CHAPTER 22

RUNUP RECIPE FOR PERIODIC WAVES ON UNIFORMLY SLOPING BEACHES

by

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ABSTRACT

The shoaling enhancement of small-amplitude, dispersive wave trains traveling over uniform, impermeable slopes was observed in a specially-constructed wave channel, where the reproducible wave elevation measurement accuracy was about .0005-in. These observations substantially support the enhancement predicted from linear theory (conservation of energy flux) except in very shallow water and on very steep slopes, where accelerative effects become important.

On the hypothesis that small-amplitude runup theory might be similarly valid for periodic waves of finite height, provided that the positive incident wave amplitude is replaced by the local crest height above still water, this theory was modified to include the effect of the superelevation under a wave crest due to profile asymmetry. The modified theory is shown to agree acceptably with runup observations of larger waves previously reported - both for breaking and non-breaking waves.

Because solutions to the modified theory cannot conveniently be obtained by manual calculation, a nomograph chart is included, from which runup predictions can be easily made, given only the wave height, period, and water depth a wavelength or so from shore, and the beach slope.

INTRODUCTION

This paper describes a theoretical and experimental investigation of the mechanism of waterwave enhancement in shoaling water up to the point of maximum forward excursion (runup) on beaches of arbitrary slope.[2] Only the case of wave propagation normal to

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[2]This paper is a substantial abridgement of a contract report. Readers are referred to Reference 1 for a more detailed discussion.

shore is considered. The offshore slope is assumed to be uniform and impermeable, and the incident wave characteristics known at some point far enough offshore so that linear wave theory applies.

This investigation might best be viewed as an attempt to bridge the gulf between the small-amplitude runup theory and the results of numerous experiments with waves of finite amplitude; first, by conducting a very careful series of experiments with small-amplitude waves in a domain wherein the boundary conditions assumed by theory can be reasonably justified, and, second, by attempting to re-interpret previous large-amplitude experimental results in terms of small-amplitude theory with appropriate corrections. The results so-obtained might better be regarded as 'recipes' rather than general mathematical solutions to the runup problem. Nevertheless, the results appear to be closer to reality than those heretofore obtained, and should provide a basis for further theoretical work.

WAVE ENHANCEMENT EXPERIMENTS

Experimental justification for the validity of the small-amplitude theory for wave enhancement was provided by a series of precise experiments with impulsively-generated, dispersive wave systems of very small amplitude, propagating in a rectangular channel (length 90', width 16", depth 14"), and incident upon smooth plate-glass slopes within the range $1/32 < S < 1/4$. The wave trains were generated at one end of the channel by the rapid immersion and withdrawal of a cylindrical wave generator having a parabolic bottom profile. Wave amplitudes within the range $0 < \eta < \pm \frac{1}{4}$-in. were measured to an accuracy of $\pm$.0005-in. in the uniform-depth section of the channel and at several positions over the slopes by sensitive electrical strain-gage transducers.[3]

The results of these experiments indicated that the enhancement observed over uniform impermeable slopes is adequately predicted by the linear theory of geometric optics to within a half-wavelength or so of the breaking point. It was also shown that, within the non-breaking region, a fairly complicated wave spectrum offshore can be resolved into discreet Fourier components, each of which can be propagated independently over the slope to obtain the time history of the elevation change at any point.

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[3]Although it was initially intended to measure runup in these experiments, this was found to be impossible because surfactant added to the water to reduce surface tension also produced a contaminating surface film that effectively dissipated all waves of interest before they reached the shore.

Figure 1 shows a typical example of the history of such a wave train over a 1/32 slope. The upper left curve in this figure shows the train at the toe of the slope in water depth d = 12-in. at a distance of 27-ft from the generator. The three succeeding curves show the wave train as observed at places where the local depths were 6-in, 2-in, and 0.4-in, respectively. The solid- and dashed-line envelopes drawn around these latter wave trains show the enhancement computed from the theory of geometric optics with and without dissipation corrections for viscous boundary dissipation on the sides, bottom and free-surface, respectively. The general agreement between the solid curves and the observed wave trains confirms both the enhancement theory and the dissipative corrections.

