

CHAPTER 3

THE SURFACE WAVE IN A TWO-DIMENSIONAL VORTEX LAYER

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ABSTRACTS

Progressive surface wave in a two-dimensional vortex layer is theoretically treated. Dynamical equations and free surface conditions are shown by using the two-dimensional stream functions of wave and vortex.

Then the perturbation equations are given by assuming that the ratio of length scale of vortices and wave is fairly small. The first approximate solution of wave has a usual form of an irrotational progressive wave. Vortices are assumed to be steady and to have simplified Fourier-Stieltjes form. Then the interaction of this primary wave and the vortices are examined. To satisfy the free surface condition of the second order, existent waves are formed. In the second order term of the free surface elevation, these secondary waves offset the effect of the above mentioned interaction, and so the surface profile of the primary wave is not altered by the existence of inner vortices of high frequency.

Some pictures of irregular surface waves in a turbulent flow are shown to verify this property.

DERIVATION OF PERTURBATION EQUATIONS

We use the two-dimensional co-ordinate system. x-axis is taken horizontally at the still water surface, y-axis is vertically upwards. The depth of water is assumed to be infinite and the motion is considered inviscid.

Then we suppose the motion is consisted both of the surface gravity wave and vortices in a steady uniform current and we assume that the motion of surface wave can be linearized. In the following equations suffix 1 is concerned to vortices and suffix 2 to wave, and u_0 is the horizontal velocity of steady uniform current.

Equations of motion are

$$\begin{aligned} \frac{\partial u_1}{\partial t} + \frac{\partial u_2}{\partial t} + (u_0 + u_1 + u_2) \frac{\partial}{\partial x} (u_1 + u_2) + (v_1 + v_2) \frac{\partial}{\partial y} (u_1 + u_2) \\ = - \frac{1}{\rho} \frac{\partial p}{\partial x} \end{aligned} \quad (1)$$

THE SURFACE WAVE IN A TWO-DIMENSIONAL
VORTEX LAYER

$$\begin{aligned} & \frac{\partial v_1}{\partial t} + \frac{\partial v_2}{\partial t} + (u_0 + u_1 + u_2) \frac{\partial}{\partial x} (v_1 + v_2) + (v_1 + v_2) \frac{\partial}{\partial y} (v_1 + v_2) \\ & = -g - \frac{1}{\rho} \frac{\partial p}{\partial y} \end{aligned} \quad (2)$$

(1) and (2) are linearized as to u_2 , v_2 , and

$$\begin{aligned} & \frac{\partial u_1}{\partial t} + \frac{\partial u_2}{\partial t} + (u_0 + u_1 + u_2) \frac{\partial u_1}{\partial x} + (u_0 + u_1) \frac{\partial u_2}{\partial x} + (v_1 + v_2) \frac{\partial u_1}{\partial y} \\ & + v_1 \frac{\partial u_2}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} \end{aligned} \quad (3)$$

$$\begin{aligned} & \frac{\partial v_1}{\partial t} + \frac{\partial v_2}{\partial t} + (u_0 + u_1 + u_2) \frac{\partial v_1}{\partial x} + (u_0 + u_1) \frac{\partial v_2}{\partial x} + (v_1 + v_2) \frac{\partial v_1}{\partial y} \\ & + v_1 \frac{\partial v_2}{\partial y} = -g - \frac{1}{\rho} \frac{\partial p}{\partial y} \end{aligned} \quad (4)$$

In incompressible fluid, the equation of continuity is

$$\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{\partial v_2}{\partial y} = 0 \quad (5)$$

Here we put the following boundary condition of v_1 at $y=0$

$$v_1 = 0 \quad \text{at } y=0 \quad (6)$$

and so $\frac{\partial v_1}{\partial t} = \frac{\partial v_2}{\partial x} = 0 \quad \text{at } y=0 \quad (7)$

Kinematical boundary condition at surface is

$$\frac{\partial \eta}{\partial t} + (u_0 + u_1 + u_2) \frac{\partial \eta}{\partial x} = v_1 + v_2 \quad \text{at } y = \eta \quad (8)$$

This is linearized as to surface wave, and using (6)

$$\frac{\partial \eta}{\partial t} + (u_0 + u_1) \frac{\partial \eta}{\partial x} = v_2 \quad \text{at } y=0 \quad (9)$$

Dynamical boundary condition at surface is

$$\frac{\partial p}{\partial t} + u_0 \frac{\partial p}{\partial x} + u_1 \frac{\partial p}{\partial x} + u_2 \frac{\partial p}{\partial x} + v_1 \frac{\partial p}{\partial y} + v_2 \frac{\partial p}{\partial y} = 0 \quad \text{at } y = \eta \quad (10)$$

Multiplying (3) by u_2 , and linearizing as to u_2 , v_2 , (11) is obtained

$$u_2 \frac{\partial u_1}{\partial t} + (u_0 + u_1) u_2 \frac{\partial u_1}{\partial x} + v_1 u_2 \frac{\partial u_1}{\partial y} = -\frac{1}{\rho} u_2 \frac{\partial p}{\partial x} \quad (11)$$

Multiplying (4) by v_2 and linearizing as to u_2 , v_2 , (12) is obtained

$$v_2 \frac{\partial v_1}{\partial t} + (u_0 + u_1) v_2 \frac{\partial v_1}{\partial x} + v_1 v_2 \frac{\partial v_1}{\partial y} = -v_2 g - \frac{1}{\rho} v_2 \frac{\partial p}{\partial y} \quad (12)$$

Inserting (11), (12) into (10),

$$\begin{aligned} & \frac{\partial p}{\partial t} + u_0 \frac{\partial p}{\partial x} + u_1 \frac{\partial p}{\partial x} + v_1 \frac{\partial p}{\partial y} - \rho u_2 \frac{\partial u_1}{\partial t} - \rho (u_0 + u_1) u_2 \frac{\partial u_1}{\partial x} \\ & - \rho v_1 u_2 \frac{\partial u_1}{\partial y} - \rho v_2 \frac{\partial v_1}{\partial t} - \rho (u_0 + u_1) v_2 \frac{\partial v_1}{\partial x} - \rho v_1 v_2 \frac{\partial v_1}{\partial y} \\ & - \rho v_2 g = 0 \end{aligned} \quad \text{at } y=0 \quad (13)$$

COASTAL ENGINEERING

Using (6) and (7), (13) is simplified as

$$\frac{\partial p}{\partial t} + u_0 \frac{\partial p}{\partial x} + u_1 \frac{\partial p}{\partial x} - \rho u_2 \frac{\partial u_1}{\partial t} - \rho(u_0 + u_1) u_2 \frac{\partial u_1}{\partial x} - \rho v_2 g = 0 \quad \text{at } y=0 \quad (14)$$

