CHAPTER 90

USE OF A COMPUTATIONAL MODEL FOR TWO-DIMENSIONAL TIDAL FLOW

Jan J. Leendertse*
The RAND Corporation
Santa Monica, California

INTRODUCTION

The numerical solution of tidal hydraulic problems has been greatly facilitated in the last decade by the development of high-speed large-memory computers. Problems which only could be studied by use of hydraulic models can now readily be studied with mathematical models if the computational techniques for solutions are developed.

Unfortunately the difficulties in formulating usable methods for these complicated problems in fluid dynamics are formidable, particularly if the problems are multidimensional in space. As a result the numerical solution approach to these hydraulic problems has generally been limited and has not kept pace with the incressed capabilities of the presently available computers.

In this paper a new approach is given to the solution of two-dimensional tidal flow in shallow water.

THE DIFFERENTIAL EQUATIONS REPRESENTING TIDAL FLOW

The basis of the computational model for tidal flow is the long-period water-wave equations (ref. 1):

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} - fV + g\frac{\partial \zeta}{\partial x} + g\frac{U(U^2+V^2)^{\frac{1}{2}}}{C^2(h+\zeta)} = 0 \qquad (1)$$

$$\frac{\partial V}{\partial t} + U\frac{\partial V}{\partial x} + V\frac{\partial V}{\partial y} + fU + g\frac{\partial \zeta}{\partial y} + g\frac{V(U^2+V^2)^{\frac{1}{2}}}{C^2(h+\zeta)} = 0 \qquad (2)$$

$$\frac{\partial \zeta}{\partial t} + \frac{\partial[(h+\zeta)U]}{\partial x} + \frac{\partial[(h+\zeta)V]}{\partial y} = 0 \qquad (3)$$

Symbols are defined as follows:

- U, V : vertically integrated velocity components in the x and y direction respectively
- ζ : elevation of the free surface over the undisturbed level
- h : depth
- g : acceleration due to gravity
- f : Coriolis parameter
- C : Chezy coefficient

*This research is sponsored by the United States Air Force under Project RAND. This is an abridgment of RAND Memorandum RM-5294-PR. The views, conclusions, and recommendations expressed herein do not necessarily reflect the official views or policies of the United States Air Force.

THE SYSTEM OF THE FINITE-DIFFERENCE EQUATIONS IN TWO DIMENSIONS

The following notation is used in the approximation of the differential equations:

$$u^{(n)}_{j,k} \equiv U(j\Delta x,\ k\Delta y,\ n\Delta t); \quad \Delta x = \Delta y = \Delta s \tag{4}$$

where

$(x,y) = (j\Delta x,\ k\Delta y)$ and is a spatial grid point

$j,k = 0,\ \pm\tfrac{1}{2},\ \pm 1,\ \pm 3/2,\ \ldots$

$n = 0,\ \tfrac{1}{2},\ 1,\ 3/2,\ 2$

Furthermore, the following notation of averages and differences shown here for ζ, is introduced:

$$\overline{\zeta}^{x}_{j,k} \equiv \tfrac{1}{2}\,(\zeta_{j+\frac{1}{2},k} + \zeta_{j-\frac{1}{2},k}) \tag{5}$$

$$\zeta_{x} \equiv (\zeta_{j+\frac{1}{2},k} - \zeta_{j-\frac{1}{2},k}) \tag{6}$$

$$\overline{\zeta}^{y}_{j,k} \equiv \tfrac{1}{2}\,(\zeta_{j,k+\frac{1}{2}} + \zeta_{j,k-\frac{1}{2}}) \tag{7}$$

$$\zeta_{y} \equiv (\zeta_{j,k+\frac{1}{2}} - \zeta_{j,k-\frac{1}{2}}) \tag{8}$$

$$\overline{\zeta}_{j,k} \equiv \tfrac{1}{4}\,(\zeta_{j-\frac{1}{2},k-\frac{1}{2}} + \zeta_{j-\frac{1}{2},k+\frac{1}{2}} + \zeta_{j+\frac{1}{2},\,k-\frac{1}{2}} + \zeta_{j+\frac{1}{2},k+\frac{1}{2}}) \tag{9}$$

A space-staggered scheme is used where velocities, water levels, and depth are described at different grid points (see Fig. 1). This scheme has the advantage that for the variable operated upon in time, there is a centrally located spatial derivative.

In time, a multistep operation is used in such a manner that the terms containing space derivatives and the Coriolis force are generally taken alternating forward and backward.

The individual operations each have two time levels. The first operation is taken from time n to time n+½, and the second operation is taken from time n+½ to time n+1. Values of the fields of ζ, u, and v at time n+½ are obtained from the fields of ζ, u, and v at time n by an operation which is implicit in ζ and u and explicit in v. Then the fields of ζ, u, and v at time n+1 are computed from the fields of ζ, u, and v at time n+½ by an operation which is implicit in ζ and v and explicit in u. The two sets of difference equations of this multioperation method are now written with the equation of continuity aa the second equation of each set (Eqs. 11, 14), using an integer value for j and k and maintaining the velocity gradients in the convective-inertia terms in differential form within angle brackets < >. The effects of bottom roughness are indicated by a function R. Thus, for the first operation the equations are:

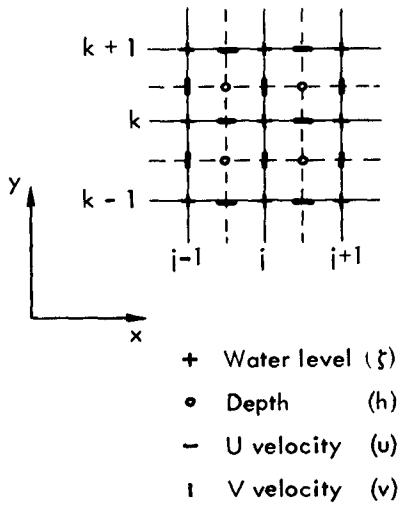

Fig. 1—Space-staggered scheme

$$u^{(n+\frac{1}{2})} = u(n) + \tfrac{1}{2}\Delta t\, f\bar{\bar{v}}^{(n)} - \tfrac{1}{2}\Delta t\, u^{(n+\frac{1}{2})} \langle\!\langle \frac{\partial u}{\partial x} \rangle\!\rangle^{(n)} - \tfrac{1}{2}\Delta t\, \bar{\bar{v}}^{(n)} \langle\!\langle \frac{\partial u}{\partial y} \rangle\!\rangle^{(n)}$$

