \tag{13}$$

The Jacobian of the <u>inverse direction function</u> is also needed. The result is defined by equation (14). This function can be approximated to a considerable degree of accuracy by finite differences from an isoplethed drawing of equation (13).

$$\frac{\partial \theta_F}{\partial \theta_R} = \frac{\partial \Theta^*(\mu, \theta_R)}{\partial \theta_R} = \Gamma(\mu, \theta_R) \tag{14}$$

<u>The power spectrum after refraction</u> - After these definitions, our problem is to find the power spectrum which represents the waves at the point of interest after refraction. We just multiply the power spectrum of the waves by the spectrum amplification function. The result is still a function of μ and θ_F. The substitution of equation (13), the inverse direction function, then expresses the above product in terms of μ and θ_R. The result is squeezed together as a function of μ and θ_R for low values of μ and it must be properly amplified by multiplication by $\Gamma(\mu, \theta_R)$

in order to preserve the correct value of the average square of the record. The result is then the power spectrum of the waves at the point of observation in the refraction zone. This power spectrum is then given by equation (15).

$$\left[A_{2RH}(\mu,\theta_R)\right]^2 = \left[A_2(\mu,\Theta^*(\mu,\theta_R))\right]^2 \cdot \left[K_H D(\mu,\Theta^*(\mu,\theta_R))\right]^2 \cdot \Gamma(\mu,\theta_R) \tag{15}$$

The waves after refraction.- It then follows that the waves in the vicinity of the point of observation in the refraction zone are represented by a power integral over the power spectrum defined in equation (15). The phases are still to be picked at random from a rectangular probability distribution. This is why it was not necessary to treat the phase change when the refraction of a simple harmonic progressive wave was considered. The representation of the waves at the new point of interest is then given by equation (16).

$$\eta_R(x_R,y_R,t)=\int_{-\pi}^{\pi}\int_0^{\infty}\cos\left[\frac{\mu^2 I(\mu,H)}{g}(x_R\cos\theta_R+y_R\sin\theta_R)-\mu t+\psi(\mu,\theta_R)\right]\sqrt{[A_{2RH}(\mu,\theta_R)]^2 d\mu\, d\theta_R}$$

Equation (16) can be approximated by a partial sum just as equation (2) was approximated by a partial sum. For a large number of terms in the partial sum, it can be shown that the result is the same as the result of refracting the individual terms in the partial sum from deep water as was done in the section entitled, the refraction of a partial sum.

The equation of a wave record in the refraction zone - It can be shown that equation (16), if observed as a function of time at the point of interest, can be given by equation (17). Equation (17) would represent a wave record made with, say, a step resistance gage such as the one described by Caldwell (1948) in the refraction zone. A pressure record would have to have its power spectrum corrected for the effect of depth by a correct amplification factor point for point over the entire range of μ before it would represent the free surface power spectrum (see Pierson and Marks (1952)).

$$\eta_R(t) = \int_0^{\infty}\cos(\mu t + \psi(\mu t+\psi(\mu))\sqrt{[A_{RH}(\mu)]^2 d\mu} \tag{17}$$

The function, $[A_{RH}(\mu)]^2$, can be found in either of the two ways defined by equations (18) and (19). Equation (19) shows that $\Gamma(\mu,\theta_R)$ need not be found if simply the wave record at one point as a function of time is needed.

$$[A_{2RH}(\mu)]^2 = \int_{-\pi}^{\pi}[A_{2RH}(\mu,\theta_R)]^2 d\theta_R \tag{18}$$

$$[A_{RH}(\mu)]^2 = \int_{-\pi}^{\pi} [A_2(\mu,\theta_F)]^2 [K_H D(\mu,\theta_F)]^2 d\theta_F \tag{19}$$

Additional properties - It can also be proved that the properties expressed by equations (20), (21), (22), (23) and (24) hold true at the point of observation in the refraction zone. The average over y_R can be restricted to only a few feet and the results would still be valid. The average square of the record as observed in the refraction zone is usually not the same as the average square of the record in deep water. These results only hold out beyond the point where non-linear effects become important, and for this reason, the wave record is still Gaussian as equation (24) states.

