CHAPTER 53

The Dissipation of Wave Energy by Turbulence

by

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1. Abstract

This paper first gives a brief review of the existing research works on the laws governing the dissipation of wave energy by turbulence. Starting from the general theory of turbulent motion and the writer's suggestion in regard to the mixing length of water particles in two-dimensional flow and making use of the principle of dimensional analysis and the trochoidal wave theory, a formula has been derived to compute the mean dissipation per unit time and per unit horizontal area of wave energy due to turbulence. The formula takes the horizontal and vertical gradients of both the horizontal and vertical velocity fields into consideration. Coefficient in the formula has been determined through laboratory experiments.

2. A Brief Review of Former Research Works

It is not far since the presentation of the suggestion that fluid turbulence plays an important role also in wave motion. The scientific researches on the laws governing the dissipation of wave energy by turbulence were started in the late forties of this century, but only in and after the fifties of this century had more research works been gradually done.

There are generally three different methods to study the problem of turbulent dissipation of wave energy. The first one bases solely on the principle of dimensional analysis, making use of the $n$-theorem. The advantages of this method lie in the simplicity of the process of derivation, but the selection of the independent variables and the determination of the formula patterns are to a certain degree arbitrary. The typical example applying this method of analysis can be found in literature (7).

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The second method utilizes the theoretical relationship of viscous dissipation of wave energy, replacing the coefficient of kinematic viscosity by means of the coefficient of kinematic eddy viscosity and then finds the functional relationship between the latter coefficient and the relevant physical quantities characterizing wave motion, applying the principle of dimensional analysis. The virtues and defects of this method are basically the same as that of the first method. Its typical example of application can be found in literature (2).

The third method makes use of the theory of turbulent flow. This method proceeds from the internal structure of the current, giving a deeper insight into the essence of the phenomenon and therefore has been widely used. Nevertheless, the existing theories take only the vertical gradients of the horizontal velocity field into consideration. But in wave motion, the magnitudes of the horizontal and vertical gradients of both the horizontal and vertical velocity fields are of the same order. They should be considered simultaneously. Literatures (3) and (4) can be referred to as the examples of application of this method.

The discrepancies among the results of the existing research works are very great. For instance, according to Тктиев (5), the average rate of energy dissipation of wave motion due to turbulence is 108 times as great as that computed by means of the formula suggested by Копейков (1). Similarly, if one uses the results of Щепелев (6) and Амброзов (3), the calculated values of this quantity will be several times to nearly one hundred times as great as that of Копейков for flat waves and steep waves respectively.

3. Theoretical Analysis

For two-dimensional turbulent flow, one may assume (7), (8)

\[
\begin{align*}
\tau_{xy} &= \rho \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \\
\tau_{xx} &= -C_1 + 2\rho \frac{\partial^2 u}{\partial x^2} \\
\tau_{yy} &= -C_2 + 2\rho \frac{\partial^2 v}{\partial y^2}
\end{align*}
\]

(1)

in which: \( \tau_{xy} \) ——- y component of the turbulent stress acting on a surface element, the outward normal of which is parallel to the x-axis; \( \tau_{xx} \) and \( \tau_{yy} \) have a similar meaning;

\( \rho \) —— mass density of liquid;

\( C_1 \) and \( C_2 \) —— coefficient of kinematic eddy viscosity;
U, V —— component velocities averaged over time in the X and Y directions;
C, C' —— constants.

Referring to Prandtl’s suggestion (8), in the case of two-dimensional flow, one may put

\[ E = |\mathbf{F}| \]  \hspace{1cm} (2)

where

\[ F^2 = 2\left(\frac{\partial U}{\partial X}\right)^2 + 2\left(\frac{\partial V}{\partial Y}\right)^2 + \left(\frac{\partial V}{\partial X} + \frac{\partial U}{\partial Y}\right)^2 \]  \hspace{1cm} (3)

\( l \) is the mixing length of fluid particles.

According to the trochoidal wave theory of deep water (7), the Cartesian coordinates of water particles are:

\[ X = a + \frac{1}{2}h e^{\lambda t} \sin \varphi \]
\[ Y = b - \frac{1}{2}h e^{\lambda t} \cos \varphi \]  \hspace{1cm} (4)

where:
- \( \varphi = \kappa t \) — wave length;
- \( T \) — wave period;
- \( t \) — time;
- \( h \) — wave height;
- \( a, b \) — the Lagrangian coordinates of a water particle, \( b = 0 \) at free surface;
- \( e \) — base of the natural logarithm;
- \( X, Y \) — the horizontal and vertical coordinates of water particle \((a, b)\) at time \( t \); \( X \)-axis coincides with the central line of water surface and its positive direction is opposite to that of wave propagation; \( Y \)-axis is vertical and positive upwards. It passes wave trough at \( t = 0 \).