$$- \tfrac{1}{2}\frac{\Delta t}{\Delta s} g\zeta_x^{(n+\frac{1}{2})} - R(x)^{(n)} \qquad \text{at } j + \tfrac{1}{2},\ k \qquad (10)$$

$$\zeta^{(n+\frac{1}{2})} = \zeta^{(n)} - \tfrac{1}{2}\frac{\Delta t}{\Delta s}\left[(\bar{h}^y + \zeta^x)^* u\right]_x^{(n+\frac{1}{2})} - \tfrac{1}{2}\frac{\Delta t}{\Delta s}\left[(\bar{h}^x + \zeta^y) v\right]_y^{(n)}$$

$$\text{at } j,\ k \qquad (11)$$

$$v^{(n+\frac{1}{2})} = v^{(n)} - \tfrac{1}{2}\Delta t\, f\bar{\bar{u}}^{(n+\frac{1}{2})} - \tfrac{1}{2}\Delta t\, \bar{\bar{u}}^{(n+\frac{1}{2})} \langle\!\langle \frac{\partial v}{\partial x} \rangle\!\rangle^{(n)}$$

$$-\tfrac{1}{2}\Delta t v^{(n+\frac{1}{2})} \langle\!\langle \frac{\partial v}{\partial x} \rangle\!\rangle^{(n)} - \tfrac{1}{2}\frac{\Delta t}{\Delta s} g\zeta_y^{(n)} - R(y)^{(n+\frac{1}{2})}{}_{j,k+\frac{1}{2}} \qquad (12)$$

For the second operation, the equations are:

$$u^{(n+1)} = u^{(n+\frac{1}{2})} + \tfrac{1}{2}\Delta t\, f\bar{\bar{v}}^{(n+1)} - \tfrac{1}{2}\Delta t\, u^{(n+1)} \langle\!\langle \frac{\partial u}{\partial x} \rangle\!\rangle^{(n+\frac{1}{2})}$$

$$- \tfrac{1}{2}\Delta t\, \bar{\bar{v}}^{(n+1)} \langle\!\langle \frac{\partial u}{\partial y} \rangle\!\rangle^{(n+\frac{1}{2})} - \tfrac{1}{2}\frac{\Delta t}{\Delta s} g\zeta_x^{(n+\frac{1}{2})} - R(x)^{(n+1)}$$

$$\text{at } j + \tfrac{1}{2},\ k \qquad (13)$$

$$\zeta^{(n+1)} = \zeta^{(n+\frac{1}{2})} - \tfrac{1}{2}\frac{\Delta t}{\Delta s}\left[(\bar{h}^y + \zeta^x) u\right]_x^{(n+\frac{1}{2})} - \tfrac{1}{2}\frac{\Delta t}{\Delta s}\left[(\bar{h}^x + \zeta^y)^* v\right]_y^{(n+1)}$$

$$\text{at } j,k \qquad (14)$$

$$v^{(n+1)} = v^{(n+\frac{1}{2})} - \tfrac{1}{2}\Delta t\, f\bar{\bar{u}}^{(n+\frac{1}{2})} - \tfrac{1}{2}\Delta t\, \bar{\bar{u}}^{(n+\frac{1}{2})} \langle\!\langle \frac{\partial v}{\partial x} \rangle\!\rangle^{(n+\frac{1}{2})}$$

$$- \tfrac{1}{2}\Delta t\, v^{(n+1)} \langle\!\langle \frac{\partial v}{\partial y} \rangle\!\rangle^{(n+\frac{1}{2})} - \tfrac{1}{2}\frac{\Delta t}{\Delta s} g\zeta_y^{(n+1)} - R(y)^{(n+\frac{1}{2})}$$

$$\text{at } j,\ k + \tfrac{1}{2} \qquad (15)$$

where

$$\langle\!\langle \frac{\partial u}{\partial x} \rangle\!\rangle_{j+\frac{1}{2},k} = \frac{1}{2\Delta s}\left(u_{j+\frac{3}{2},k} - u_{j-\frac{1}{2},k}\right) \qquad (16)$$

$$\langle\!\langle \frac{\partial u}{\partial y} \rangle\!\rangle_{j+\frac{1}{2},k} = \frac{1}{2\Delta s}\left(u_{j+\frac{1}{2},k+1} - u_{j+\frac{1}{2},k-1}\right) \qquad (17)$$

$$\langle\!\langle \frac{\partial v}{\partial x} \rangle\!\rangle_{j,k+\frac{1}{2}} = \frac{1}{2\Delta s}\left(v_{j+1,k+\frac{1}{2}} - v_{j-1,k+\frac{1}{2}}\right) \qquad (18)$$

$$\langle\!\langle \frac{\partial v}{\partial y} \rangle\!\rangle_{j,k+\frac{1}{2}} = \frac{1}{2\Delta s}\left(v_{j,k+\frac{3}{2}} - v_{j,k-\frac{1}{2}}\right) \qquad (19)$$

$$R(x)^{(n)} = \frac{1}{2}\,\Delta t\, g u^{(n)} \frac{\sqrt{(u^{(n)})^2 + (\bar{\bar{v}}^{(n)})^2}}{(\bar{h}^y + \zeta^{x(n)})(C^x)^2} \qquad \text{at } j + \tfrac{1}{2},\ k \tag{20}$$

$$R(y)^{(n+\frac{1}{2})} = \frac{1}{2}\,\Delta t\, g v^{(n+\frac{1}{2})} \frac{\sqrt{(\bar{\bar{u}}^{(n+\frac{1}{2})})^2 + (v^{(n)})^2}}{(\bar{h}^x + \zeta^{y(n+\frac{1}{2})})(C^y)^2} \qquad \text{at } j,\ k + \tfrac{1}{2} \tag{21}$$

$$R(x)^{(n+1)} = \frac{1}{2}\,\Delta t\, g u^{(n+1)} \frac{\sqrt{(u^{(n+\frac{1}{2})})^2 + (\bar{\bar{v}}^{(n+1)})^2}}{(\bar{h}^y + \zeta^{x(n+1)})(C^x)^2} \qquad \text{at } j + \tfrac{1}{2},\ k \tag{22}$$

$$R(y)^{(n+\frac{1}{2})} = \frac{1}{2}\,\Delta t\, g v^{(n+\frac{1}{2})} \frac{\sqrt{(\bar{\bar{u}}^{(n+\frac{1}{2})})^2 + (v^{(n+\frac{1}{2})})^2}}{(\bar{h}^x + \zeta^{y(n+\frac{1}{2})})(C^y)^2} \qquad \text{at } j,\ k + \tfrac{1}{2} \tag{23}$$

A special, additional technique will be employed for the computation of the nonlinear terms marked with an asterisk.