$$\lim_{\substack{\bar{T}\to\infty \\ \bar{y}_R\to\infty}} \frac{1}{\bar{T}\,\bar{y}_R} \int_{t^*}^{t^*+\bar{T}} \int_{y_R^*}^{y_R^*+\bar{y}_R} [\eta_R(x_R,y_R,t)]^2 dy_R\, dt = \frac{1}{2}\int_0^{\infty}\int_{-\pi}^{\pi} [A_{2RH}(\mu,\theta_R)]^2 d\theta_R\, d\mu \tag{20}$$

$$\lim_{\bar{T}\to\infty} \frac{1}{\bar{T}} \int_{t^*}^{t^*+\bar{T}} [\eta(t)]^2 dt = \frac{1}{2}\int_0^{\infty} [A_{RH}(\mu)]^2 d\mu \tag{21}$$

$$\int_0^{\infty}\int_{-\pi}^{\pi} [A_{2RH}(\mu,\theta_R)]^2 d\theta\, d\mu = E_R \tag{22}$$

$$E_R \neq E \tag{23}$$

$$P(-\infty < \eta_R(t) < K) = \frac{1}{\sqrt{\pi E_R}} \int_{-\infty}^{K} e^{-\xi^2/E_R}\, d\xi \tag{24}$$

SOME GENERAL COMMENTS

The above theoretical derivation is rather complicated. It suggests, at least, that ocean waves are far more complicated and far more intricate in their properties and construction than current theories and practices would admit. Waves are complicated, and oversimplifications at the start of a theoretical consideration of their properties must eventually lead to erroneous predicted results. The complete power spectrum, $[A_2(\mu,\theta)]^2$ of a wave system has never been determined. Its exact functional form is unknown. Arthur (1949) has shown that waves from a storm propagate out of the storm at angles to the direction of the wind such that they arrive at points they could not possibly reach if they traveled only in the direction of the average wind. Wave spectra as a function of μ alone determined electronically without an intensity scale have been reported by Klebba (1946), Deacon (1949), Rudnick (1951) and Barber and Ursell (1948). These spectra and a few determined by more precise techniques all show that μ can vary from $2\pi/20$ to $2\pi/6$ or over a range of periods from twenty seconds to six seconds, and that the various spectral components are all important. Since the records are pressure records in fairly deep water, there is even reason to believe that periods less than six seconds are also of importance.

If the available evidence suggests that "sea" waves have a power spectrum which varies over a wide range of μ and θ in a storm, and if the theory of refraction presented above is correct, then it is of interest to assume some functional form for the power spectrum and to find out the spectrum of the waves after refraction. The result will be that interesting features predicted by the model wave system will be obtained which will show that caution must be employed in the interpretation of wave records which are currently obtained along the coasts of the United States. In particular, the results will show that wave records obtained at Long Branch, New Jersey, do not represent wave conditions at nearby points on the New Jersey coast.

APPLICATION TO THE NORTHERN NEW JERSEY COAST

A MODEL STORM

In order to discover some of the consequences of the above theory and in order to provide an example of the techniques to be employed in forecasting waves according to the properties of their power spectra, a model storm was constructed over the Atlantic Ocean. Winds in the storm were assumed to be blowing from east to west over an area 566 km long and 550 km wide for a total duration of 24 hours. The center of the forward edge of the storm area was located 872 km due east of Cape Hatteras or at latitude 35°N and longitude 64°W. The time, t equal to zero, was referred to the start of the winds over the storm areas. The assumed functional form of the power spectrum was based upon the observations and results cited above. The center of the forward edge of the model storm was located 825 km from Long Branch, New Jersey.

Given these assumptions, the power spectrum at various times and places outside of the storm area can be forecasted according to the methods described by Pierson (1952). The wave conditions in deep water just offshore from Long Branch, New Jersey, were forecasted by these techniques and the different power spectra at this point were found.

The power spectrum of the waves off the New Jersey coast varies very, very slightly over distances comparable to the distance from Asbury Park to Sandy Hook which is 7 nautical miles. It can therefore be assumed to be the same in form for all points in deep water along this section of the coast.

WAVE REFRACTION DATA

Source- Our interest in this particular paper is to find out the effects of refraction on these waves as they move from deep water to the coast. Data prepared for the northern New Jersey coast in a study by Pierson, Martineau, James and Pocinki (1951) were available and these data were worked up in more detail for three points along the coast for a depth offshore of 20 feet mean sea level.

The points which were chosen were at the base of Sandy Hook, near Ship Ahoy Inn, at latitude 40°22'N; at Long Branch, near the North End Beach Club, at latitude 40°18'N; and near Asbury Park*, at latitude 40°15'N. The point at Sandy Hook is four nautical miles north of Long Branch and the point near Asbury Park is three nautical miles south of Long Branch. These differences in distance are negligible compared to the scale of the wave forecasting problem.