We use stream functions Ψ_1 , Ψ_2 and define them as

$$u_1 = -\frac{\partial \Psi_1}{\partial y}, \quad u_2 = -\frac{\partial \Psi_2}{\partial y}, \quad v_1 = \frac{\partial \Psi_1}{\partial x}, \quad v_2 = \frac{\partial \Psi_2}{\partial x} \quad (15)$$

The equation of continuity (5) is clearly satisfied by (15). The vorticity of vortex motion is

$$\zeta_1 = \frac{\partial v_1}{\partial x} - \frac{\partial u_1}{\partial y} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \Psi_1 \quad (16)$$

The vorticity of surface wave is

$$\zeta_2 = \frac{\partial v_2}{\partial x} - \frac{\partial u_2}{\partial y} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \Psi_2 \quad (17)$$

If surface wave does not exist, it becomes $\Psi_2 = 0$ and in the present inviscid condition

$$\frac{\partial}{\partial t} \left\{ \frac{\partial^2 \Psi_1}{\partial x^2} + \frac{\partial^2 \Psi_1}{\partial y^2} \right\} + (u_0 - \frac{\partial \Psi_1}{\partial y}) \frac{\partial}{\partial x} \left\{ \frac{\partial^2 \Psi_1}{\partial x^2} + \frac{\partial^2 \Psi_1}{\partial y^2} \right\} + \frac{\partial \Psi_1}{\partial x} \frac{\partial}{\partial y} \left\{ \frac{\partial^2 \Psi_1}{\partial x^2} + \frac{\partial^2 \Psi_1}{\partial y^2} \right\} = 0 \quad (18)$$

If Ψ_2 exists and it is a function of x , y , t , we cannot deduce (18) from the motion of $\Psi_1 + \Psi_2$, and we assume that the condition (18) is maintained even in this case. In other words, we assume that the vorticity (16) of vortex motion is steady and it is not influenced by the existence of surface wave. From (3), (4), eliminating pressure p and using Ψ_1 , Ψ_2 of (15) we obtain the equation concerned to the vorticity of the motion of $\Psi_1 + \Psi_2$. Subtracting (18) from the equation, we have

$$\begin{aligned} & \frac{\partial}{\partial t} \left(\frac{\partial^2 \Psi_2}{\partial x^2} + \frac{\partial^2 \Psi_2}{\partial y^2} \right) + (u_0 - \frac{\partial \Psi_1}{\partial y}) \frac{\partial}{\partial x} \left(\frac{\partial^2 \Psi_2}{\partial x^2} + \frac{\partial^2 \Psi_2}{\partial y^2} \right) \\ & + \frac{\partial \Psi_1}{\partial x} \frac{\partial}{\partial y} \left(\frac{\partial^2 \Psi_2}{\partial x^2} + \frac{\partial^2 \Psi_2}{\partial y^2} \right) - \frac{\partial}{\partial x} \left(\frac{\partial^2 \Psi_1}{\partial x^2} + \frac{\partial^2 \Psi_1}{\partial y^2} \right) \cdot \frac{\partial \Psi_2}{\partial y} \\ & + \frac{\partial}{\partial y} \left(\frac{\partial^2 \Psi_1}{\partial x^2} + \frac{\partial^2 \Psi_1}{\partial y^2} \right) \cdot \frac{\partial \Psi_2}{\partial x} = 0 \end{aligned} \quad (19)$$

Here we suppose the primary surface wave and its period, length and celerity are given by T_2 , L_2 and C_2 ($L_2 = C_2 T_2$) and stillmore the length L_1 is taken as a representative length of Ψ_1 . By making use of these representative values, we set the following dimensionless values.

$$\begin{aligned} \text{They are } & \Psi_1 = C_2 L_1 \Psi'_1, \quad \Psi_2 = C_2 L_2 \Psi'_2, \quad \eta = L_2 \eta', \\ & p = L_2 \rho g \rho', \quad \tau = T_2 \tau', \quad x = L_2 x', \quad y = L_2 y' \\ & \text{and } u_0 = C_2 u'_0. \end{aligned}$$

THE SURFACE WAVE IN A TWO-DIMENSIONAL
VORTEX LAYER

Using these values, equation (19) and surface conditions (9) and (14) are transformed to dimensionless.

$$\begin{aligned} & \frac{\partial}{\partial t'} \left(\frac{\partial^2 \Psi_2'}{\partial x'^2} + \frac{\partial^2 \Psi_2'}{\partial y'^2} \right) + \left(u_0' - \frac{L_1}{L_2} \frac{\partial \Psi_1'}{\partial y'} \right) \frac{\partial}{\partial x'} \left(\frac{\partial^2 \Psi_2'}{\partial x'^2} + \frac{\partial^2 \Psi_2'}{\partial y'^2} \right) \\ & + \frac{L_1}{L_2} \frac{\partial \Psi_1'}{\partial x'} \frac{\partial}{\partial y'} \left(\frac{\partial^2 \Psi_2'}{\partial x'^2} + \frac{\partial^2 \Psi_2'}{\partial y'^2} \right) - \frac{L_1}{L_2} \frac{\partial}{\partial x'} \left(\frac{\partial^2 \Psi_1'}{\partial x'^2} + \frac{\partial^2 \Psi_1'}{\partial y'^2} \right) \cdot \frac{\partial \Psi_2'}{\partial y'} \\ & + \frac{L_1}{L_2} \frac{\partial}{\partial y'} \left(\frac{\partial^2 \Psi_1'}{\partial x'^2} + \frac{\partial^2 \Psi_1'}{\partial y'^2} \right) \cdot \frac{\partial \Psi_2'}{\partial x'} = 0 \end{aligned} \quad (20)$$

$$\frac{\partial \eta'}{\partial t'} + \left(u_0' - \frac{L_1}{L_2} \frac{\partial \Psi_1'}{\partial y'} \right) \frac{\partial \eta'}{\partial x'} = \frac{\partial \Psi_2'}{\partial x'} \quad \text{at } y'=0 \quad (21)$$

$$\begin{aligned} & g \frac{\partial p'}{\partial t'} + u_0' g \frac{\partial p'}{\partial x'} - \frac{L_1}{L_2} g \frac{\partial \Psi_1'}{\partial y'} \frac{\partial p'}{\partial x'} - \frac{L_1}{L_2} \frac{C_2}{T_2} \frac{\partial \Psi_2'}{\partial y'} \frac{\partial^2 \Psi_1'}{\partial y' \partial t'} \\ & - \frac{C_2}{T_2} \left(u_0' - \frac{L_1}{L_2} \frac{\partial \Psi_1'}{\partial y'} \right) \frac{\partial \Psi_2'}{\partial y'} \frac{L_1}{L_2} \frac{\partial^2 \Psi_1'}{\partial x' \partial y'} - g \frac{\partial \Psi_2'}{\partial x'} = 0 \quad \text{at } y'=0 \quad (22) \end{aligned}$$

We put $L_1/L_2 = \alpha$, $C_2/T_2 = \beta$, and $\beta/g = \gamma$ has clearly order of 100, but α can be taken arbitrary. Here we consider the case in which L_1 is fairly small compared with L_2 , and so $\alpha < 1$. We use the method of perturbation concerned to α .