RUNUP HYPOTHESIS

The fact that the geometric theory works quite well even when the wave amplitude is fairly large and the water depth quite small implies that the group velocity is relatively insensitive to substantial variations in the relative wave height. It was therefore hypothesized that the linear standing-wave theory for wave runup should be similarly valid for waves of finite height, if one took into account the local superelevation under the crests of waves in shallow water. To test this hypothesis, an extensive series of data on runup behavior of periodic waves obtained experimentally by Savage (Ref 2) was reanalyzed according to the following procedure:

1) The linear shallow water theory (Ref 3) conventionally gives the vertical extent of the runup R as the product of the incident wave amplitude offshore H/2, an enhancement factor A, and a slope factor $(2\pi/\alpha)^{\frac{1}{2}}$:

$$R = \frac{H}{2} A(2\pi/\alpha)^{\frac{1}{2}} \tag{1}$$

Where α is the beach slope in radians. The enhancement factor A depends upon the bottom profile and the wave period T. For the case of a horizontal bottom of depth d terminating in an upward slope, the runup has the implicit form

$$R = H \left[J_0^2 (2\gamma/\alpha) + J_1^2 (2\gamma/\alpha)\right]^{-\frac{1}{2}} \tag{2}$$

Where $\gamma = 2\pi/T(g/d)^{\frac{1}{2}}$ is the dimensionless wave frequency. Comparison of (2) with Savage's original data indicates good agreement for the runup of the smallest waves observed ($H \simeq 0.1$ d), but with increasing error as the amplitudes became larger. It was thus suspected that this disagreement might be due to increasing asymmetry of the larger waves, and that the small-amplitude theory might still give acceptable results if this asymmetry could be accounted for. The simpliest hypothesis is that the effective crest amplitude responsible for the runup is no longer equal to H/2, but instead is just the vertical crest elevation above the still water level. This is tantamount to the inclusion of an additional factor in (2): $\Delta = (1 + \delta H/H)$, and δH is the wave superelevation given by (Ref 1)

$$\delta H_s = \frac{H^2}{8d} \frac{\sigma}{\tanh \sigma} \cdot [1 + 3/2 \sinh^2 \sigma] \qquad U \leq 100 \tag{3a}$$

for Stokes waves, or by

$$\delta H_c = (2 + H/d) \left[\frac{1}{k^2} (1 - E/K) - \tfrac{1}{2}\right] \qquad U > 100 \tag{3b}$$

for cnoidal[4] waves, where K(k) and E(k) are the first and second complete elliptic integrals of modulus k respectively, and $\sigma = 2\pi d/\lambda$ is the dimensionless number (λ = wavelength). The distinction between these types of waves is given by the local value of the Ursell parameter, $U = (H/d)\,(2\pi/\sigma)^2$.

2) In a small-amplitude theory, the runup is limited by instability to the range governed by the inequality $HA/2d \leq (\alpha/\gamma)^2$ which, upon introduction of the superelevation factor, becomes

$$\Delta \frac{R}{d} \leq (\alpha/\gamma)^2 \tag{4}$$

3) For breaking waves, wherein the stability criterion is unsatisfied, the runup appears to be adequately given by the existing empirical relation (Ref 4)

$$\frac{R_o}{H_o} = (L_o/H_o)^{\frac{1}{2}} \tan \alpha \div (2\pi d/H_o)^{\frac{1}{2}} (\alpha/\gamma) \tag{5}$$

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[4]The factor in square brackets is the conventional expression for the crest superelevation of a small-amplitude (first order) cnoidal wave. The factor (2 + H/d) is proposed in Ref 1 as a useful engineering formula that fits a higher approximation to solitary wave theory.

where H_o and L_o are the wave height and length in deep water, and we have substituted α for tan α to avoid implying infinite runup for infinite slope. In terms of the incident progressive wave height H in depth d, and assuming conservation of energy flux for a wave propagating from deep water to a region of uniform depth, the runup for breaking waves can be written

$$\frac{R}{H} = (2\pi d/H)^{\frac{1}{2}} (H/H_o)^{\frac{1}{2}} (\alpha/\gamma)$$

$$= (2\pi d/H)^{\frac{1}{2}} (\alpha/\gamma) [\tanh \sigma (1 + 2\sigma/\sinh 2\sigma)]^{\frac{1}{4}} \qquad (6)$$