The spectrum amplification function and the direction function

The effect of refraction at these three points is quite different. Figure 1 shows the spectrum amplification function for Ship Ahoy Inn, and Figure 2 shows the direction function. Figure 3 shows the spectrum amplification function for Long Branch, and Figure 4 shows the direction function. Figure 5 shows the spectrum amplification function for the point near Asbury Park, and Figure 6 shows the direction function.

The angular variables on these six figures are labeled in two different ways. One way shows the direction toward which the elemental components are traveling labeled in degrees from north. In such a system, the angle increases in a clockwise direction, and the notation in the derivation does not provide for such an angular system. The other way shows the angle, θ_F, where θ_F varies in a counterclockwise direction. The angle, θ_F, is zero for waves traveling from east to west. It is equal to ten degrees for waves traveling toward 260° (from north). These values for θ_F are shown in parenthesis on the figures. Since the coast runs very nearly north-south in the vicinity of Long Branch, the problem can be treated simply in terms of θ_F, but for coasts which are not north-south, sometimes another change of angular variable helps.

These six figures have features in common, and yet they are quite different. They show that it is practically impossible for spectral components with periods greater than 14 seconds to reach the northern corner of the state of New Jersey. The data have been analyzed by extrapolation for μ less than $2\pi/14$. At a depth of 20 feet, waves with a wavelength of 40 feet are unaffected by the bottom. Thus for μ greater than or equal to $2\pi/2.8$ the spectrum amplification function is essentially one everywhere. Even for μ equal to $2\pi/4$, the waves are affected by only a narrow strip of depths along the coast, and the spectrum amplification function for all three places is essentially the same. Between the values for μ equal to $2\pi/4$ and $2\pi/6$, all three spectrum amplification functions

*Actually the point is about two miles north of Asbury Park, proper.

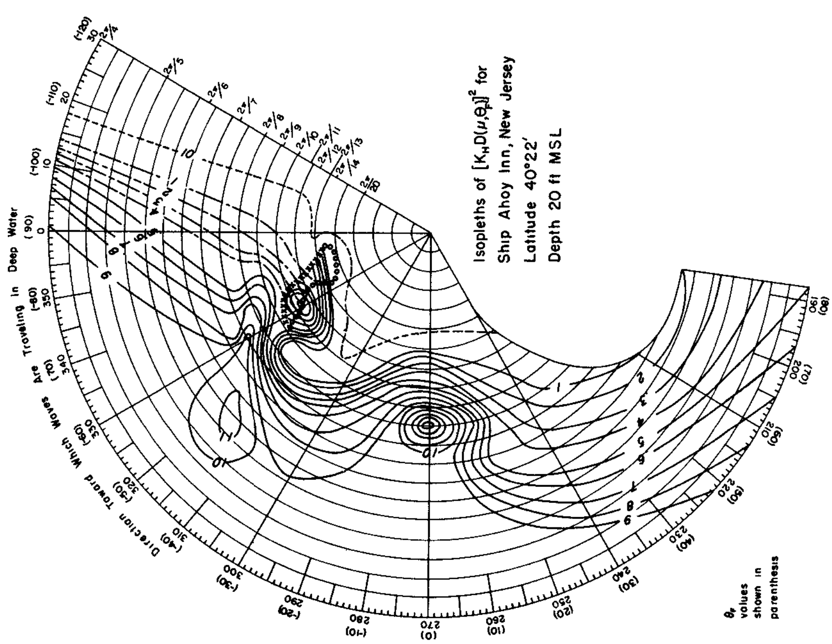

Fig. 1. Spectrum amplification function for Ship Ahoy Inn.