Ψ_2' , η' , p' can be expanded as

$$\left. \begin{aligned} \Psi_2' &= \Psi_{20}' + \alpha \Psi_{21}' + \alpha^2 \Psi_{22}' + \alpha^3 \Psi_{23}' + \dots \\ \eta' &= \eta_0' + \alpha \eta_1' + \alpha^2 \eta_2' + \alpha^3 \eta_3' + \dots \\ p' &= p_0' + \alpha p_1' + \alpha^2 p_2' + \alpha^3 p_3' + \dots \end{aligned} \right\} \quad (23)$$

Inserting (23) into (20), (21) and (22) and comparing terms of same power of α , the following perturbed equations may be obtained.

From (20)

$$\frac{\partial}{\partial t'} \nabla^2 \Psi_{20}' + u_0' \frac{\partial}{\partial x'} \nabla^2 \Psi_{20}' = 0 \quad (24-1)$$

$$\begin{aligned} & \frac{\partial}{\partial t'} \nabla^2 \Psi_{21}' + u_0' \frac{\partial}{\partial x'} \nabla^2 \Psi_{21}' - \frac{\partial \Psi_1'}{\partial y'} \frac{\partial}{\partial x'} \nabla^2 \Psi_{20}' + \frac{\partial \Psi_1'}{\partial x'} \frac{\partial}{\partial y'} \nabla^2 \Psi_{20}' \\ & - \frac{\partial}{\partial x'} \nabla^2 \Psi_{11}' \cdot \frac{\partial \Psi_{20}'}{\partial y'} + \frac{\partial}{\partial y'} \nabla^2 \Psi_{11}' \cdot \frac{\partial \Psi_{20}'}{\partial x'} = 0 \end{aligned} \quad (24-2)$$

$$\begin{aligned} & \frac{\partial}{\partial t'} \nabla^2 \Psi_{22}' + u_0' \frac{\partial}{\partial x'} \nabla^2 \Psi_{22}' - \frac{\partial \Psi_1'}{\partial y'} \frac{\partial}{\partial x'} \nabla^2 \Psi_{21}' + \frac{\partial \Psi_1'}{\partial x'} \frac{\partial}{\partial y'} \nabla^2 \Psi_{21}' \\ & - \frac{\partial}{\partial x'} \nabla^2 \Psi_{11}' \cdot \frac{\partial \Psi_{21}'}{\partial y'} + \frac{\partial}{\partial y'} \nabla^2 \Psi_{11}' \cdot \frac{\partial \Psi_{21}'}{\partial x'} = 0 \end{aligned} \quad (24-3)$$

Here $\nabla^2 = \frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2}$

From (21), at $y'=0$

$$\frac{\partial \eta_0'}{\partial t'} + u_0' \frac{\partial \eta_0'}{\partial x'} = \frac{\partial \Psi_{20}'}{\partial x'} \quad (25-1)$$

COASTAL ENGINEERING

$$\frac{\partial \eta'_1}{\partial t'} + u'_0 \frac{\partial \eta'_1}{\partial x'} - \frac{\partial \Psi'_1}{\partial y'} \frac{\partial \eta'_0}{\partial x'} = \frac{\partial \Psi'_{21}}{\partial x'} \quad (25-2)$$

$$\frac{\partial \eta'_2}{\partial t'} + u'_0 \frac{\partial \eta'_2}{\partial x'} - \frac{\partial \Psi'_1}{\partial y'} \frac{\partial \eta'_1}{\partial x'} = \frac{\partial \Psi'_{22}}{\partial x'} \quad (25-3)$$

From (22), at $y'=0$

$$\frac{\partial \beta'_0}{\partial t'} + u'_0 \frac{\partial \beta'_0}{\partial x'} - \frac{\partial \Psi'_{20}}{\partial x'} = 0 \quad (26-1)$$

$$\begin{aligned} \frac{\partial \beta'_1}{\partial t'} + u'_0 \frac{\partial \beta'_1}{\partial x'} - \frac{\partial \Psi'_1}{\partial y'} \frac{\partial \beta'_0}{\partial x'} - \gamma \frac{\partial^2 \Psi'_1}{\partial y' \partial x'} \frac{\partial \Psi'_{20}}{\partial y'} - \gamma u'_0 \frac{\partial \Psi'_{20}}{\partial y'} \frac{\partial^2 \Psi'_1}{\partial x' \partial y'} \\ - \frac{\partial \Psi'_{21}}{\partial x'} = 0 \end{aligned} \quad (26-2)$$

$$\begin{aligned} \frac{\partial \beta'_2}{\partial t'} + u'_0 \frac{\partial \beta'_2}{\partial x'} - \frac{\partial \Psi'_1}{\partial y'} \frac{\partial \beta'_1}{\partial x'} - \gamma \frac{\partial^2 \Psi'_1}{\partial y' \partial t'} \frac{\partial \Psi'_{21}}{\partial y'} - \gamma u'_0 \frac{\partial \Psi'_{21}}{\partial y'} \frac{\partial^2 \Psi'_1}{\partial x' \partial y'} \\ + \gamma \frac{\partial \Psi'_1}{\partial y'} \frac{\partial \Psi'_{20}}{\partial y'} - \frac{\partial \Psi'_{22}}{\partial x'} = 0 \end{aligned} \quad (26-3)$$