Differentiating Eq. (4) with respect to \( t \) yields

\[ U = \frac{\partial X}{\partial t} = \frac{1}{2}h e^{\lambda t} \cos \varphi = -\sigma (Y-b) \]
\[ V = \frac{\partial Y}{\partial t} = \frac{1}{2}h e^{\lambda t} \sin \varphi = \sigma (X-a) \]  \hspace{1cm} (5)

Differentiating Eqs. (4) and (5) with respect to \( X \) and \( Y \) successively and simplifying the resulting equations gives

\[ \frac{\partial U}{\partial X} = -\frac{\partial \varphi}{\partial X} \frac{\sigma \sin \varphi}{1 - \frac{\sigma^2}{\lambda^2}} , \quad \frac{\partial U}{\partial Y} = \frac{\partial \varphi}{\partial Y} \frac{\sigma \cos \varphi}{1 - \frac{\sigma^2}{\lambda^2}} \]
\[ \frac{\partial V}{\partial X} = -\frac{\partial \varphi}{\partial X} \frac{\sigma \cos \varphi}{1 - \frac{\sigma^2}{\lambda^2}} , \quad \frac{\partial V}{\partial Y} = \frac{\partial \varphi}{\partial Y} \frac{\sigma \sin \varphi}{1 - \frac{\sigma^2}{\lambda^2}} \]  \hspace{1cm} (6)
Based on the theory of turbulent flow (9), the energy loss \( \psi \) of turbulent motion per unit time and per unit liquid volume is

\[
\psi = \sum \frac{\partial u}{\partial x} + \sum \frac{\partial v}{\partial y} + \sum \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)
\]

Substituting Eq. (6) and (8) into Eq. (9) yields

\[
\psi = \sum \frac{\partial u}{\partial x} + \sum \frac{\partial v}{\partial y} + \sum \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)
\]

If one imagines two vertical planes to be drawn at unit crest width apart, parallel to the direction of wave propagation and extended from water surface to bottom, the total turbulent dissipation \( E_\lambda \) per unit time and per wave length of the fluid between these planes is

\[
E_\lambda = \int \psi \, dx \, dy
\]

According to the rule of changing of variables (10), it follows immediately that

\[
\int \psi \, dx \, dy = \int \frac{\partial (x,y)}{\partial (a,b)} \, da \, db
\]

in which \( \left| \frac{\partial (x,y)}{\partial (a,b)} \right| \) is the functional determinant of \( x,y \) with respect to \( a,b \).

Differentiating Eq. (11) with respect to \( a \) and \( b \) successively, substituting the results into the expression of \( \left| \frac{\partial (x,y)}{\partial (a,b)} \right| \) and simplifying leads to

\[
E_\lambda = \int \int \psi \left| \frac{\partial (x,y)}{\partial (a,b)} \right| \, da \, db
\]

From Eqs. (10)-(13) it may be seen that

\[
E_\lambda = \int \int \psi \left| \frac{\partial (x,y)}{\partial (a,b)} \right| \, da \, db
\]

In two-dimensional flow, the writer suggests that

\[
E_\lambda = \int \int \psi \left| \frac{\partial (x,y)}{\partial (a,b)} \right| \, da \, db
\]

In accordance with the principle of dimensional analysis, putting

\[
\ell = \left| \frac{\partial F}{\partial x} \right| \left( \frac{\partial F}{\partial y} \right)
\]

where \( \delta \) is a dimensionless constant, one obtains

\[
\beta_1 = 1, \quad \beta_2 + \beta_3 = -1
\]
Thus

\[ L = 2 \left| \frac{\alpha F^2}{\beta Y} \left( \frac{\alpha F^2}{\beta Y} \right) \right| \]  

(16)

Differentiating Eq. (7) with respect to \( X \) and \( Y \) respectively and substituting the results into Eq. (16) gives

\[ L = \frac{\alpha (1 - \alpha)^2}{K^2 (1 + \alpha)(1 + \alpha \cos \varphi)} \left[ -\alpha \sin \varphi \right] \]  

(17)

\( \beta_2 \) is determined from the following conditions: at bottom \((\varphi = 0), L = 0\); at water surface \((b = 0), L = L \). Based on these conditions, one may put \( \alpha = -1 \). Hence,

\[ \beta_2 = \frac{\alpha^2 R^2 (1 - \alpha)^2 \sin \varphi}{K^2 (1 + \alpha)(1 + \alpha \cos \varphi)^2} \]  

(18)

Setting \( X = \lambda \) and \( 0 \) successively in the first part of Eq. (18) and subtracting yields

\[ \lambda - \alpha X = \lambda - \alpha X = \lambda \]  