The fact that so many equations and conditions are required to describe the runup for waves of finite height is a testimonial of the inadequacy of present theory to describe the transformation of waves in shoaling water. What these equations say, in principle, is that as periodic waves advance from deep water into a horizontal region of finite depth, the change in amplitude is given by the theory of geometric optics assuming no reflections. In the constant-depth regime one may have either sinusoidal Stokes waves of second or higher order, or cnoidal waves, depending upon the value of the Ursell parameter. The runup, in our hypothesis, will differ because the superelevation is different for each of these classes of waves. Figure 2 is a plot of the parameter Δ versus Ursell number for various local values of the ratio H/d. The figure is divided into two regions by the vertical line U = 100. The region to the left of this line is occupied by Stokes waves, while that to the right applies to cnoidal waves[5]. It is apparent, from consideration of these curves, that as a periodic wave moves into shoaling water the Ursell parameter will progressively increase, and an individual wave will trace out a trajectory in the Ursell diagram, progressively crossing lines of increasing H/d as the local wave height and Ursell parameter increase. The heavy dashed lines in this figure correspond to two such trajectories. The upper curve (a), having an initially higher amplitude in deep water, never escapes from the Stokes region, but will break as the crest elevation increases to about 75% of the local water depth. The lower curve (b) is initially of such small amplitude that it crosses into the cnoidal wave

[5] In his analysis of higher-order cnoidal waves (Ref 5) Laitone postulates that this dividing line should occur at about U = 48. From the behavior of the functions shown, however, the value U = 100 appears more reasonable.

region before breaking. Perhaps significantly, no matter how small the offshore amplitude is a wave that is initially sinusoidal can never reach the asymptotic limit of cnoidal waves and become a solitary wave, and such waves must be regarded as a laboratory curiosity that can only be generated by the net addition of fluid to the region. It is also significant that in the Stokes region the factor Δ is relatively constant as the Ursell parameter increases, but once in the cnoidal wave region the superelevation increases very rapidly, which explains the often-observed very rapid change in wave elevation as a wave moves into shallower water.

Figure 3 illustrates in a general way the striking difference between the runup characteristics of non-breaking - as opposed to breaking waves. The ascending curve is a plot of equation (2) showing the relative runup R/H for small-amplitude non-breaking waves as a function of the dimensionless frequency/beach angle ratio (γ/α). For small values of this ratio (high frequencies and steep slopes), optical reflection occurs, while for larger values the relative runup increases as $(\gamma/\alpha)^{\frac{1}{2}}$. According to this hypothesis, the relative runup for waves of finite height is obtained by multiplying the ordinate values for the small-amplitude theory by the factor Δ, which includes the effect of crest superelevation. Since Δ is a function of the relative wave height H/d, the runup will be different for each value of H/d.

The descending curves H_o/d = constant give the relative runup R/H_o for breaking waves (Equation 5). Strictly speaking, the change in the ordinate scale from R/H to R/H_o precludes presentation of these curves in the same figure, but the ratio H/H_o is not large except at relatively low-frequencies, and this figure is therefore useful for illustrating the general behavior of these functions, interpreted as follows. The relative runup tends to increase with increasing values of (γ/α) up to the point where the instability limit given by equation 4 results in wave breaking, beyond which point relative runup decreases rapidly in proportion to $(\gamma/\alpha)^{-1}$. The point of breaking instability is again governed by the local value of the ratio H/d. It is apparent from this figure that the runup can (in principle) be very large for waves of very small steepness over very small slopes, but in this case the runup will be physically limited by dissipative processes which become very large for small slopes. Some tendency towards this limiting condition is exhibited by the abnormally large runup of tsunamis on gradual continental slopes.

COMPARISON WITH EXPERIMENTS

As a test of this runup hypothesis, equations 2-6 were programmed for computer computation and the results compared with some 254 individual runup observations reported by Savage. These results are shown by the normal regression curves of Figure 4 (non-breaking waves and 5 (breaking waves). In these figures the observed runup R was compared with that computed by the above method R*, separately for each slope tested. The degree of correlation is given by the closeness of fit of the computed ratios to the 45° line drawn in each figure. The overall RMS error for all cases is less than 12%, which is of the order of accuracy reported in these experiments, and contrasts to errors as large as 400% for individual data when the superelevation is neglected. Since the above experiments covered a range of 9 slopes, 12 frequencies, 4 water depths, and a wide variety of incident wave heights, the runup hypothesis appears to be adequate for most prediction purposes. There is always the possibility, however, that laboratory experiments may incorporate scale effects not observed in prototype conditions, and verification of this runup model must await application to prototype observations.