Besides these perturbed equations, we need equation of motion in x (or y) direction to determine β' , and we may use the equation of motion in x direction. Its perturbed form is

$$-\frac{\partial^2 \Psi'_{20}}{\partial t' \partial y'} - u'_0 \frac{\partial^2 \Psi'_{20}}{\partial x' \partial y'} + \frac{1}{\gamma} \frac{\partial \beta'_0}{\partial x'} = 0 \quad (27-1)$$

$$\begin{aligned} -\frac{\partial^2 \Psi'_1}{\partial t' \partial y'} - \frac{\partial^2 \Psi'_{21}}{\partial t' \partial y'} - u'_0 \frac{\partial^2 \Psi'_1}{\partial x' \partial y'} + \frac{\partial \Psi'_{20}}{\partial y'} \frac{\partial^2 \Psi'_1}{\partial x' \partial y'} - u'_0 \frac{\partial^2 \Psi'_{21}}{\partial x' \partial y'} \\ + \frac{\partial \Psi'_1}{\partial y'} \frac{\partial^2 \Psi'_{20}}{\partial x' \partial y'} - \frac{\partial \Psi'_{20}}{\partial x'} \frac{\partial^2 \Psi'_1}{\partial y'^2} - \frac{\partial \Psi'_1}{\partial x'} \frac{\partial^2 \Psi'_{20}}{\partial y'^2} + \frac{1}{\gamma} \frac{\partial \beta'_1}{\partial x'} = 0 \end{aligned} \quad (27-2)$$

$$\begin{aligned} -\frac{\partial^2 \Psi'_{22}}{\partial t' \partial y'} + \frac{\partial \Psi'_1}{\partial y'} \frac{\partial^2 \Psi'_1}{\partial x' \partial y'} + \frac{\partial \Psi'_{21}}{\partial y'} \frac{\partial^2 \Psi'_1}{\partial x' \partial y'} - u'_0 \frac{\partial^2 \Psi'_{22}}{\partial x' \partial y'} + \frac{\partial \Psi'_1}{\partial y'} \frac{\partial^2 \Psi'_{21}}{\partial x' \partial y'} \\ - \frac{\partial \Psi'_1}{\partial x'} \frac{\partial^2 \Psi'_1}{\partial y'^2} - \frac{\partial \Psi'_{21}}{\partial x'} \frac{\partial^2 \Psi'_1}{\partial y'^2} - \frac{\partial \Psi'_1}{\partial x'} \frac{\partial^2 \Psi'_{21}}{\partial y'^2} + \frac{1}{\gamma} \frac{\partial \beta'_2}{\partial x'} = 0 \end{aligned} \quad (27-3)$$

By making use of (24), (25), (26), (27), we may proceed to higher order approximation of Ψ'_2 for given Ψ'_1 .

THE SURFACE WAVE IN A TWO-DIMENSIONAL
VORTEX LAYER

THE FIRST AND SECOND APPROXIMATION

(1) The first approximation

We put Ψ'_{20} , η'_0 , ρ'_0 as follows.

$$\Psi'_{20} = \varphi'_{20}(\mathcal{Y}') e^{i2\pi(x'-t')} \quad (28-1)$$

$$\eta'_0 = A'_0 e^{i2\pi(x'-t')} \quad (28-2)$$

$$\rho'_0 = D'_0(\mathcal{Y}') e^{i2\pi(x'-t')} - \mathcal{Y}' \quad (28-3)$$

From (28-1)

$$\nabla^2 \Psi'_{20} = \left\{ -4\pi^2 \varphi'_{20}(\mathcal{Y}') + \frac{\partial^2 \varphi'_{20}(\mathcal{Y}')}{\partial \mathcal{Y}'^2} \right\} e^{i2\pi(x'-t')}$$

Inserting this into (24-1)

$$\left\{ -4\pi^2 \varphi'_{20}(\mathcal{Y}') + \frac{\partial^2 \varphi'_{20}(\mathcal{Y}')}{\partial \mathcal{Y}'^2} \right\} (u'_0 - 1) = 0$$

In general, $u'_0 - 1$ is not zero, and so

$$\frac{\partial^2 \varphi'_{20}(\mathcal{Y}')}{\partial \mathcal{Y}'^2} - 4\pi^2 \varphi'_{20}(\mathcal{Y}') = 0 \quad (29)$$

Here $\Psi'_{20} \rightarrow 0$ is physically demanded at $\mathcal{Y}' \rightarrow -\infty$.
From (29)

$$\varphi'_{20}(\mathcal{Y}') = B'_{20} e^{2\pi \mathcal{Y}'} \quad (30)$$

From (29) $\nabla^2 \Psi'_{20} = 0$. This means the motion given by Ψ'_{20} is irrotational. Then inserting (28-2) and (30) into surface condition (25-1)

$$B'_{20} = A'_0 (u'_0 - 1) \quad (31)$$

$D'_0(\mathcal{Y}')$ in (28-3) may be determined by (27-1)

$$D'_0(\mathcal{Y}') = 2\pi \gamma A'_0 (u'_0 - 1)^2 e^{2\pi \mathcal{Y}'} \quad (32)$$

γ is determined by the surface dynamical condition (26-1).

$$\gamma = \frac{1}{2\pi(u'_0 - 1)^2} \quad (33)$$

By these computations, (28-1), (28-2), (28-3) may be rewritten

$$\Psi'_{20} = A'_0 (u'_0 - 1) e^{2\pi \mathcal{Y}' + i2\pi(x'-t')} \quad (34-1)$$

COASTAL ENGINEERING

$$\eta'_0 = A'_0 e^{i2\pi(x'-t')} \quad (34-2)$$

$$p'_0 = A'_0 e^{2\pi y' + i2\pi(x'-t') - y'} \quad (34-3)$$

We use the real part of these expressions.

(2) The second approximation

We select the simplest expression of Ψ_1 , which satisfies both the vorticity equation (18) and surface conditions (6), (7). Naming such Ψ_1 as Ψ_{10} , we put its representative component as follows.

$$\Psi_{10} = -\frac{U_{100}}{k_1} \cos k_3(x - u_0 t) \sin k_3 y \quad (35)$$

Here $k_1 = \frac{2\pi}{L_1}$, $k_3 = \frac{2\pi}{L_3}$, $\frac{L_1}{L_2} = \alpha$, $\frac{L_3}{L_2} = \alpha'$

and so

$$u_{10} = \frac{U_{100}}{k_1} k_3 \cos k_3(x - u_0 t) \cos k_3 y$$

$$v_{10} = \frac{U_{100}}{k_1} k_3 \sin k_3(x - u_0 t) \sin k_3 y$$

$$\zeta_{10} = \frac{\partial v_{10}}{\partial x} - \frac{\partial u_{10}}{\partial y} = \frac{2U_{100}}{k_1} k_3^2 \cos k_3(x - u_0 t) \sin k_3 y$$

Therefore Ψ_{10} satisfies (18).