Solutions of the Equations (10-15) are found by direct computation or by solving systems of equations which relate water levels and velocities on lines.

In each of Eqs. (10) and (11) there are three values at the time level $(n + \frac{1}{2})$, which are all situated at the line k and have to be computed. Writing the continuity equation first and omitting the subscript k, we can write Eqs. (10) and (11) in the following form:

$$-\frac{1}{2}\frac{\Delta t}{\Delta s}\left[(\bar{h}^y + \zeta^x)^* u\right]_{j-\frac{1}{2}}^{(n+\frac{1}{2})} = \zeta_j^{(n+\frac{1}{2})} + \frac{1}{2}\frac{\Delta t}{\Delta s}\left[(\bar{h}^y + \zeta^x)^* u\right]_{j+\frac{1}{2}}^{(n+\frac{1}{2})} = A_j^{(n)} \tag{24}$$

$$-\frac{1}{2}\frac{\Delta t}{\Delta s}\, g\zeta_j^{(n+\frac{1}{2})} + \left(1 + \frac{1}{4}\frac{\Delta t}{\Delta s}\left(u_{j+\frac{1}{2}}^{(n)} - u_{j-\frac{1}{2}}^{(n)}\right)\right) u_{j+\frac{1}{2}}^{(n+\frac{1}{2})} + \frac{1}{2}\frac{\Delta t}{\Delta s}\, g\zeta_{j+1}^{(n+\frac{1}{2})} = B_{j+\frac{1}{2}}^{(n)} \tag{25}$$

where

$$A_j^{(n)} = \zeta^{(n)} - \frac{1}{2}\frac{\Delta t}{\Delta s}\left[(\bar{h}^x + \zeta^y)v\right]_y^n \qquad \text{at } j,\ k \tag{26}$$

$$B_{j+\frac{1}{2}}^{(n)} = u^{(n)} + \frac{1}{2}\left[\Delta t\, f - \frac{\Delta t}{2\Delta s}\left(u_{j+\frac{1}{2},k+1}^{(n)} - u_{j+\frac{1}{2},k-1}^{(n)}\right)\right]\bar{\bar{v}}^{(n)}$$

$$- \frac{1}{2}\,\Delta t\, g u^{(n)} \frac{\sqrt{(u^{(n)})^2 + (\bar{\bar{v}}^{(n)})^2}}{(\bar{h}^y + \zeta^{x(n)})(C^x)^2} \qquad \text{at } j + \tfrac{1}{2},\ k \tag{27}$$

Thus, one equation with three unknowns exists for each velocity field point $(u_{j+\frac{1}{2}})$ and each water level field point (ζ_j) on a line k. If a row of N water level points is on the line and velocities are given at the boundaries outside of the water levels concerned, N water levels and N - 1 velocities at time $(n + \frac{1}{2})$ must be solved from 2N - 1 equations.

We now introduce

$$r_{j-\frac{1}{2}} = \frac{1}{2}\frac{\Delta t}{\Delta s}\,(\bar{h}^{y} + \zeta^{x})^{*}_{j-\frac{1}{2}};\quad r_{j+\frac{1}{2}} = \frac{1}{2}\frac{\Delta t}{\Delta s}\,(\bar{h}^{y} + \zeta^{x})^{*}_{j+\frac{1}{2}},\ \ldots \tag{28}$$

$$r_{j} = \frac{1}{2}\frac{\Delta t}{\Delta s}\,g;\quad r_{j+1} = \frac{1}{2}\frac{\Delta t}{\Delta s}\,g;\ \ldots \tag{29}$$

Equations (24) and (25) can be written in matrix form for a line k, assuming that $u_{J-\frac{1}{2}}$ is a given velocity at the lower boundary and $u^{(n+\frac{1}{2})}_{I+\frac{1}{2}}$ is a given velocity at the upper boundary:

$$\begin{bmatrix} 1 & r_{J+\frac{1}{2}} & 0 & 0 & . & 0 \\ -r_{J} & a_{J+\frac{1}{2}} & r_{J+1} & 0 & . & 0 \\ 0 & -r_{J+\frac{1}{2}} & 1 & r_{J+\frac{3}{2}} & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & 0 & \ldots & -r_{I-\frac{1}{2}} & 1 \end{bmatrix} \begin{bmatrix} \zeta_{J} \\ u_{J+\frac{1}{2}} \\ \zeta_{J+1} \\ . \\ . \\ . \\ \zeta_{I} \end{bmatrix}^{(n+\frac{1}{2})} = \begin{bmatrix} A_{J} \\ B_{J+\frac{1}{2}} \\ A_{J+1} \\ . \\ . \\ . \\ A_{I} \end{bmatrix}^{(n)} + \begin{bmatrix} r_{J-\frac{1}{2}}\,u_{J-\frac{1}{2}} \\ 0 \\ 0 \\ 0 \\ . \\ . \\ r_{I+\frac{1}{2}}\,u_{I+\frac{1}{2}} \end{bmatrix}^{(n+\frac{1}{2})} \tag{30}$$

where

$$a_{J+\frac{1}{2}} = 1 + \frac{1}{4}\frac{\Delta t}{\Delta s}\left(u^{(n)}_{J+\frac{3}{2}} - u^{(n)}_{J-\frac{1}{2}}\right) \tag{31}$$

The values of the vector $(\zeta_J,\ u_{J+\frac{1}{2}},\ \zeta_{J+1},\ \ldots,\ \zeta_I)$ at the $n + \frac{1}{2}$ time level can be solved with a limited number of operations by a process of elimination of unknowns. Starting with the first equation, the water level $\zeta^{n+\frac{1}{2}}_{J}$ is expressed as a function of the unknown velocity $u_{J+\frac{1}{2}}$:

$$\zeta^{(n+\frac{1}{2})}_{J} = -P_{J}u^{(n+\frac{1}{2})}_{J+\frac{1}{2}} + Q_{J} \tag{32}$$

where

$$P_{J} = r_{J+\frac{1}{2}} \tag{33}$$

$$Q_{J} = A^{(n)}_{J} + r_{J-\frac{1}{2}}\,u^{(n+\frac{1}{2})}_{J-\frac{1}{2}} \tag{34}$$