NOMOGRAPH FOR WAVE PREDICTION

Because the above system of equations is inconvenient to solve by hand computation methods, Figures 6 and 7 comprise a set of nomographic diagrams which can be entered with the offshore wave height, period, water depth, and beach slope as independent determinable variables, and the runup rapidly determined by graphical interpolation for any particular case of interest. These diagrams are included here for illustrative purposes only, and are too small to be of practical use. A reproduction on a much larger scale is included in Reference 1.

REFERENCES

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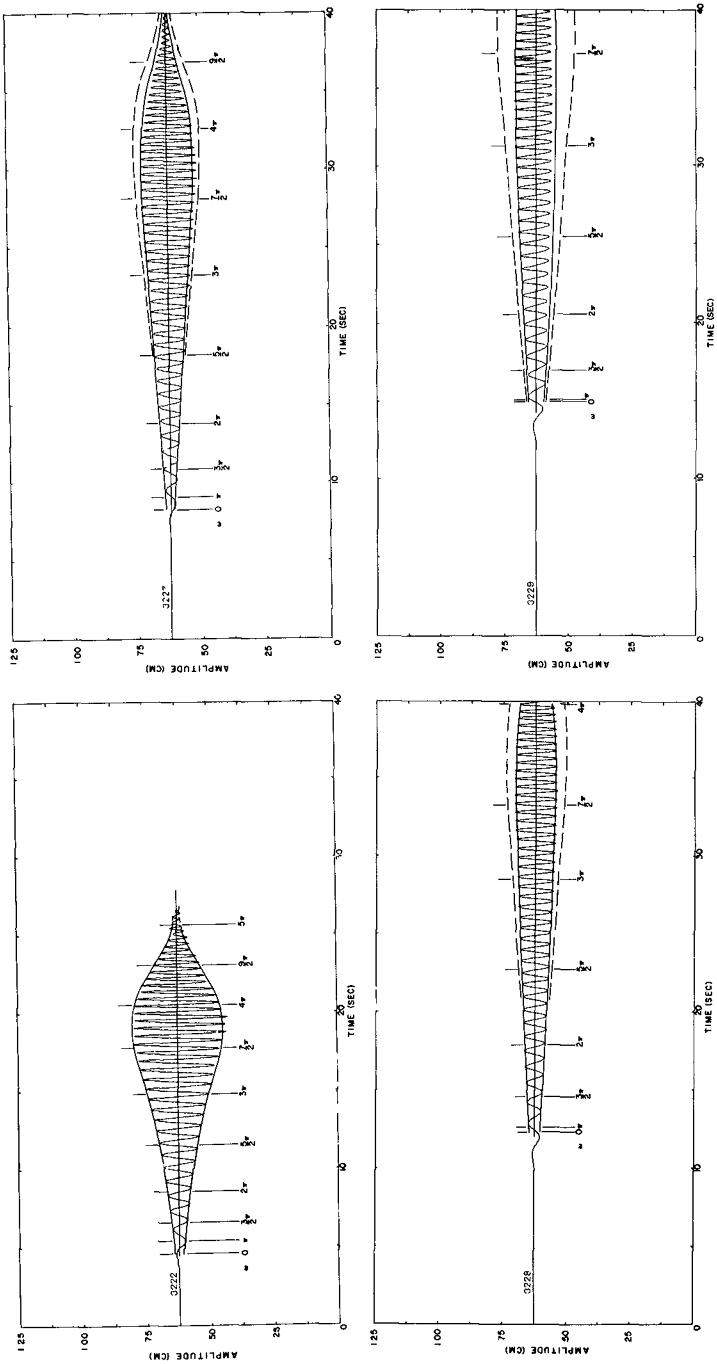

Fig. 1. Dispersive wave train advancing over 1/32 slope.