Evidently, \( \alpha X = \lambda = \lambda \) is a solution satisfying Eq. (19). Utilizing this relationship and resolving rational fraction into simpler partial fractions, one gets

from Eq. (18) by integration

\[ \int_{\alpha x = \lambda}^{\alpha x = \lambda} \beta dx = \frac{\pi \alpha^2 R^2 (1 - \alpha)^2}{K^2 (1 + \alpha) \times} \]  

(20)

Substituting Eq. (20) into Eq. (14) and then integrating, neglecting the minor terms and applying the relationship \( \varphi = \alpha X \) (c is wave celerity), one finds the average dissipation \( E_T \) of wave energy due to turbulence per unit time and per unit horizontal area as follows:

\[ E_T = \frac{E}{\lambda} = \frac{4 \pi^2 R^2 (1 - \alpha)^2}{\lambda c^2 \left( \frac{\pi h}{2} \right)^2} \]  

(21)

\[ U. \text{ Experimental Results} \]

In order to find the value of the constant \( \alpha \) in Eq. (21), experiments were conducted in hydraulic laboratory. The ratio of water depths to wave lengths were controlled in these experiments in such a manner that the condition of deep water wave was fulfilled and thus the dissipation due to bottom friction may not enter.

The wave tank is 62.140 m long, 0.80m wide and 1.80m deep and has glass panels on both sides throughout its length. It is provided with an end slag mound of slope 1t0 to avoid wave reflection. Two wave gauges of resistance type were used. They were placed along the center line of the tank in measuring sections 1 and 2 which were 18.00m apart. The equation of balance of wave energy between sections 1 and 2 is

\[ E_1 - E_2 = L (E_1 + E_2 + E_3) \]  

(22)

where
$E_{1}, E_{2}$ are the average quantities of energy transmitted by waves per unit time and per unit crest width in the direction of wave propagation through sections 1 and 2 respectively.

$E_{d}$ is the rate of dissipation due to fluid viscosity, negligible in comparison with that of turbulent dissipation.

$E_{a}$ is the average dissipation per unit time and per unit horizontal area in the boundary surface between air and liquid when wave propagates in calm air.

$E_{w}$ is the average dissipation per unit time and per unit horizontal area caused by the friction of the sides walls of the wave tank.

$L$ is the horizontal distance between sections 1 and 2, $L = 18.00$ m.

In accordance with the trochoidal wave theory, it is well known that

$$E_{1} = \frac{F}{16} \left(1 - \frac{h_{1}^2}{2L}ight)$$

$$E_{2} = \frac{F}{16} \left(1 - \frac{h_{2}^2}{2L}ight)$$

(23)

in which $h_{1}$ and $h_{2}$ are the wave heights in sections 1 and 2 respectively, and $g$ is the acceleration of gravity. $E_{d}$ is calculated by means of the formula (6), which was derived on the basis of wind tunnel tests.

$$E_{d} = \frac{f_{d} \rho A}{g}$$

(24)

where $\rho$ is the density of air and $X$ is a dimensionless coefficient. As for the value of $E_{w}$, Hunt's result (II) is applied.

$$E_{w} = \frac{F}{45} \left(\frac{B^{2}}{\mu}ight)$$

(25)

where $B$ is the width of the wave tank and $\mu$ is the dynamic viscosity of water.

From Eqs. (21)-(25), the value of $\alpha^2$ can be computed with the help of the measuring data. The result of computation is shown in Table I.

From Table I, the mean value of $\alpha^2$ can be calculated to be $\bar{\alpha}^2 = 0.0376$, its standard deviation $\sigma = \sqrt{\frac{\sum (\alpha^2 - \bar{\alpha}^2)^2}{n}} = 0.0209$ and its coefficient of variation $C_V = 55.5\%$.

Substituting $\alpha^2$ into Eq. (21) yields finally

$$E_{1} = \frac{32}{105} \left(\frac{\rho}{\mu}\right)^{1} \left(1 - \frac{f_{d} h_{1}^{2}}{X}ight)$$

(26)

5. Conclusions

Three methods of studying the problem of energy dissipation of wave motion due to turbulence have been reviewed and their advantages and disadvantages briefly discussed.
<table>
<thead>
<tr>
<th>Run No.</th>
<th>Water Depth H (cm)</th>
<th>Wave Period T (sec)</th>
<th>Wave Celerity C (cm/sec)</th>
<th>Wave Length λ (cm)</th>
<th>$H + \lambda$</th>
<th>Wave Height H</th>
<th>Water Temperature t (°C)</th>
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<table>
<thead>
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<th>Run No.</th>
<th>Atmospheric Temperature t (°C)</th>
<th>Atmospheric Pressure mm Hg (Column)</th>
<th>Density of Air $\rho' \times 10^2$ (g/cm³)</th>
<th>Dynamic Viscosity of Water $\mu \times 10^2$ (cP = dyne-sec/cm²)</th>
<th>$E_1 \times 10^4$ (g-cm/sec²)</th>
<th>$E_2 \times 10^3$ (g-cm/sec²)</th>
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<td>1,578</td>
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### Table I