Ψ_{10} is transformed to

$$\Psi_{10} = -\frac{C_2 L_1 U'_{100}}{2\pi} \cos \frac{2\pi}{\frac{L_3}{L_2}}(x' - u'_0 t') \sin \frac{2\pi}{\frac{L_3}{L_2}} y'$$

and its dimensionless form is

$$\Psi'_{10} = -\frac{U'_{100}}{2\pi} \cos \frac{2\pi}{\alpha'}(x' - u'_0 t') \sin \frac{2\pi}{\alpha'} y'$$

$$= -\frac{U'_{100}}{2\pi} \cos k'(x' - u'_0 t') \sin k' y'$$

$$\left(\frac{2\pi}{\alpha'} = k' \right)$$

The general expression of Ψ'_{10} is

$$\begin{aligned} \Psi'_{10} &= \int_{-\infty}^{\infty} -\frac{dU'_{100}(k')}{8\pi i} (\text{sgn } k') e^{-i k' u'_0 t'} e^{i k'(x'+y')} \\ &+ \int_{-\infty}^{\infty} \frac{dU'_{100}(k')}{8\pi i} (\text{sgn } k') e^{-i k' u'_0 t'} e^{i k'(x'-y')} \end{aligned} \quad (36)$$

THE SURFACE WAVE IN A TWO-DIMENSIONAL
VORTEX LAYER

Here uncorrelated increment $dU'_{100}(k')$ satisfies $dU'_{100}(k') = \overline{dU'_{100}(-k')}$ (bar indicates the complex conjugate), and its argument is related to x' variable. The form of spectrum given by Ψ'_{10} must not be inconsistent with the method of perturbed derivation in the preceding paragraph.

From (36)

$$U'_{10} = -\frac{\partial \Psi'_{10}}{\partial y'} = \int_{-\infty}^{\infty} \frac{dU'_{100}(k')}{8\pi} k'(\text{sgn } k') e^{-ik' u'_0 t'} e^{ik'(x'+y')} + \int_{-\infty}^{\infty} \frac{dU'_{100}(k')}{8\pi} k'(\text{sgn } k') e^{-ik' u'_0 t'} e^{ik'(x'-y')} \quad (37)$$

$$v'_{10} = \frac{\partial \Psi'_{10}}{\partial x'} = \int_{-\infty}^{\infty} -\frac{dU'_{100}(k')}{8\pi} k'(\text{sgn } k') e^{-ik' u'_0 t'} e^{ik'(x'+y')} + \int_{-\infty}^{\infty} \frac{dU'_{100}(k')}{8\pi} k'(\text{sgn } k') e^{-ik' u'_0 t'} e^{ik'(x'-y')} \quad (38)$$

$$\zeta'_{10} = \nabla^2 \Psi'_{10} = \int_{-\infty}^{\infty} \frac{dU'_{100}(k')}{4\pi i} k'^2(\text{sgn } k') e^{-ik' u'_0 t'} e^{ik'(x'+y')} - \int_{-\infty}^{\infty} \frac{dU'_{100}(k')}{4\pi i} k'^2(\text{sgn } k') e^{-ik' u'_0 t'} e^{ik'(x'-y')} \quad (39)$$

(putting $2\pi = k'_0(k'_0 > 0)$)

From (34-1), (34-2) and (34-3)

$$\Psi'_{20} = \frac{A'_0(u'_0 - 1)}{2} e^{k'_0 y'} \left\{ e^{ik'_0(x'-t')} + e^{-ik'_0(x'-t')} \right\} \quad (40)$$

$$\eta'_0 = \frac{A'_0}{2} \left\{ e^{ik'_0(x'-t')} + e^{-ik'_0(x'-t')} \right\} \quad (41)$$

$$\beta'_0 = \frac{A'_0}{2} e^{k'_0 y'} \left\{ e^{ik'_0(x'-t')} + e^{-ik'_0(x'-t')} \right\} - y' \quad (42)$$

By making use of (24-2) and $\nabla^2 \Psi'_{20} = 0$,

$$\frac{\partial}{\partial t'} \nabla^2 \Psi'_{21} + u'_0 \frac{\partial}{\partial x'} \nabla^2 \Psi'_{21} - \frac{\partial}{\partial x'} \nabla^2 \Psi'_{11} \frac{\partial \Psi'_{20}}{\partial y'} + \frac{\partial}{\partial y'} \nabla^2 \Psi'_{11} \frac{\partial \Psi'_{20}}{\partial x'} = 0 \quad (43)$$

Inserting $\nabla^2 \Psi'_{11}$, Ψ'_{20} given by (39), (40) into (43), $\nabla^2 \Psi'_{21}$ ($= \zeta'_{21}$) may be expressed by

$$\zeta'_{21} = \int_{-\infty}^{\infty} \frac{\sqrt{2} A'_0 dU'_{100}(k') (\text{sgn } k')}{8\pi} k'^3 \left[e^{i(\frac{5}{4}\pi + 2n\pi)} e^{(k'_0 + ik') y'} \cdot e^{i(k'_0 + k') x'} e^{-i(k'_0 + k' u'_0) t'} + e^{i(\frac{3}{4}\pi + 2n\pi)} e^{(k'_0 + ik') y'} \right]$$

COASTAL ENGINEERING

$$\begin{aligned}
 & \cdot e^{-i(k'_0 - k')x'} e^{-i(k'u'_0 - k'_0)t'} + e^{i(\frac{3}{4}\pi + 2n\pi)} e^{(k'_0 - ik')y'} \\
 & \cdot e^{i(k'_0 + k')x'} e^{-i(k'_0 + k'u'_0)t'} + e^{i(\frac{5}{4}\pi + 2n\pi)} e^{(k'_0 - ik')y'} \\
 & \cdot e^{-i(k'_0 - k')x'} e^{-i(k'u'_0 - k'_0)t'} \Big]
 \end{aligned} \tag{44}$$