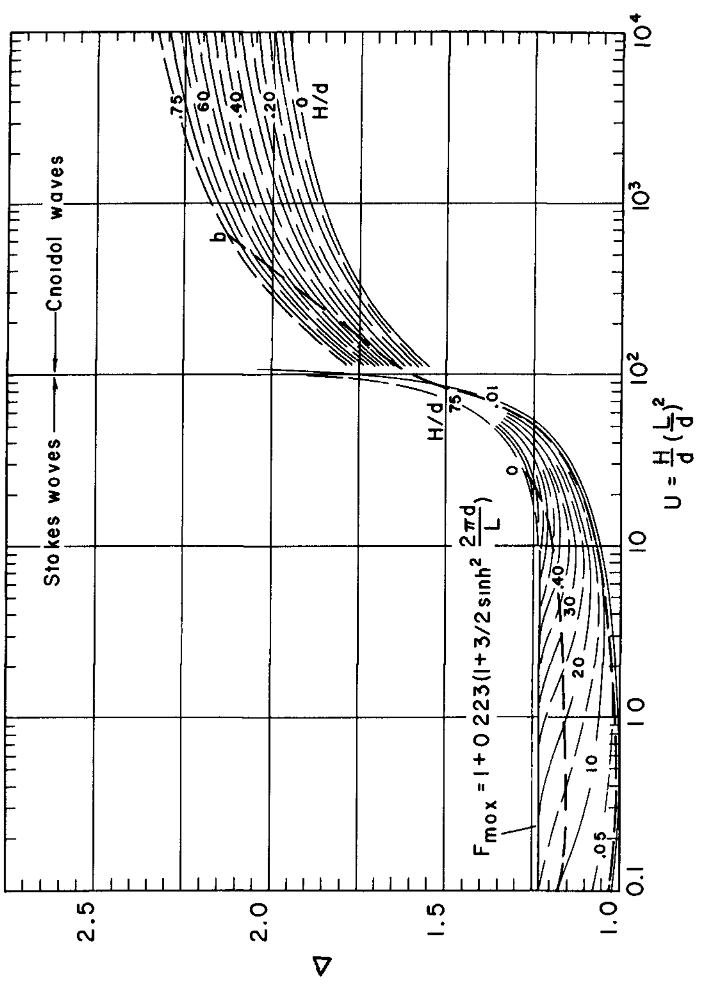

Fig. 2. Ursell diagram, showing crest elevation above still water Δ as a function of the Ursell parameter U, for stable waves of finite height. Dashed lines a) and b) show trajectories of two example waves through this diagram as the depth changes.