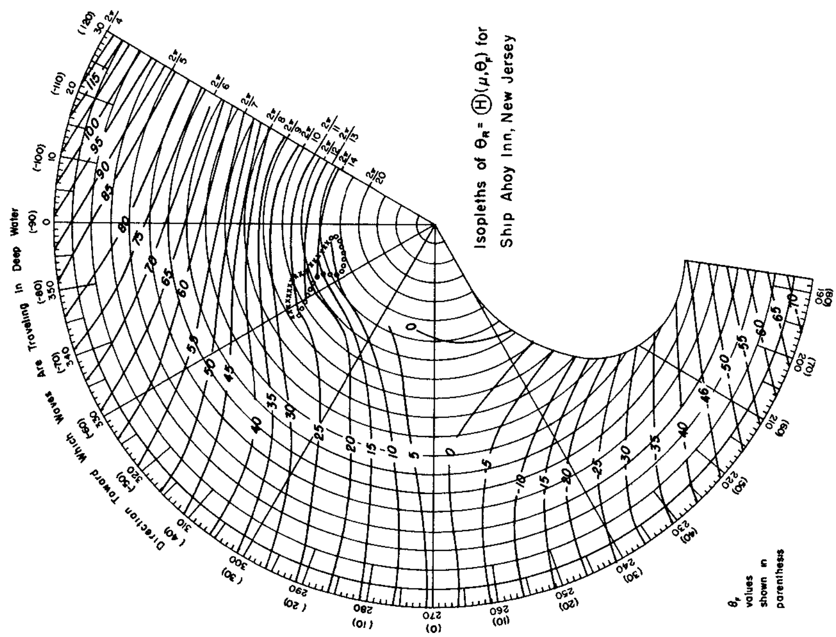

Fig. 2. Direction function for Ship Ahoy Inn.

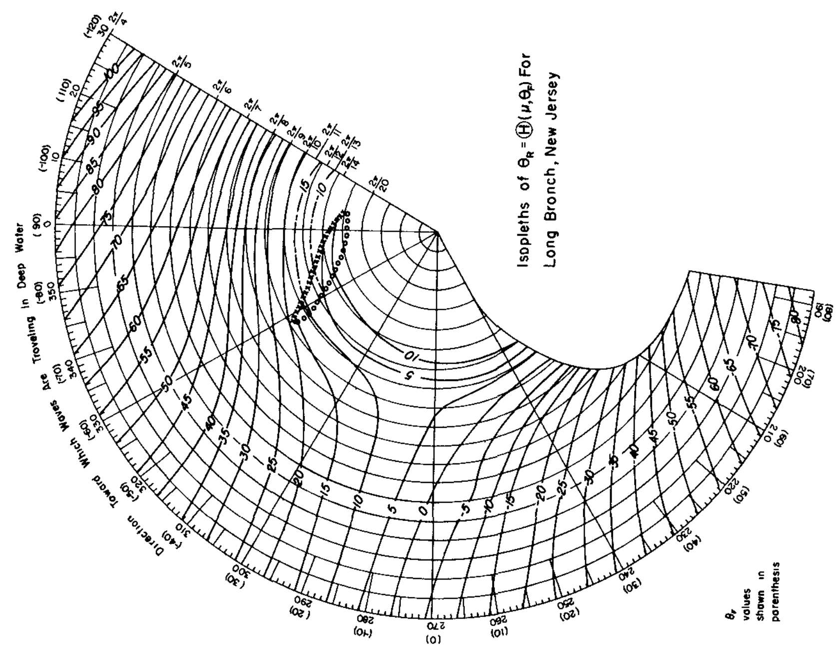

Fig. 4. Direction function for Long Branch.

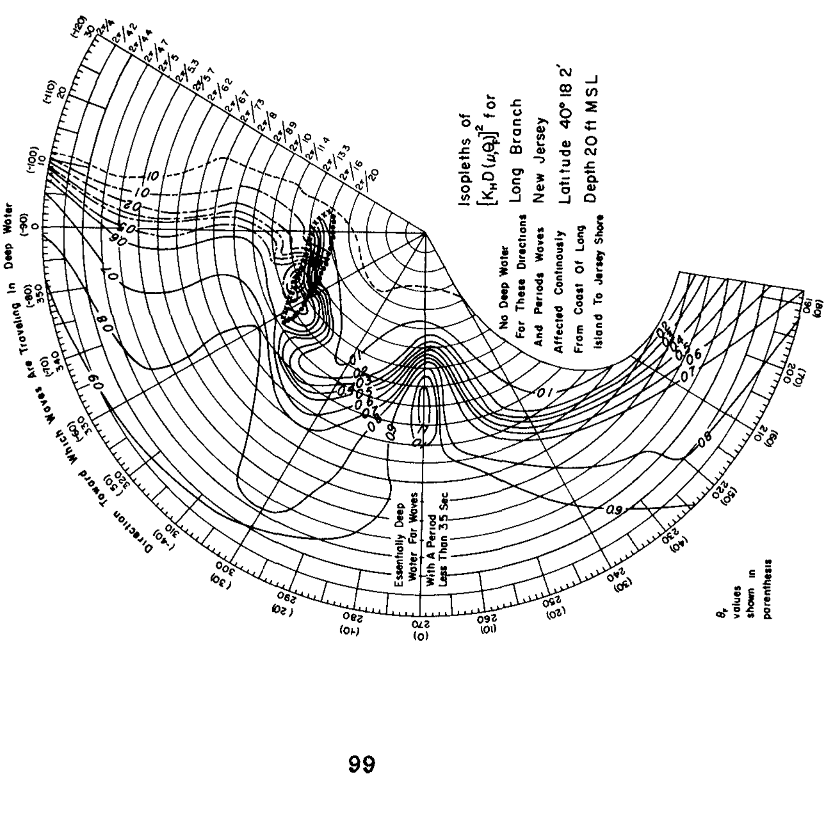

Fig. 3. Spectrum amplification function for Long Branch.

have values greater than 0.8 over a wide range of directions.