Substitution of Eq. (32) into the second equation of (30) gives

$$-r_{J}(-P_{J}u^{(n+\frac{1}{2})}_{J+\frac{1}{2}} + Q_{J}) + \left(1 + \frac{1}{4}\frac{\Delta t}{\Delta s}(u^{(n)}_{J+\frac{3}{2}} - u^{(n)}_{J-\frac{1}{2}})\right) u^{(n+\frac{1}{2})}_{J+\frac{1}{2}} + r^{(n+\frac{1}{2})}_{J+1}\,\zeta^{(n+\frac{1}{2})}_{J+1} = B^{(n)}_{J+\frac{1}{2}} \tag{35}$$

or expressing $u^{(n+\frac{1}{2})}_{J+\frac{1}{2}}$ as a function of $\zeta^{(n+\frac{1}{2})}_{J+1}$ gives

$$u_{J+\frac{1}{2}}^{(n+\frac{1}{2})} = -R_J \zeta_{J+1}^{(n+\frac{1}{2})} + S_J \tag{36}$$

where

$$R_J = \frac{r_{J+1}}{\left[1 + r_J P_J + \frac{1}{4}\frac{\Delta t}{\Delta s}\left(u_{J+\frac{3}{2}}^{(n)} - u_{J-\frac{1}{2}}^{(n)}\right)\right]} \tag{37}$$

$$S_J = \frac{B_{J+\frac{1}{2}}^{(n)} + r_J Q_J}{\left[1 + r_J P_J + \frac{1}{4}\frac{\Delta t}{\Delta s}\left(u_{J+\frac{3}{2}}^{(n)} - u_{J-\frac{1}{2}}^{(n)}\right)\right]} \tag{38}$$

Again, the water level can be expressed as a function of the next velocity:

$$\zeta_{J+1} = -P_{J+1} u_{J+\frac{3}{2}}^{n+\frac{1}{2}} + Q_{J+1} \tag{39}$$

where

$$P_{J+1} = \frac{r_{J+\frac{3}{2}}}{1 + r_{J+\frac{1}{2}} R_J} \tag{40}$$

$$Q_{J+1} = \frac{A_{j+1}^{(n)} + r_{J+\frac{1}{2}} S_J}{1 + r_{J+\frac{1}{2}} R_J} \tag{41}$$

Generally, the following recursion formulas can be written

$$\zeta_j^{(n+\frac{1}{2})} = -P_j u_{j+\frac{1}{2}}^{(n+\frac{1}{2})} + Q_j \tag{42}$$

$$u_j^{(n+\frac{1}{2})} = -R_{j-1} \zeta_j^{(n+\frac{1}{2})} + S_{j-1} \tag{43}$$

where

$$P_j = r_{j+\frac{1}{2}} / (1 + r_{j-\frac{1}{2}} R_{j-1}) \tag{44}$$

$$Q_j = (A_j^{(n)} + r_{j-\frac{1}{2}} S_{j-1}) / (1 + r_{j-\frac{1}{2}} R_{j-1}) \tag{45}$$

$$R_j = r_{j+1} / \left[1 + r_j P_j + \frac{1}{4}\frac{\Delta t}{\Delta s}\left(u_{j+\frac{3}{2}}^{(n)} - u_{j-\frac{1}{2}}^{(n)}\right)\right] \tag{46}$$

$$S_j = (B_{j+\frac{1}{2}}^{(n)} + r_j Q_j) / \left[1 + r_j P_j + \frac{1}{4}\frac{\Delta t}{\Delta s}\left(u_{j+\frac{3}{2}}^{(n)} - u_{j-\frac{1}{2}}^{(n)}\right)\right] \tag{47}$$

The recursion factors P, Q, R, and S can be computed in succession until the other bound is reached. If $u_{I+\frac{1}{2}}^{(n+\frac{1}{2})}$ is a given velocity, the last two factors computed are P_I and Q_I. Since Eq. (42) expresses ζ_I as a known function of the velocity $u_{I+\frac{1}{2}}^{(n+\frac{1}{2})}$, ζ_I can be computed, and all water levels and velocities can be found in descending order by use of Eqs. (42) and (43).

The nonlinear terms in the continuity equation which are marked with an asterisk can be computed on their proper time level by an iteration procedure. The first estimate of $\zeta^{(n+\frac{1}{2})}$ is made by the implicit procedure of taking the nonlinear term at the time level n. Next, the value $\zeta^{(n+\frac{1}{2})}$ thus computed is used in Eq. (11) for the actual computation. This iteration can be repeated several times. Use of the mathematical model, however, indicated that no gain in accuracy is obtained.

The velocity in the other direction at the time level $n + \frac{1}{2}$ can be found explicitly from Eq. (12), since the velocity $u^{(n+\frac{1}{2})}$ in the Coriolis term is already known. Eq. (12) can then be written:

$$v^{(n+\frac{1}{2})} = \frac{\left\{v^{(n)} - \frac{1}{2}\left[\Delta t\, f + \frac{1}{2}\frac{\Delta t}{\Delta s}\left(v^{(n)}_{j+1,k+\frac{1}{2}} - v^{(n)}_{j-1,k+\frac{1}{2}}\right)\bar{\bar{u}}^{(n+\frac{1}{2})}\right] - \frac{1}{2}\frac{\Delta t}{\Delta s}\, g\zeta^{(n)}_y\right\}}{\left[1 + \frac{1}{2}\Delta t\, g\frac{\sqrt{(\bar{\bar{u}}^{(n+\frac{1}{2})})^2 + (v^{(n)})^2}}{(\bar{n}^x + \bar{\zeta}^{y(n+\frac{1}{2})})(\bar{C}^y)^2} + \frac{1}{4}\frac{\Delta t}{\Delta s}\left(v^{(n)}_{j,k+\frac{3}{2}} - v^{(n)}_{j,k-\frac{1}{2}}\right)\right]}$$

at j, k + ½ (48)

For the second operation, going from the time level $n + \frac{1}{2}$ to n + 1, Eqs. (14) and (15) are solved implicitly in the same manner as described for Eq. (10) and (11). Finally, the velocity in the x-direction can be computed explicitly from Eq. (13). It can be seen that no more than two successive fields have to be stored in the computer memory, as only the latest information available is necessary to compute the next step.