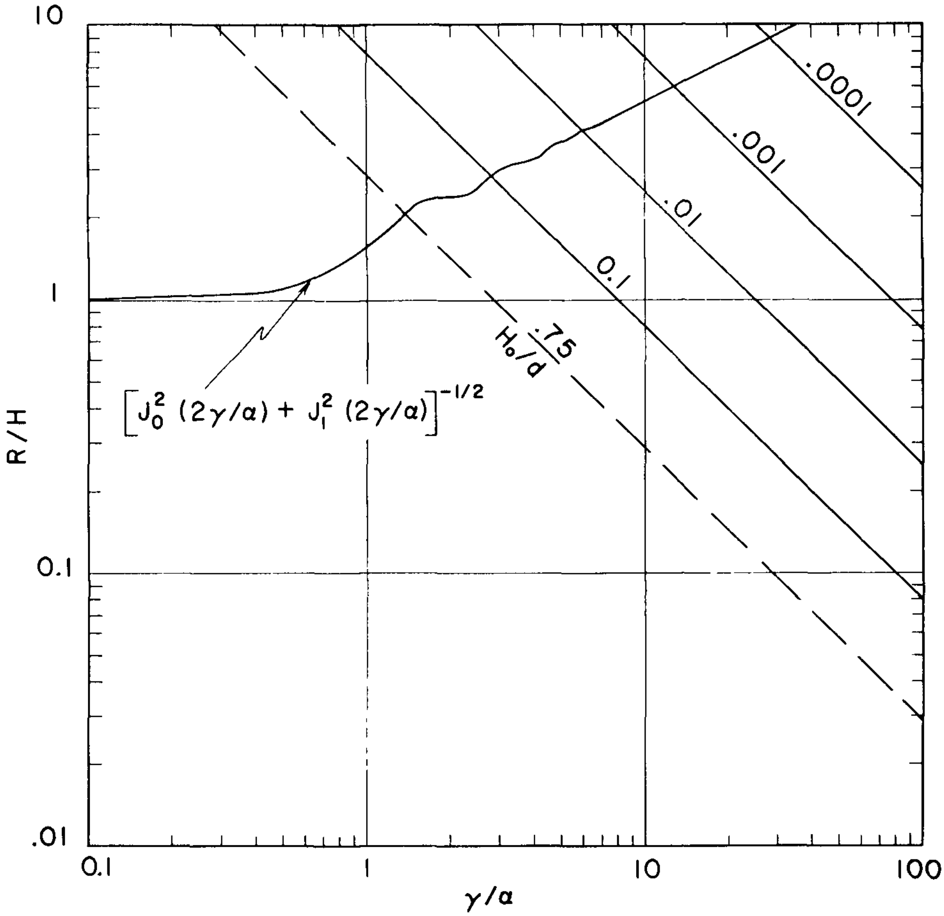

Fig. 3. Relative runup versus ratio of frequency to beach angle for breaking (empirical) and non-breaking (small-amplitude theory) waves.

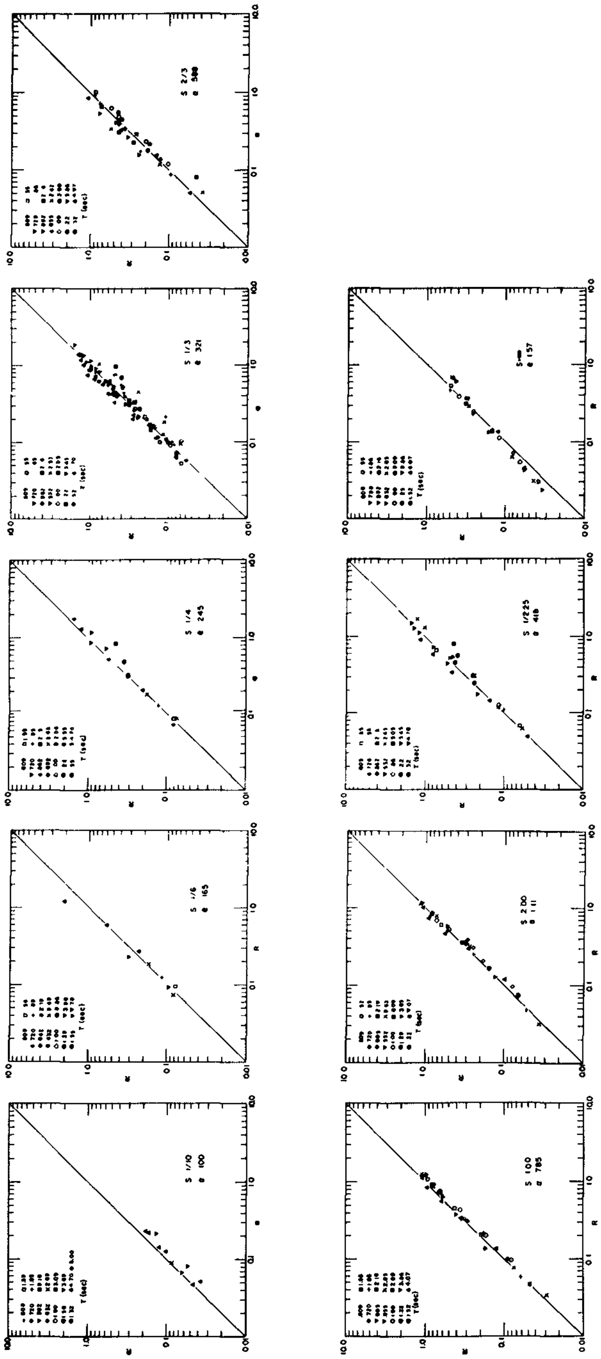

Fig. 4. Observed (R) versus computed (R*) runup for non-breaking waves.