Each figure shows a narrow winding band of isopleths such that a small change in μ results in a large variation in the spectrum amplification function. The band is located at different places in the three figures. If the band is toward the center of the diagram, then low values of μ (corresponding to high periods) can be observed at that point if they come from those particular directions. If the band is toward high values of μ, then low frequencies do not show up at that point.

The little area bounded by circles and crosses enclosed an area where two heights and two directions exist for one sine wave in deep water. It indicates the presence of caustics in the orthogonal pattern. For further information, see Pierson (1951a).

In particular, compare the spectrum amplification function for Ship Ahoy Inn and Asbury Park for spectral components traveling toward directions between 300° and 310°. The 0.1 contour at Ship Ahoy Inn cuts down all spectral components for μ less than $2\pi/8$ for this range of angles. The 0.1 contour at Asbury Park affects only values of μ less than $2\pi/16$. In general over a wide range of θ and μ for waves traveling toward the northwest, the spectrum amplification function for Asbury Park amplifies low values of μ, (high period) far more than the spectrum amplification for Ship Ahoy Inn.

THE WAVE POWER SPECTRUM IN DEEP WATER AS FORECASTED FROM THE MODEL STORM

In the model which was constructed, it was possible to forecast the theoretical spectrum for deep water at six hour intervals. For example, at t equal to 54 hours, the spectrum was found to have the form shown in Figure 7 as a function of μ and θ. Figure 7 shows that the spectra consist of elemental frequencies which vary from $2\pi/18.2$ to (in this figure) $2\pi/5.3$. The sharp sides of the spectrum are due to the approximations used in the forecasting theory, and in actuality the edges and sides would be rounded. Later spectra included even lower values for the period. The waves in deep water would appear to be traveling toward approximately 300°. They would be quite short crested and the elemental spectral components would be present for all directions from 287° to 311°. The lowest frequency, $2\pi/20$ for some of the first spectra, was chosen to correspond with the observed maximum period found by Barber and Ursell (1918) in a storm with a wind velocity of 45 knots. The integral over θ of the deep water μ, θ power spectrum of course yields the power spectrum, $[A(\mu)]^2$, of the waves in deep water. Such a power spectraum could be evaluated from, say, a twenty-five minute record made with the spark plug type spar buoy wave recorder constructed by the Beach Erosion Board. The variation of $[A_2(\mu, \theta)]^2$ as a function of θ is much more difficult to determine in practice. If the deep water waves would have been picked to be traveling more toward 330°, the results which would have been obtained upon refraction would have been even more pronounced.

Some additional recent theoretical evidence

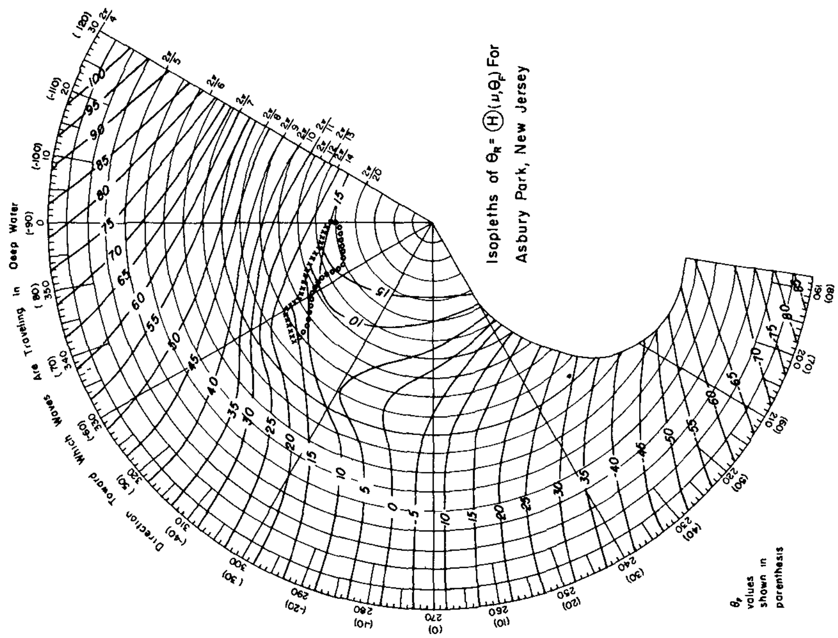

Fig. 6. Direction function for Asbury Park.