The coefficients for the recursion formulas in the y-direction are

$$P_k = r_{k+\frac{1}{2}}/(1 + r_{k-\frac{1}{2}}R_{k-1}) \tag{49}$$

$$Q_k = (A_k^{(n+\frac{1}{2})} + r_{k-\frac{1}{2}}S_{k-1})/(1 + r_{k-\frac{1}{2}}R_{k-1}) \tag{50}$$

$$R_k = r_{k+1}/\left[1 + r_kP_k + \frac{1}{4}\frac{\Delta t}{\Delta s}\left(v^{(n+\frac{1}{2})}_{k+\frac{3}{2}} - v^{(n+\frac{1}{2})}_{k-\frac{1}{2}}\right)\right] \tag{51}$$

$$S_k = (B^{(n+\frac{1}{2})}_{k+\frac{1}{2}} + r_kQ_k)/\left[1 + r_kP_k + \frac{1}{4}\frac{\Delta t}{\Delta s}\left(v^{(n+\frac{1}{2})}_{k+\frac{3}{2}} - v^{(n+\frac{1}{2})}_{k-\frac{1}{2}}\right)\right] \tag{52}$$

where

$$A_k^{(n+\frac{1}{2})} = \zeta^{(n+\frac{1}{2})} - \frac{1}{2}\frac{\Delta t}{\Delta s}\left[(\bar{n}^y + \bar{\zeta}^x)u\right]_x^{n+\frac{1}{2}} \quad \text{at j, k} \tag{53}$$

$$B_{k+\frac{1}{2}}^{(n+\frac{1}{2})} = v^{(n+\frac{1}{2})} - \frac{1}{2}\left[\Delta t\, f + \frac{1}{2}\frac{\Delta t}{\Delta s}\left(v_{j+1,k+\frac{1}{2}}^{(n+\frac{1}{2})} - v_{j-1,k+\frac{1}{2}}^{(n+\frac{1}{2})}\right)\right]\bar{u}^{(n+\frac{1}{2})}$$

$$- \frac{1}{2}\Delta t\, g v^{(n+\frac{1}{2})}\frac{\sqrt{\left(\bar{u}^{(n+\frac{1}{2})}\right)^2 + \left(v^{(n+\frac{1}{2})}\right)^2}}{\left(\bar{h}^{x} + \bar{\zeta}^{y(n+\frac{1}{2})}\right)\left(C^{y}\right)^2} \qquad \text{at } j,\ k+\frac{1}{2} \tag{54}$$

The recursion formulas in the y-direction are

$$\zeta_k^{(n+1)} = -P_k v_{k+\frac{1}{2}}^{(n+1)} + Q_k \tag{55}$$

$$v_{k-\frac{1}{2}}^{(n+1)} = -R_{k-1}\zeta_k^{(n+1)} + S_{k-1} \tag{56}$$

The explicit operation for the u velocity becomes

$$u^{(n+1)} =$$

$$\frac{\left\{u^{(n+\frac{1}{2})} + \frac{1}{2}\left[\Delta t\, f - \frac{1}{2}\frac{\Delta t}{\Delta s}\left(u_{j+\frac{1}{2},k+1}^{(n+\frac{1}{2})} - u_{j+\frac{1}{2},k-1}^{(n+\frac{1}{2})}\right)\right]\bar{v}^{(n+1)} - \frac{1}{2}\frac{\Delta t}{\Delta s} g\zeta_x^{(n+\frac{1}{2})}\right\}}{\left[1 + \frac{1}{2}\Delta t\, g\frac{\sqrt{\left(u^{(n+\frac{1}{2})}\right)^2 + \left(\bar{v}^{(n+1)}\right)^2}}{\left(\bar{h}^{y} + \bar{\zeta}^{x(n+1)}\right)\left(C^{x}\right)^2} + \frac{1}{4}\frac{\Delta t}{\Delta s}\left(u_{j+\frac{3}{2},k}^{(n+\frac{1}{2})} - u_{j-\frac{1}{2},k}^{(n+\frac{1}{2})}\right)\right]} \tag{57}$$

The formulas developed above cannot be used directly in this form for electronic computation. Coding in FORTRAN presents problems, as coordinate description of the variables can be made only on integers, and integer-and-one-half values do appear in the formulas developed. Multiplication of all coordinate descriptions of the points by a factor of two would eliminate this problem; however, such a lattice system is uneconomical from the point of view of memory use of the computer because not all locations of an array u, v, ζ and h would be used.

It is possible to give each of the variables u, v, ζ and h a separate coordinate system, which would result in a good use of memory and would also allow coordinate description on integers. The computational formulas for these special coordinate systems and the computation program are presented in Ref. 2.

STABILITY, DISSIPATION AND DISPERSION

One of the requirements of the numerical solutions of the equations is that the obtained solutions converge to the solutions of differential equations. Also, the solutions should be stable. This can be investigated by following a Fourier expansion of a line of errors as time progresses. If this line of errors (error wave) grows with time, the computation

becomes unstable and dominates the solution. If, on the other hand, error waves will be damped heavily if computation progresses, then, components of the waves which we try to represent by the numerical procedures will be damped also and possibly in a degree which is not in agreement with the solutions of the differential equations. Such a numerical scheme is dissipative. An investigation[2] of simplified sets of difference equations (10) through (15), indicates that the multioperation method is unconditionally stable and not dissipative.

Numerical procedures however, have influence on the phase speed of the wave. This effect is called dispersive. The discreetness of the representation of the waves in time and in two spatial dimensions, influences this effect. Fig. 2 shows the computed wave velocity versus the wave velocity of the partial differential equations as function of spatial representation for a given ratio of time and spatial gridsize. The direction of the wave compared to the grid influences this ratio. If more than thirty points are used to represent one wavelength on the spatial grid, then the difference between computed wave velocity and the wave velocity of solutions of the partial differential equations are less than 1%.

NUMERICAL EXPERIMENTS

The model was tested by comparing measurements with computational data. The mouth of the Haringvliet in The Netherlands was used (Fig. 3) since detailed measurement of tidal levels and currents were available, by The Netherlands Rijkswaterstaat, and in addition this area had large variations in depth over short distances. A grid size of 400 m with a computational array of 31 by 55 points was chosen. As an aid in the evaluation of the results, a plot program was developed for the display of currents and water levels at certain times of the computation (Fig. 4). The values of the Chezy coefficients were then adjusted until a good agreement was found between measured and computed velocities and water levels (Fig. 5). A detailed description of these experiments can be found elsewhere[2].

As a final check of the general computational procedures developed during the experiments with the Haringvliet area, a tidal computation was made of the southern North Sea (see Fig. 3). The gridsize used is 5600 m.

The northern boundary of the model was described at four locations as a time function of the water levels. Other points of this boundary were computed by linear interpolation or extrapolation (Fig. 6). The southern boundary in the English Channel was described at the coasts as a time function of the water levels, and the intermediate points were computed by interpolation.