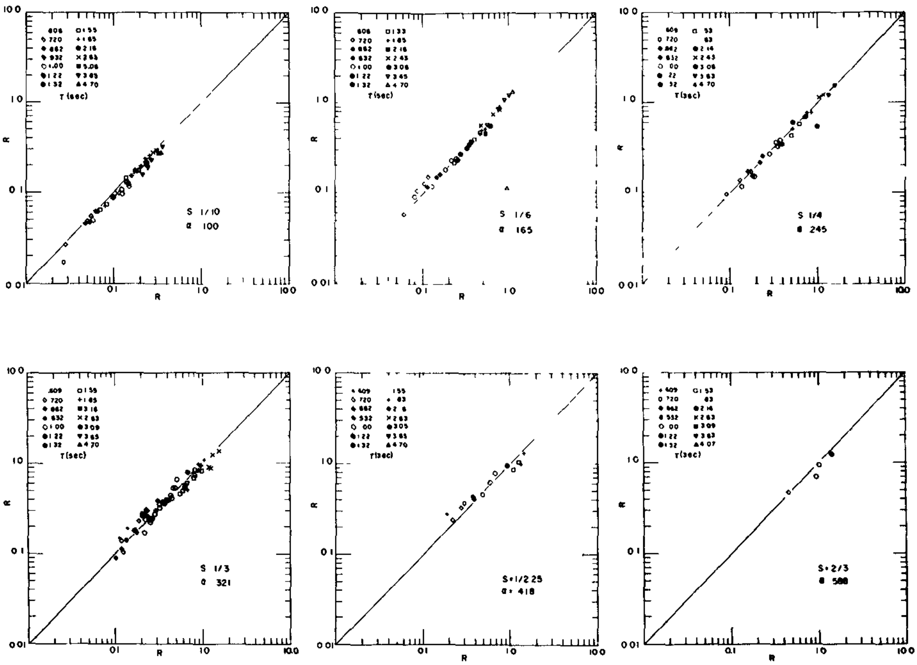

Fig. 5. Observed (R) versus computed (R*) runup for breaking waves.