Homogeneous equation $\nabla^2 \psi'_{21} = 0$ has a following form of solution

$$\psi'_{21} = \sum A'_{\beta'}(t) e^{\pm i\beta'x' + \beta'y'} \tag{45}$$

here $R(\beta') > 0$,
The general solution of ψ'_{21} is

$$\begin{aligned}
 \psi'_{21} = & \sum_{R(\beta') > 0} A'_{\beta'}(t) e^{\pm i\beta'x' + \beta'y'} \\
 & + \int_{-\infty}^{\infty} \frac{\sqrt{2} A'_0 dU'_{10}(k') (\text{sgn } k')}{16\pi} k'^2 \left[\frac{1}{ik'_0 - k' - k'_0} e^{i(\frac{5}{4}\pi + 2n\pi)} \right. \\
 & \cdot e^{(k'_0 + ik')y'} e^{-i(k'_0 + k')x'} e^{-i(k'_0 + k'u'_0)t'} + \frac{1}{ik'_0 - k' + k'_0} \\
 & \cdot e^{i(\frac{3}{4}\pi + 2n\pi)} e^{(k'_0 + ik')y'} e^{-i(k'_0 - k')x'} e^{-i(k'u'_0 - k'_0)t'} \\
 & + \frac{1}{-ik'_0 - k' - k'_0} e^{i(\frac{3}{4}\pi + 2n\pi)} e^{(k'_0 - ik')y'} e^{i(k'_0 + k')x'} \\
 & \cdot e^{-i(k'_0 + k'u'_0)t'} + \frac{1}{-ik'_0 - k' + k'_0} e^{i(\frac{5}{4}\pi + 2n\pi)} \\
 & \cdot e^{(k'_0 - ik')y'} e^{-i(k'_0 - k')x'} e^{-i(k'u'_0 - k'_0)t'} \Big] \tag{46}
 \end{aligned}$$

Using (46), p'_i may be expressed from (27-2).

$$\begin{aligned}
 p'_i = & \gamma \sum_{R(\beta') > 0} \left\{ \frac{1}{\pm i} \frac{\partial A'_{\beta'}(t)}{\partial t'} + \beta'u'_0 A'_{\beta'}(t) \right\} e^{\pm i\beta'x' + \beta'y'} \\
 & + \int_{-\infty}^{\infty} \frac{\gamma A'_0 dU'_{10}(k') (\text{sgn } k')}{8\pi} (u'_0 - 1) k' k_0'^2 \left[\frac{-1}{ik'_0 - k' - k'_0} e^{i(k'_0 + k')x'} \right. \\
 & \cdot e^{(k'_0 + ik')y'} e^{-i(k'u'_0 + k'_0)t'} + \frac{1}{ik'_0 - k' + k'_0} e^{-i(k' - k'_0)x'} e^{(k'_0 + ik')y'} \\
 & \cdot e^{-i(k'u'_0 - k'_0)t'} - \frac{1}{-ik'_0 - k' - k'_0} e^{i(k'_0 + k')x'} e^{(k'_0 - ik')y'} e^{-i(k'u'_0 + k'_0)t'} \\
 & + \frac{1}{-ik'_0 - k' + k'_0} e^{i(k' - k'_0)x'} e^{(k'_0 - ik')y'} e^{-i(k'u'_0 - k'_0)t'} \Big] \tag{47}
 \end{aligned}$$

THE SURFACE WAVE IN A TWO-DIMENSIONAL
VORTEX LAYER

$A'_p(t')$ is determined by the surface dynamical condition (26-2), when we insert Ψ'_{21} , P'_1 represented by (46), (47) into (26-2). After some computations, the final forms of Ψ'_{21} , P'_1 are

$$\begin{aligned}
 \Psi'_{21} = & \int_{-\infty}^{\infty} A_1(k') e^{i(k'_0+k')x'} e^{|k'_0+k'|y'} e^{-i(k'u'_0+k'_0)t'} \\
 & + \int_{-\infty}^{\infty} A_2(k') e^{i(k'-k'_0)x'} e^{|k'-k'_0|y'} e^{-i(k'u'_0-k'_0)t'} + \int_{-\infty}^{\infty} \frac{A'_0 du'_{10}(k')(\text{sgn } k')}{16\pi} k'^2 \\
 & \cdot \left[\frac{-1-i}{ik'_0-k'-k'_0} e^{i(k'_0+k')x'} e^{(k'_0+i k')y'} e^{-i(k'_0+k'u'_0)t'} + \frac{-1+i}{ik'_0-k'+k'_0} \right. \\
 & \cdot e^{i(k'-k'_0)x'} e^{(k'_0+i k')y'} e^{-i(k'u'_0-k'_0)t'} + \frac{-1+i}{-ik'_0-k'-k'_0} e^{i(k'_0+k')x'} \\
 & \cdot e^{(k'_0-i k')y'} e^{-i(k'_0+k'u'_0)t'} + \frac{-1-i}{-ik'_0-k'+k'_0} e^{i(k'-k'_0)x'} e^{(k'_0-i k')y'} \\
 & \left. \cdot e^{-i(k'u'_0-k'_0)t'} \right] \tag{48}
 \end{aligned}$$

$$\begin{aligned}
 P'_1 = & -\gamma \int_{-\infty}^{\infty} \left\{ \{\text{sgn}(k'_0+k')\} (k'u'_0+k'_0) - |k'_0+k'|u'_0 \right\} A_1(k') \\
 & \cdot e^{i(k'_0+k')x'} e^{|k'_0+k'|y'} e^{-i(k'u'_0+k'_0)t'} \\
 & - \gamma \int_{-\infty}^{\infty} \left\{ \{\text{sgn}(k'-k'_0)\} (k'u'_0-k'_0) - |k'-k'_0|u'_0 \right\} A_2(k') e^{i(k'-k'_0)x'} \\
 & \cdot e^{|k'-k'_0|y'} e^{-i(k'u'_0-k'_0)t'} + \int_{-\infty}^{\infty} \gamma \frac{A'_0 du'_{10}(k')(\text{sgn } k')}{8\pi} (u'_0-1) k'^2 k'^2 \\
 & \cdot \left[\frac{-1}{ik'_0-k'-k'_0} e^{i(k'_0+k')x'} e^{(k'_0+i k')y'} e^{-i(k'u'_0+k'_0)t'} \right. \\
 & + \frac{1}{ik'_0-k'+k'_0} e^{i(k'-k'_0)x'} e^{(k'_0+i k')y'} e^{-i(k'u'_0-k'_0)t'} - \frac{1}{-ik'_0-k'-k'_0} \\
 & \cdot e^{i(k'_0+k')x'} e^{(k'_0-i k')y'} e^{-i(k'u'_0+k'_0)t'} + \frac{1}{-ik'_0-k'+k'_0} \\
 & \left. \cdot e^{i(k'-k'_0)x'} e^{(k'_0-i k')y'} e^{-i(k'u'_0-k'_0)t'} \right] \tag{49}
 \end{aligned}$$

COASTAL ENGINEERING

Here

$$A_1(k') = - \frac{A'_0 d u'_{100}(k') (\text{sgn } k')}{8 \pi} \frac{2 \delta (u'_0 - 1)^2 k' k'_0 (k' + k'_0) + k' (2 k'_0{}^3 + 2 k'_0{}^2 k' - k'^3)}{(2 k'_0{}^2 + 2 k'_0 k' + k'^2) \left[(\text{sgn}(k' + k'_0)) \frac{k'_0{}^2}{2\pi} - (k'_0 + k') \right]} \quad (50)$$

$$A_2(k') = - \frac{A'_0 d u'_{100}(k') (\text{sgn } k')}{8 \pi} \frac{2 \delta (u'_0 - 1)^2 k' k'_0 (k' - k'_0) - k' (2 k'_0{}^3 - 2 k'_0 k'^2 + k'^3)}{(2 k'_0{}^2 - 2 k'_0 k' + k'^2) \left[(\text{sgn}(k' - k'_0)) \frac{k'_0{}^2}{2\pi} - (k' - k'_0) \right]} \quad (51)$$

The first and the second integrals of (48) indicate the irrotational surface waves which are determined by the surface dynamical condition (26-2), and their celerities are quite different from those derived by usual surface dynamical conditions.