In the southwestern part of The Netherlands, currents in the different parts of the estuaries of the Rhine and Schelde were used for the boundary. Tidal data for the period from 0.00 hr (Middle European Time), September 13, 1958, were used for computation. Some adjustments were made at the beginning and end of this period in order to make all tidal curves a complete cycle over this period. The half time step of each operation was taken at 5 min. The C values were computed every half hour (real time) on the location of the water level as a function of the average depth at that moment.

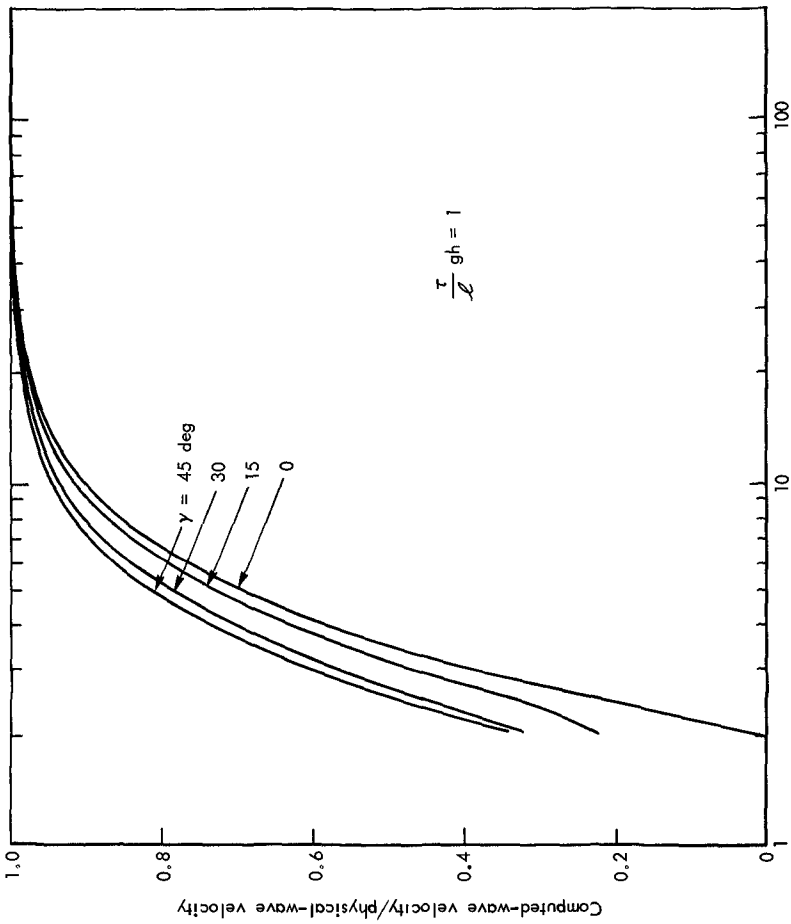

Fig. 2—Ratio of computed-wave velocity and physical-wave velocity as a function of discreetness