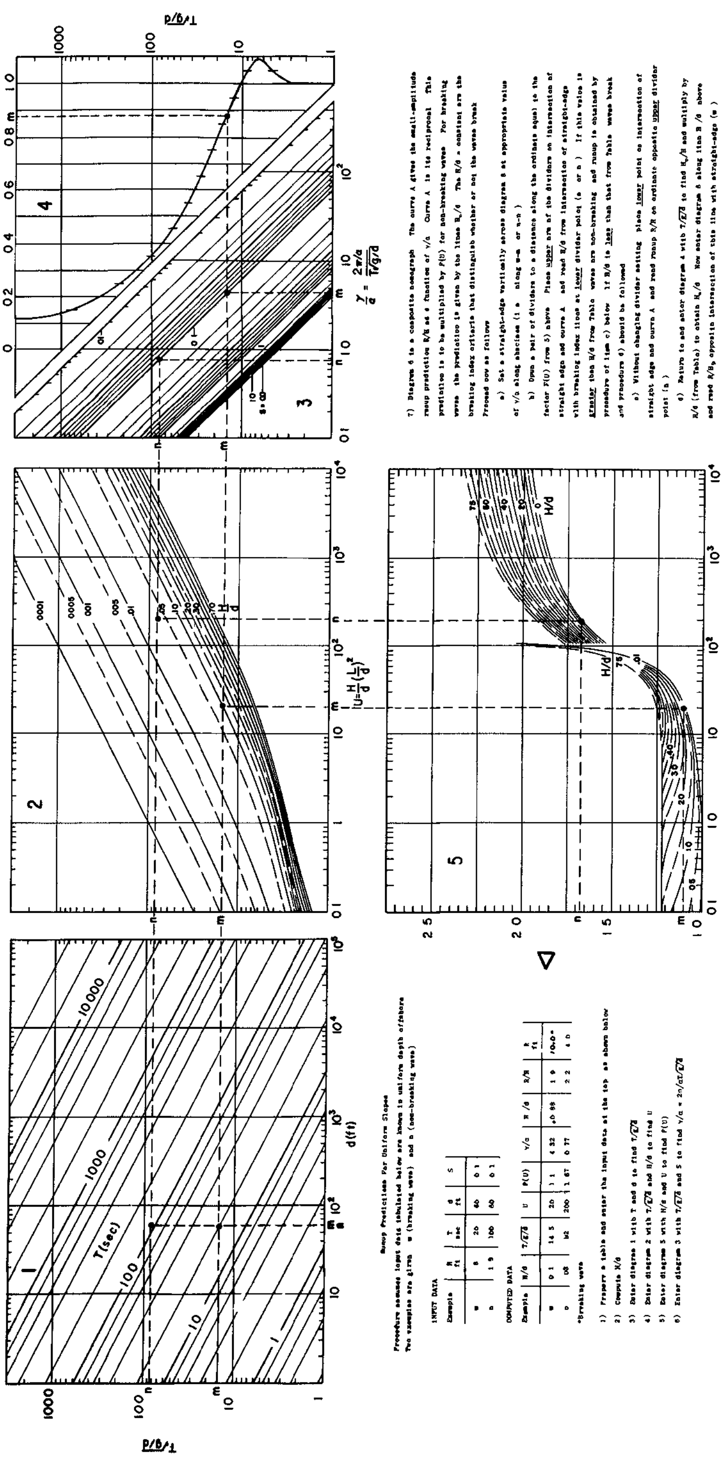

Fig. 6. Instruction and wave function nomographs.

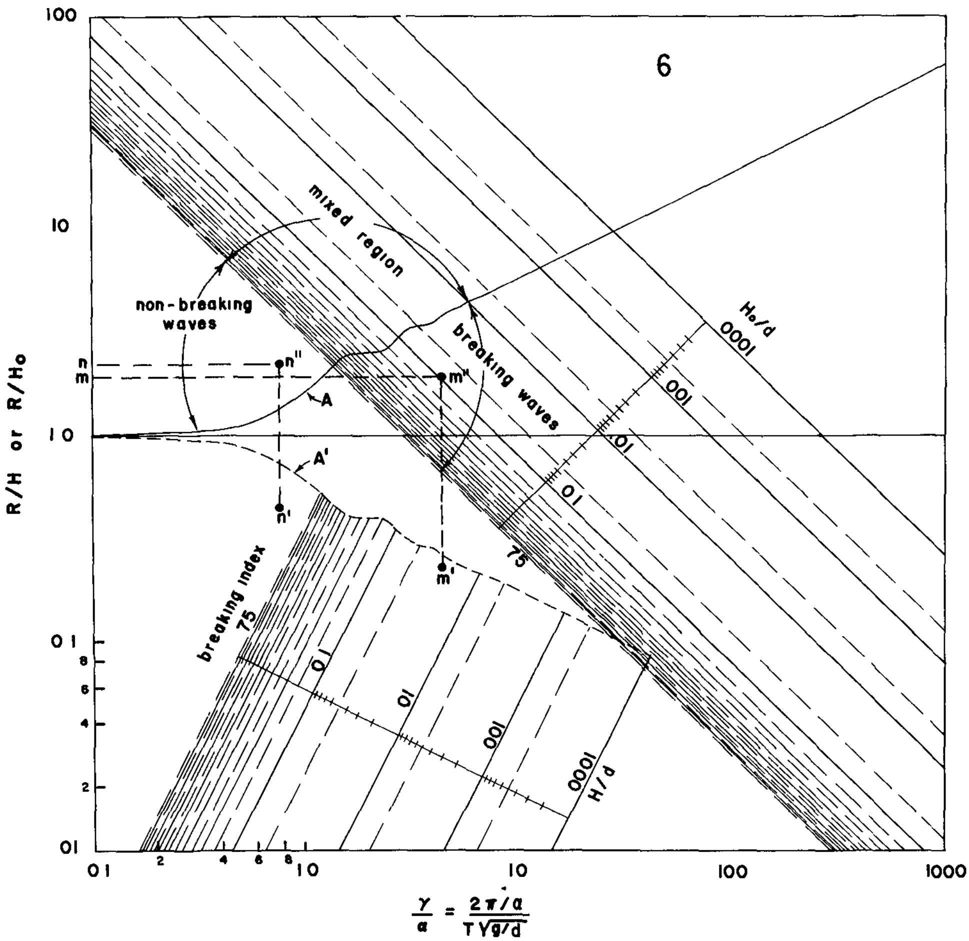

Fig. 7. Wave runup nomograph; Runup = R/H for non-breaking waves
= R/H_o for breaking waves.