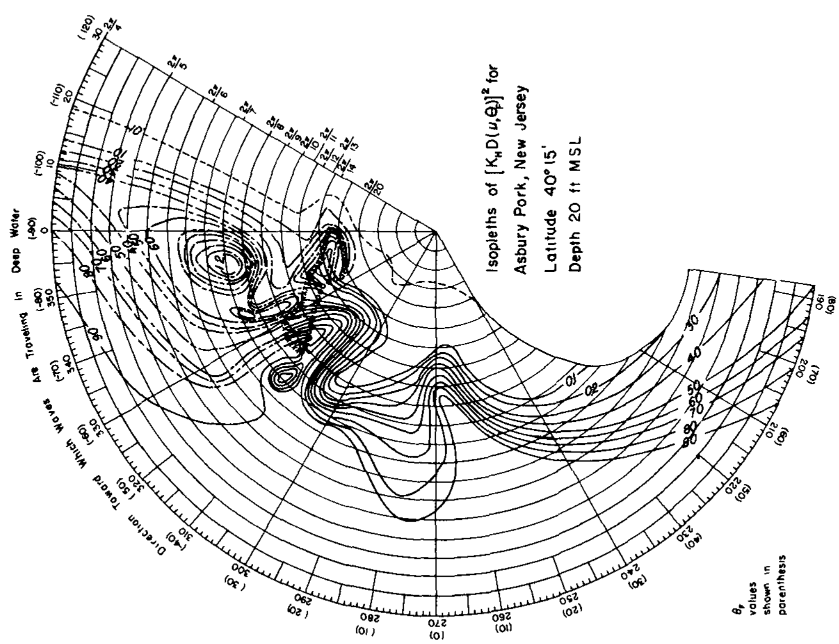

Fig. 5. Spectrum amplification function for Asbury Park.

A certain functional form was assumed for the power spectrum of the waves at the edge of the storm. At the time when this investigation was undertaken, there was not too much evidence available as to the relative power associated with each frequency. Very recent results of Darbyshire show that the power spectra in deep water should be much higher for low values of μ and much lower for high values of μ than the ones assumed in this paper. The result would be even more striking in the effects on the refracted power spectra which would result. The results predicted by our mathematical model could easily be an underestimate of the actual effect.

THE WAVE POWER SPECTRA AT THE COAST

These forecasted power spectra in deep water were multiplied by the spectrum amplification function for each of the three points of interest. Then the result was integrated numerically over θ_F to find the function $[A_{RH}(\mu)]^2$, for that time and place. By virtue of equation (19), the transformations involved in equation (15) need not be made if only $[A_{RH}(\mu)]^2$ is desired.

The forecasted power spectra, $[A(\mu)]^2$ and $[A_{RH}(\mu)]^2$, for deep water and for Ship Ahoy Inn, Long Branch, and Asbury Park are shown in Figure 8. The values of E and E_R for each spectrum are shown below the spectrum and the range of variation over μ is also shown. Note that the origin of the μ axis is not shown and that it lies progressively more to the left for the later power spectra.

Free surface wave records, produced by the above power spectra would, of course, have some significant height and period. The significant period would correspond to some value of μ near the center of the spectrum. The significant height, crest to trough, would be approximately equal to $2.88\sqrt{E/2}$. These values are also shown below the different spectra. For some important recent results on the distribution of wave heights in a wave record, see a forthcoming paper by Longuet-Higgens(1952). The results of Longuet-Higgens, which are more accurate, yield a slightly higher value for the above factor.

The first discernible swell in deep water would be observed eighteen hours after the start of the storm. Its significant period would be about 17.4 seconds, and its significant height would be about 4.18 feet. At Ship Ahoy Inn, practically no waves would be observed whereas at Asbury Park, waves with a significant period of 17.3 seconds and a height of 1.56 feet would be evident.

Thirty hours after the start of the storm, waves with a significant height of 8.27 feet and a significant period of 12.5 seconds would be present in deep water. At Ship Ahoy Inn, the significant period would appear to be about 10.5 seconds and the significant height would be about 1.46 feet. At Asbury Park, the significant period would appear to be about 11.8 seconds and the significant height about 3.97 feet.