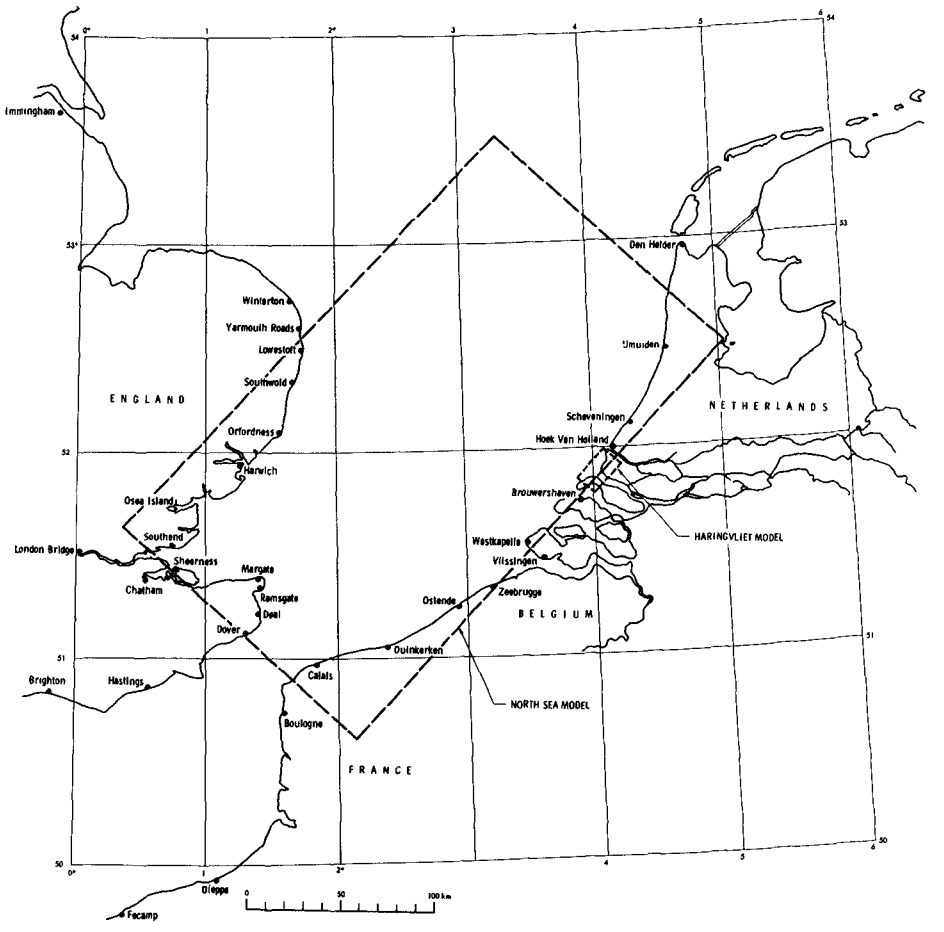

Fig. 3—Location of Haringvliet and North Sea computational models

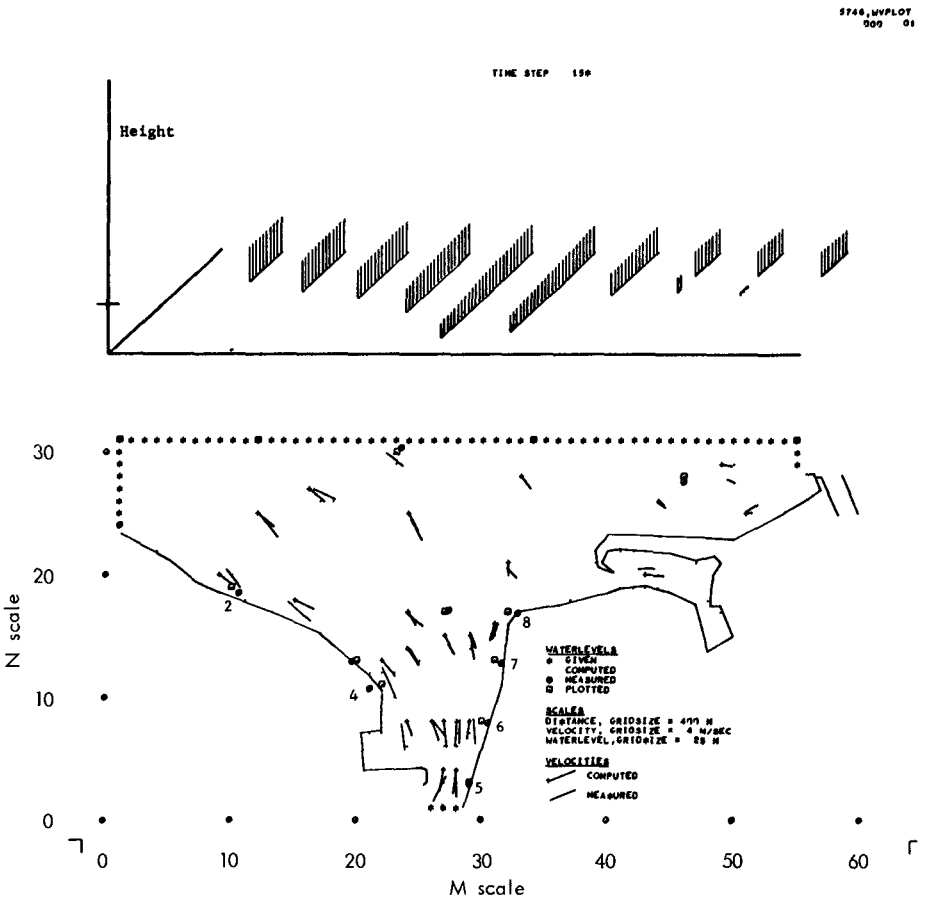

Fig. 4—Haringvliet (estuary of the Rhine)
Situation 15.6 hr (real time) after start of computation

Bottom Field of the computotion ond comparison between meosured velocities ond computed velocities. The meosured velocities ore overoge velocities of similar tidol cycles at o porticulor location, while the computed velocities ore the overage velocity in an 800-by-800-m orea of o particular tidal cycle.

Top Isometric sections representing computed water levels on every fifth line of the bottom graph.

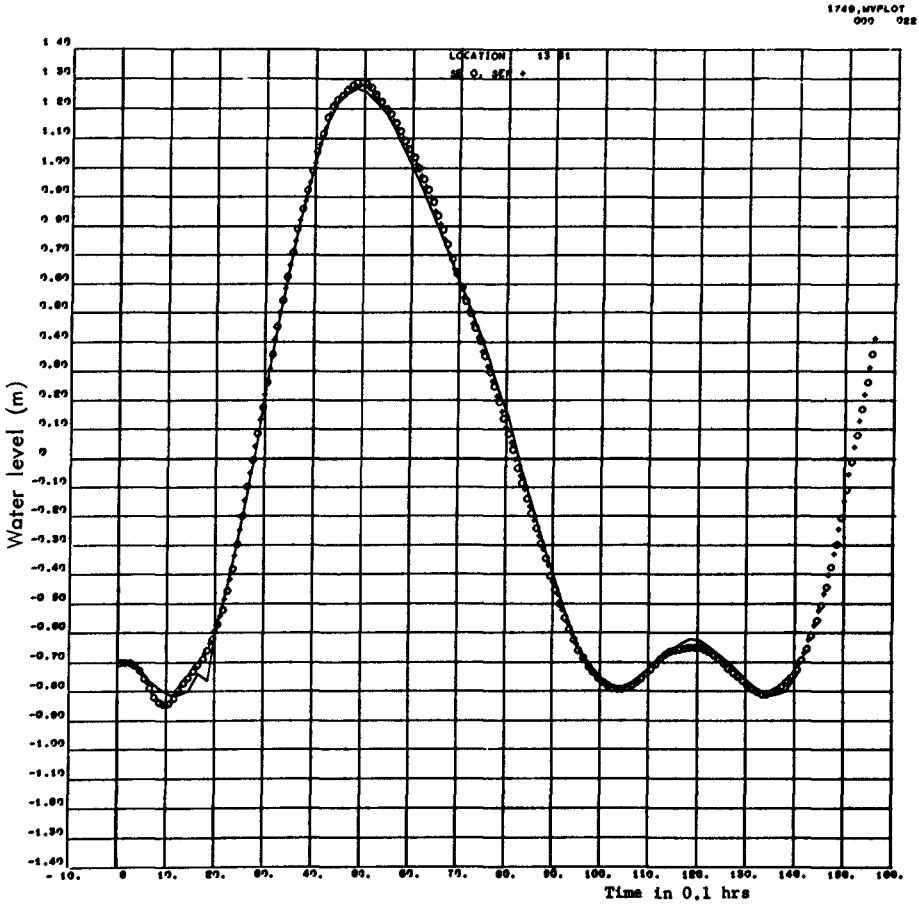

Fig. 5—Typical plot of a water-level history

Gage No. 7 Comparison between measured data (−) and computation (+0+)

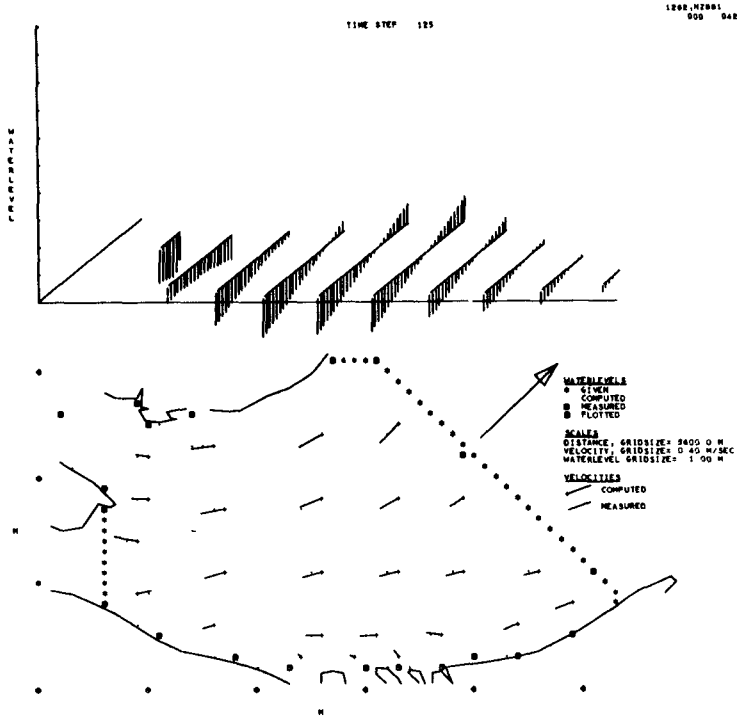

Fig. 6—North Sea computed water-levels and currents on September 12, 1958 at 20.00 hr