Experimental Results of Energy Losses due to Wave Motion

<table>
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<tr>
<th>Run No.</th>
<th>$E_I - E_2$</th>
<th>$E_a$</th>
<th>$E_w$</th>
<th>$R_T = \frac{E_I - E_2}{L}$</th>
<th>$\alpha^2$</th>
<th>$(\alpha^2 - \overline{\alpha^2})^2$ $\times 10^4$</th>
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<td>9.9</td>
<td>85.2</td>
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$\overline{\alpha^2} = 0.0376$
It seems more properly to approach the subject by making use of the theory of turbulent flow.

The importance of considering simultaneously the horizontal and vertical gradients of both the horizontal and vertical velocity fields have been pointed out. An analytical formula, i.e., the Eq. (26), has been derived theoretically, with the coefficient in it determined experimentally, which can be used to compute the rate of turbulent dissipation of wave energy.

Acknowledgements

The aid of comrades Chang Chi-chang, Tu Chin-man and Wu Tung-chang in performing the experiments is greatly appreciated.

Appendix I References

(1) Крылов, В.М., Теория и Расчет Ветровых Волн Глубокого Моря, Тр. ГОИ, вып. 26(38), 1954.
(3) Доброклонский, С. В., Турбулентная Вязкость в Поверхностном Слое Моря и Волны, Докл. АН СССР, 1947, Т. ЛУШ, №.7.
(4) Бронников, И.С., О Расчете Элементов Ветровых Волн при Переменном Ветре, Тр. ГОИ, вып. 38, 1957.
(5) Екунец, А.М., Определение Потерь Энергии Волны Земля, вызванных Действием Турбулентной и Кинематической Вязкости, Океанология, 1963, №.2
(6) Ухлебин, В.В., Теория Морских Волн, Тр. Мурского Гидрофизического Института, Т.Ж, 1956.
(7) Кочин, Н.Б., Кирьель, И.А. и Розе, Н. В., Теоретическая Гидромеханика, 1963.
Appendix 2 Notation

- $a, b$ ---- Lagrangian coordinates of liquid particles, $b = 0$ at free surface
- $B$ ---- width of wave tank
- $c$ ---- wave celerity
- $c_1, c_2$ ---- constants
- $c_v$ ---- coefficient of variation
- $D$ ---- symbol, $\left| \frac{D(x,y)}{D(a,b)} \right|$ is the functional determinant of $X, Y$ with respect to $a, b$
- $e$ ---- base of the natural logarithm

$E_f, E_g$ ---- average quantity of energy transmitted by waves per unit time and per unit crest width in the direction of wave propagation through sections 1 and 2 respectively

$E_d$ ---- average dissipation per unit time and per unit horizontal area in the boundary surface between air and liquid when wave propagates in calm air

$E_{mt}$ ---- turbulent dissipation of energy per unit time, per unit crest width and per wave length of the liquid volume extended from water surface to bottom

$E_d, E_{mt} \propto \frac{1}{\lambda}$

$E_k$ ---- rate of dissipation due to fluid viscosity

$E_w$ ---- average dissipation per unit time and per unit horizontal area caused by the friction of the side walls of a wave tank

$P$ ---- quantity defined by Eq.(3)

$g$ ---- acceleration of gravity

$h, h_1, h_2$ ---- wave heights

$K, z$ ---- wave number

$z = \frac{x}{\lambda}$

$l$ ---- mixing length of fluid particles

$L$ ---- horizontal distance between wave gauge stations

$t$ ---- time
\( T \) —— wave period
\( u \) —— component velocity in \( X \) direction
\( v \) —— component velocity in \( Y \) direction
\( X, Y \) —— Cartesian coordinates
\( \bar{X} \) —— dimensionless coefficient
\( \alpha, \beta, \gamma \beta \) —— dimensionless constants
\( \epsilon \) —— coefficient of kinematic eddy viscosity
\( \lambda \) —— wave length
\( \mu \) —— coefficient of dynamic viscosity
\( \pi = 3.142 \)
\( \rho \) —— mass density of liquid
\( \rho' \) —— mass density of air
\( \sigma \) —— radian frequency \((= \frac{2\pi}{T})\); standard deviation
\( \tau_{xy} \) —— \( Y \) component of turbulent stress acting on a surface element, the outward normal of which is parallel to \( X \)-axis
\( \tau_{xx}, \tau_{yy} \) —— similar in meaning to \( \tau_{xy} \)
\( \Phi = K\alpha + \delta t \), Eq.(4)
\( \phi \) —— energy loss of turbulent motion per unit time and per unit liquid volume