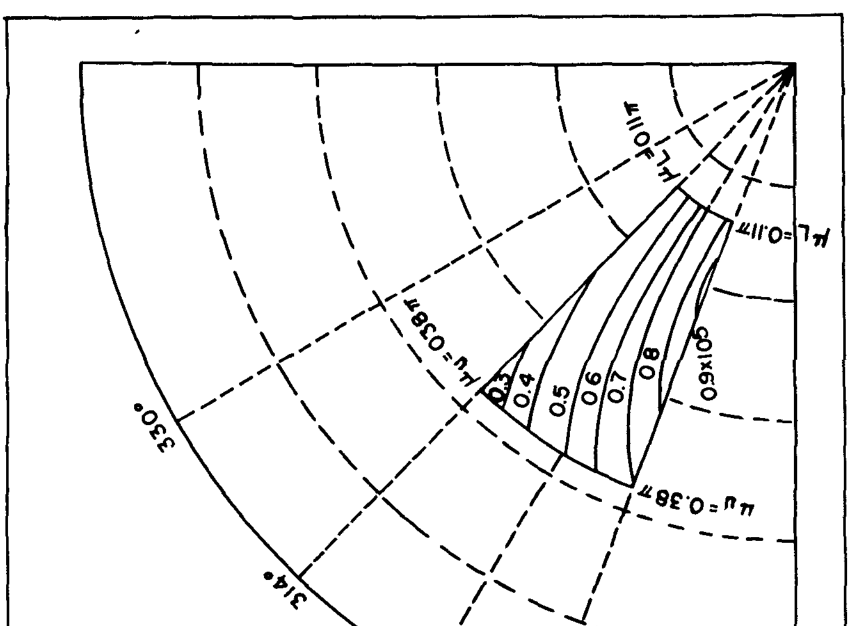

Fig. 7. Forecasted wave spectrum for a time 54 hours after the start of the storm (E equals 2,400 cm^2).

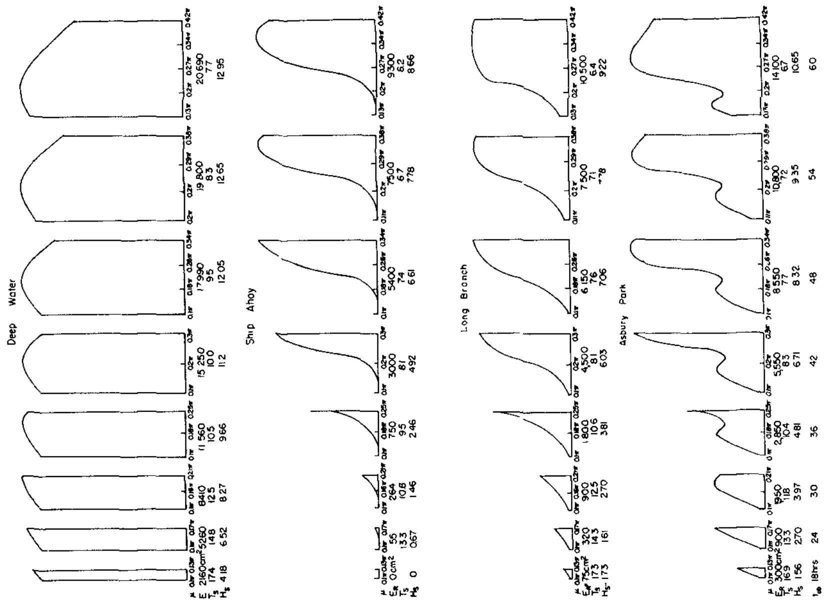

Fig. 8. Forecasted power spectra for deep water and the three points of observation.

The spectra for the three points near shore are markedly different with low frequency components containing greater energy at Asbury Park than at the other two places. The waves present would differ in fundamental ways even for those cases in which the significant periods differ by only a few tenths of a second.

Pressure recorders located at the points of interest would not record these values, and a simple computation of the free surface values by means of the "significant" height and period of the pressure record would be incorrect. The pressure record would have a higher significant period than the free surface value, and the computed free surface significant height as based upon the pressure significant period would be too low.

INTERPRETATION OF RESULTS

The reason for the results is basically the effect of the Hudson Submarine Canyon. The elemental pure sine waves in the partial sum for high periods are focused at Asbury Park and very nearly obliterated at Ship Ahoy Inn. The lower period elemental waves still show up at Ship Ahoy Inn.

Consider the particular spectra for 18 hours after the start of the storm. The results show that for a storm to the southeast of the New Jersey coast, there could be conditions such that a significant period of 9.5 seconds and a significant height of 2.46 feet would be observed at Ship Ahoy Inn and simultaneously a significant period of 10.4 seconds and a height of 4.81 feet would be observed at Asbury Park. These two points are only seven nautical miles apart.