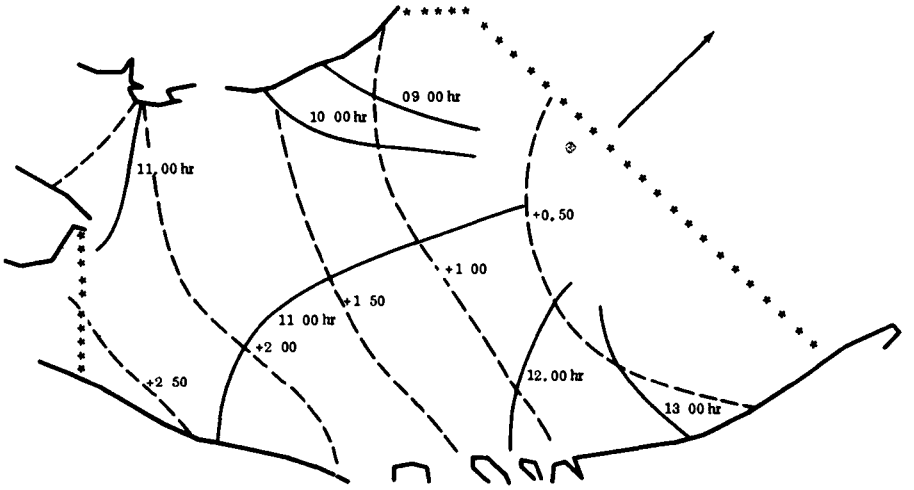

Fig. 7—Iso high-tide levels and times on September 12, 1958

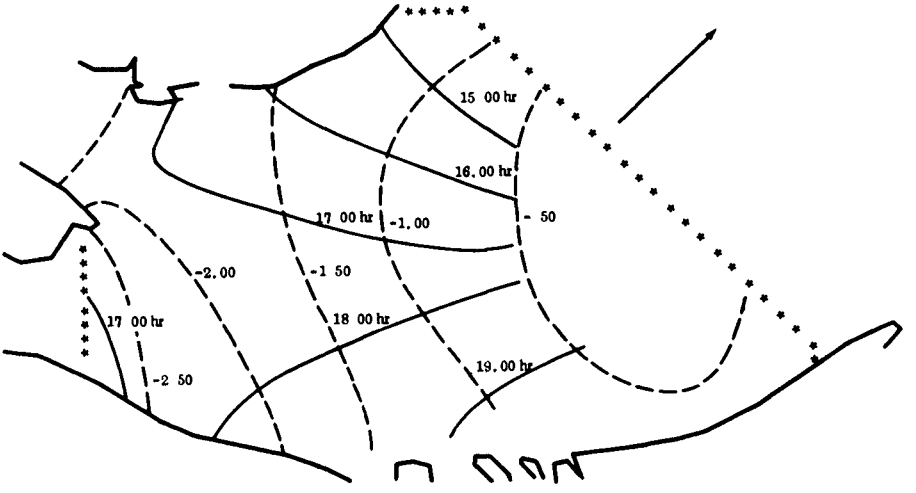

Fig. 8—Iso low-tide levels and times on September 12, 1958

Computations were started with all water levels and velocities taken at zero. The water levels and currents at the open boundaries were increased from zero with five steps into the given tidal curve. The starting disturbance disappeared after approximately 14 hr.

The locations of equal high-tide levels (co-range lines) obtained from the results of this computation are shown in Fig. 7, together with the arrival times of the highest water levels (co-tidal lines). Figure 8 shows the same for the low tides. In these two figures a counterclockwise rotation of the vertical tide may be seen. The water level information shown in Figs. 7 and 8 concurs with tidal information presented by Defant[3] as to phases and amplitudes of the tides.

The maximum magnitude of the computed currents such as shown in Fig 6 agrees with information given on tidal charts, but no detailed analysis has been made of the phase and amplitude of these currents with respect to such information. Water levels and currents along the coast are not accurate as the grid size is too large for a good representation, and also more accurate input data is needed for these computations.

CONCLUDING REMARKS

It has been shown that the propagation of long waves in coastal waters can be studied successfully by use of a numerical integration method. The multioperation method developed is characteristic of implicit methods; namely, there is no upper limit on the time step for stability reasons, as is the case with explicit methods. The multioperation method allows a direct and rapid solution of all velocities and water levels on each time level.

The multioperation method described is particularly suitable for long-wave computation in coastal waters, where water movements are introduced by changes in the water level (or currents) along the sides of the model and where the effect of bottom friction is larger than the effects of lateral eddy viscosity, which is neglected.

The contribution of the convective-inertia terms in the equation of motion is assumed to be small compared to that of other terms. These terms are represented with a lower order of accuracy.

The detailed description of computational procedures[2] permits an expedient introduction of geographic features such as water depth, boundaries, and characteristics of bottom roughness for the modeling of wave propagation in hydraulic engineering research.

Generally, information concerning the magnitude of the effect of bottom roughness is inadequate. In some cases, like the Haringvliet model, the water movements are influenced by bottom roughness to a considerable extent. In such a case, the parameter C has to be found in an iterative manner by comparing computed results with actual field measurements. The rate at which the model can be adjusted to resemble the prototype depends on the extent of available field data and on the experience of the engineer making the investigation and his insight into the physics of the wave problem and into the behavior of the method of numerical solution.

REFERENCES

1. Dronkers, J. J., (1964). Tidal Computations in Rivers and Coastal Waters, John Wiley & Sons, Inc., New York.

2. Leendertse, J. J., (1967). Aspects of a Computational Model for Long-Period Water-Wave Propagation, The RAND Corporation, Santa Monica, California, RM-5294-PR.

3. Defant, Albert, Physical Oceanography, Vol. III, The Macmillan Company, New York, 1961