Inserting (48) into (25-2),

$$\begin{aligned} \eta'_1 = & \int_{-\infty}^{\infty} \left[\frac{A'_0 d u'_{100}(k')}{k'_0 (u'_0 - 1) 8 \pi} (\text{sgn } k') k' \frac{k'^3 - 2 k' k'_0{}^2 - 2 k'_0{}^3}{k'^2 + 2 k' k'_0 + 2 k'_0{}^2} \right. \\ & \left. + \frac{A_1(k') (k'_0 + k')}{k' (u'_0 - 1)} \right] e^{i(k'_0 + k') x'} e^{-i(k' u'_0 + k'_0) t'} \\ & - \int_{-\infty}^{\infty} \left[\frac{A'_0 d u'_{100}(k')}{k'_0 (u'_0 - 1) 8 \pi} (\text{sgn } k') k' \frac{k'^3 - 2 k' k'_0{}^2 + 2 k'_0{}^3}{2 k'_0{}^2 - 2 k'_0 k' + k'^2} + \frac{A_2(k') (k' - k'_0)}{k'_0 (u'_0 - 1)} \right] \\ & \cdot e^{i(k' - k'_0) x'} e^{-i(k' u'_0 - k'_0) t'} \end{aligned} \quad (52)$$

The expansion of η'_1 in (23) indicates that $\alpha \eta'_1$ may be considered as the approximate expression of surface irregularity of wave profile due to the interaction with vortex motion.

NUMERICAL COMPUTATION OF η'_1

In the two integrals of (52), the first terms show the surface irregularities caused by the direct interaction between primary wave and the vortex motion. The second terms are contributed by the second order dynamical condition of surface. Of course they are closely connected with each other. Stillmore $|d u'_{100}(k')|$ should be sufficiently small when $|k'|$ is large, not to disturb the method of perturbation.

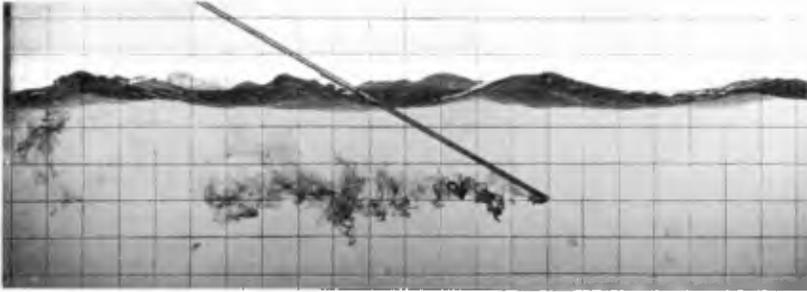
To know the essential property of η'_1 in (52), numerical computations are shown in Table-1. We summarize the first term in the first integral of (52) as $A'_0 d u'_{100}(k') m_1$, the second term in the same integral as $A'_0 d u'_{100}(k') m_2$, the first term in the second integral of (52) as $A'_0 d u'_{100}(k') n_1$, and