The waves in deep water would have a significant period of 10.5 seconds and a significant height of 9.66 feet. If a direction of 300° is assumed for the deep water waves, and if the waves from deep water are refracted according to their significant height and period, the result is a forecast of 10.5 seconds and 1.36 feet at Ship Ahoy Inn and 10.5 seconds and 3.05 feet at Asbury Park.

These values are compared in Table I. The significant height and period method when compared with the more accurate power spectrum method gives completely different results. Note that the significant period also changes from deep to shallow water in the power spectrum method of wave refraction. Of course, the computation of the significant height and period from the refracted power spectra is a step in the wrong direction because the power spectra tell us much more about the waves than these two numbers.

<u>The usefulness of coastal wave records</u>

Wave records are currently obtained at Long Branch and evaluated by the significant height and period method. If we take the significant height and period of these records and assume some one deep water wave direction, then the deep water significant height and period could be deduced from the refraction diagram. From these values, the significant

Table I. Comparison of the power spectrum method of wave refraction and the significant height and period method.

	Power Spectrum Method		Sig. Hgt. and Period Method	
	Sig. Hgt. feet	Sig. Period seconds	Sig. Hgt. feet	Sig. Period seconds
Deep Water	9.66	10.5	9.66	10.5
Ship Ahoy Inn	2.46	9.5	1.36	10.5
Asbury Park	4.81	10.4	3.05	10.5

height and period at the other two points could be forecasted. The results would be just as much in error as the refraction from deep water by the significant height and period. There is no assurance that the significant period as observed near the coast will be the same in deep water. It is necessary to conclude, therefore, if these theoretical results are correct and approximate true conditions, that wave records at Long Branch, New Jersey do not yield reliable information at nearby points along the coast or in deep water when interpreted by the significant height and period techniques.

THE EFFECT OF THE DIRECTION FUNCTION

The effect of the direction function is to make the waves in a partial sum from equation (16) travel in more nearly the same direction compared to those in equation (2) for low values of μ. This means that if the waves are relatively short crested in deep water, they will be longer crested in the shallower water after refraction. Such a phenomenon can be observed in many aerial photographs of waves undergoing refraction, and Pierson (1952) has discussed two such photographs. The complete evaluation and interpretation of this feature has not been worked out, and results of a continued study will be reported in the future.

VERIFICATION

The actual verification of these results quantitatively has not been accomplished. This paper has been written to demonstrate a theoretical example of the refraction of a wave system with properties similar to those known to be the properties of actual ocean waves. To verify the results completely, pressure wave recorders at the three points would be needed, and a method for determining the deep water conditions would be needed. Partial verification from three pressure recorders would be possible since completely different spectra are predicted for the three points for the same time.

A qualitative verification of these results based upon crude wave measurements and purely visual observations can be given. When the group at New York University first began to study waves a few years ago, a hurricane generated waves from a position roughly the same as the one assumed for the model storm in this paper. A field trip was organized to observe these waves and within a time interval of forty-five minutes or so the waves at the three points under study in this paper were observed. The result based upon these crude observations was that waves with a significant period of six to eight seconds with a significant height of three or four feet were observed at Ship Ahoy Inn. At Long Branch, the significant period was nine or ten seconds and the significant height was four or five feet. At Asbury Park, the significant period was about twelve seconds, and the waves had a significant height of six or seven feet.

The observations were doubted because it was thought that the significant period had to be the same at all points. The theory of ocean wave refraction was based solely on equation (1), and a change of period is not possible in such a theory. It was thought at the time that there was an error in the observation techniques and not in the theory.

Finally, for what it is worth, we report the experience of those who swim along the Northern New Jersey coast. Those who like to ride the breakers as they come up to the beach report that they prefer to swim at the points to the south along the coast. The rollers, they say, are higher and more regular at points to the south. Since waves are rarely of zero height at any point along the coast when waves are present at other points, this suggests that there is a difference (and a long time statistical difference at that, since otherwise it would not have been noted by swimmers simply out for pleasant recreation) in the character of the waves along the coast.

CONCLUSIONS

Wave refraction is a complex problem since actual ocean waves are not simple harmonic progressive waves. Theoretical results from model wave forecasts, and crude visual observations suggest that <u>both</u> different significant periods and significant heights can result at nearby points when a short crested Gaussian sea surface is refracted. Care must therefore be exercised in the extension of wave observations made at one point to nearby points.

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