THE SURFACE WAVE IN A TWO-DIMENSIONAL
VORTEX LAYER

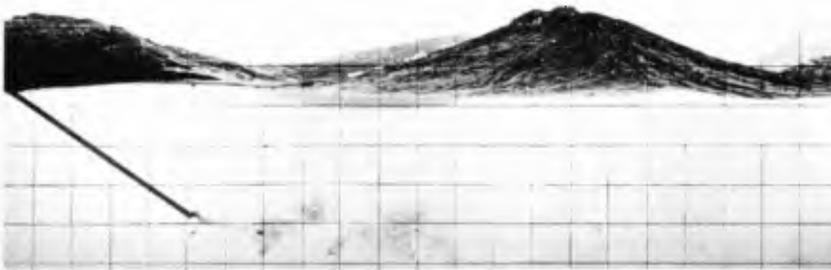
Table 1

u_0'	k'	m_1	m_2	m_1+m_2	$\frac{m_1}{m_2}$	n_1	n_2	n_1+n_2	$\frac{n_1}{n_2}$
	4π	-0.14286	-0.42857	-0.57143	+0.33333	-2.1429	∞	∞	0
	6π	-1.19748	+0.92437	-0.27311	-1.2955	-4.9286	+8.1429	+3.2143	-0.60526
	10π	-5.45367	+5.84942	+0.39575	-0.93234	-12.2899	+15.2661	+2.9762	-0.80504
	20π	-28.6300	+30.7844	+2.1544	-0.93000	-42.7700	+47.2343	+4.4643	-0.90549
	100π	-857.156	+873.586	+16.430	-0.98118	-928.556	+947.161	+18.605	-0.98036
+0.3	200π	-3500.01	+3534.31	+34.30	-0.99035	-3642.84	+3679.29	+36.45	-0.99010
	-4π	+2.1429	∞	∞	0	+0.14286	+0.42857	+0.57143	+0.33333
	-6π	+4.9286	-8.1429	-3.2143	-0.60526	+1.19748	-0.92437	+0.27311	-1.2955
	-10π	+12.2899	-15.2661	-2.9762	-0.80504	+5.45367	-5.84942	-0.39575	-0.93235
	-20π	+42.7700	-47.2343	-4.4643	-0.90549	+28.6300	-30.7844	-2.1544	-0.93002
	-100π	+928.556	-947.161	-18.605	-0.98036	+857.156	-873.586	-16.430	-0.98119
	-200π	+3642.84	-3679.29	-36.45	-0.99010	+3500.01	-3534.31	-34.30	-0.99030
	4π	-0.07692	-0.23077	-0.30769	+0.33333	-1.5385	∞	∞	0
	6π	-0.64480	+0.49774	-0.14706	-1.2955	-2.65385	+4.38462	+1.7308	-0.60526
	10π	-2.93659	+3.14968	+0.21309	-0.93234	-6.61765	+8.22023	+1.6026	-0.80504
	20π	-15.4162	+16.5762	+1.1600	-0.93000	-23.0300	+25.4338	+2.4038	-0.90549
	100π	-461.546	+470.392	+8.846	-0.98117	-499.992	+510.010	+10.018	-0.98036
-0.3	200π	-1884.63	+1903.09	+18.46	-0.99030	-1961.53	+1981.15	+19.62	-0.99010
	-4π	+1.15385	∞	∞	0	+0.07692	+0.23077	+0.30769	+0.33333
	-6π	+2.65385	-4.38462	-1.7308	-0.60526	+0.64480	-0.49774	+0.14706	-1.2955
	-10π	+6.61765	-8.22023	-1.6026	-0.80504	+2.93659	-3.14968	-0.21309	-0.93235
	-20π	+23.0300	-25.4338	-2.4038	-0.90549	+15.4162	-16.5762	-1.1600	-0.93002
	-100π	+499.992	-510.010	-10.018	-0.98036	+461.546	-470.392	-8.846	-0.98119
	-200π	+1961.53	-1981.15	-19.62	-0.99010	+1884.63	-1903.09	-18.46	-0.99030
	4π	-0.1	-0.3	-0.4	+0.33333	-1.5	∞	∞	0
	6π	-0.83824	+0.64706	-0.19118	-1.2955	-3.45	+5.70	+2.25	-0.60526
	10π	-3.8176	+4.0946	+0.2770	-0.93235	-8.6029	+10.6863	+2.0834	-0.80504
	20π	-20.041	+21.549	+1.508	-0.93002	-29.939	+33.064	+3.125	-0.90549
	100π	-600.01	+611.51	+11.50	-0.98119	-649.99	+663.01	+13.02	-0.98036
$u_0'=0$	200π	-2450.0	+2474.0	+24.0	-0.99030	-2550.0	+2575.5	+25.50	-0.99010
	-4π	+1.5	∞	∞	0	+0.1	+0.3	+0.4	+0.33333
	-6π	+3.45	-5.70	-2.25	-0.60526	+0.83824	-0.64706	+0.19118	-1.2955
	-10π	+8.6029	-10.6863	-2.0834	-0.80504	+3.8176	-4.0946	-0.2770	-0.93235
	-20π	+29.939	-33.064	-3.125	-0.90549	+20.041	-21.549	-1.5080	-0.93002
	-100π	+649.989	-663.013	-13.024	-0.98036	+600.01	-611.51	-11.50	-0.98119
	-200π	+2550.0	-2575.5	-25.50	-0.99010	+2450.0	-2474.0	-24.0	-0.99030

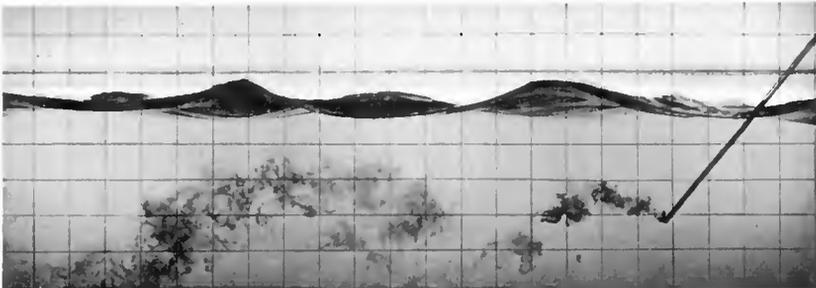
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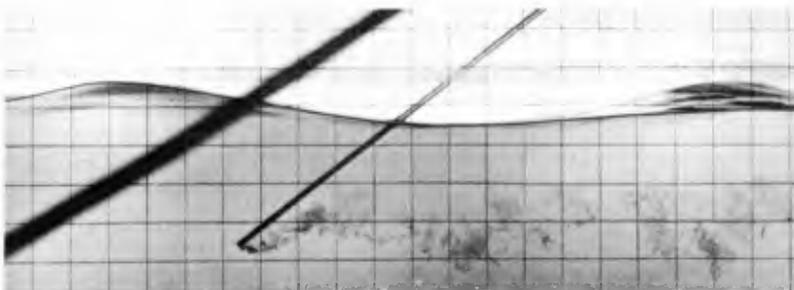
Wind wave in a turbulent flow under direct wind action.
Photo 1



Wind wave in a turbulent flow under direct wind action.
Photo 2



Wind generated wave in a turbulent flow under calm air.
Photo 3



Wind generated wave in a turbulent flow under calm air.
Photo 4

THE SURFACE WAVE IN A TWO-DIMENSIONAL VORTEX LAYER

the second term in the same integral as $A'_0 du'_{100}(k') n_2$. In the table, the numerical values of m_1 , m_2 , n_1 , and n_2 for k' ($|k'| > k'_0$) are shown. We take the numerical value of u'_0 as 0.3, 0, -0.3, and this selection may cover the proper range of u'_0 . The table shows η'_1 is a real process and m_2 , n_2 each offsets m_1 , n_1 except cases of $k' = 4\pi, -4\pi$. Referring to the expression ψ'_{21} in (48), $k' = 4\pi (= 2k'_0)$ and $-4\pi (= -2k'_0)$ are very special cases, and we follow the numerical cases of $|k'| \geq 6\pi$.

PHOTOGRAPHIC EXAMPLES

From the above-mentioned computations we generally expect that surface waves have almost smooth surface even if they coexist with the vortex motion of moderate strength. This is caused by the condition of constant pressure at the direct wave surface. If the surface pressure is disturbed by wind in the different condition, the surface profile of waves may be also different. To examine this problem experimentally, the surface profiles of wind generated waves in turbulent flows are photographed. Photo-1, 2, 3, 4 are their examples. Photo-1, 2 show the waves under direct wind action. (Averaged wind velocity 1,100cm/sec.) In Photo-1, the direction of water current consists with the direction of wave propagation and in Photo-2, they are opposite each other. We can observe surface irregularities. To understand properly these irregularities, of course, the term including the effect of surface tension should be considered in the surface dynamical condition. Photo-3, 4 show the waves under calm air. In Photo-3, the direction of water current coincides with the direction of wave propagation and in Photo-4 they are inverse. The velocity of water flow is from 13cm/sec to 26cm/sec, and its turbulent condition is shown by dye injected to water. The surfaces of waves are very smooth.

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