COMPUTATION OF PARTICLE PATHS USING THE LAGRANGIAN LONG WAVE EQUATIONS

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0. ABSTRACT

The Lagrangian long wave equations and the Lagrangian expressions for stress and strain are derived. Retaining the dominant terms the long wave equations are solved using an explicit finite difference method.

Using the numerical solution, particle paths are computed for the tidal motion in a basin connected to the ocean by a single inlet. At the open boundary particle displacements are described. Computations are carried out with snd without the Coriolis force and for linear and nonlinear bottom friction.

1. INTRODUCTION

With regard to the physical oceanography of estuaries and lagoons, the coastal engineer's interest traditionally has been with current velocities and tide levels. Only recently as a result of the development of water quality models knowledge of water particle trajectories have become important. The current method of computing particle trajectories is to first solve for the Eulerian velocity field and then to calculate the successive particle positions by numerical integration and interpolation.

This paper illustrates the calculation of particle paths by integrating the Lagrangian form of the long wave equations. The Lagrangian long wave equations describe the particle position as a function of its original particle position and time, whereas the Eulerian equations describe the velocity (flow) at a fixed position in space. In addition to the particle trajectories, the Lagrangian long wave equations yield the water level and depth associated with a traveling parcel of water.

The Lagrangian equations are often overlooked because of the severe nonlinearity of some of the terms, the difficulty in calibrating and verifying the results and the lack of appropriate boundary conditions. Nevertheless the technique has been successfully applied to 2-D vertical fluid flow problems.

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The general three-dimensional Lagrangian form of the Navier-Stokes equations can be found in various text books, e.g. Lamb [1932]. Neumann and Pierson [1966], Defant [1961] etc. Because a literature search revealed no previous studies aimed at specifically developing the Lagrangian form of the two-dimensional horizontal long wave equations a rather detailed derivation of these equations together with a derivation of the expressions for strain and stress is presented. Wherever possible, equations are derived starting from physical principles rather than applying a straight transformation from Eulerian to Lagrangian variables.

Past work utilizing the Lagrangian form of the fluid flow equations has been mainly in the area of surface gravity waves. Miche [1944] uses a perturbation technique to solve the Lagrangian equations for first and second order surface gravity waves. To a first order of approximation his results yield a wave profile similar to Gerstner's trochoidal wave, whereas in the Eulerian system this isn't possible until the third order of approximation. Goto [1979] and Shuto and Goto [1978] numerically computed tsunami run-up using the nonlinear 1-D long wave equations. In the equations bottom friction and viscosity was neglected. Brennen [1970] and Brennen and Whitney [1970] present a numerical solution to the problem of unsteady free surface gravity waves using the 2-D vertical Lagrangian equations.

Other studies of interest utilizing the Lagrangian equations are mainly in the field of turbulence, Pierson [1962], and the related problems of stirring, mixing and dispersion.

2. THE MOTION OF A FLUID ELEMENT; STRESS AND STRAIN

2.1 Coordinate System

Considered is a fixed Cartesian coordinate system x, y, z. The position of a particle is designated $s(a,b,c,t)$, $p(a,b,c,t)$ and $r(a,b,c,t)$ where s, p and r, and a, b and c are measured in respectively the x, y and z direction. $(a,b,c,0)$ represents the original position of the particle. In some instances it is convenient to use the particle positions $s'(a,b,c,t)$, $p'(a,b,c,t)$ and $r'(a,b,c,t)$ where $s' = s-a$, $p' = p-b$ and $r' = r-c$. The original position of the particle is then $s' = 0$, $p' = 0$ and $r' = 0$.

In the following only planar motion in the x, y plane will be considered.

2.2 Fluid Deformation

A fluid element subject to stresses undergoes deformation. For a rectangular element the deformation after a time Δt is shown in Fig. 1. In general the deformation is a combination of normal strains, shear strains and rotation.

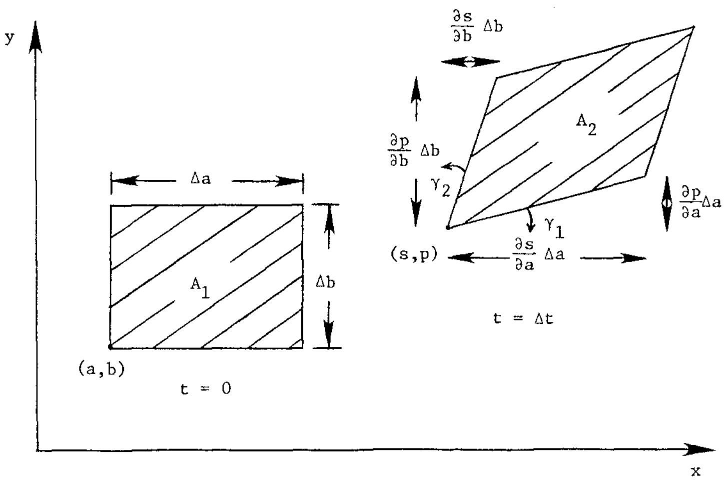


Figure 1. Deformation of a Fluid Element

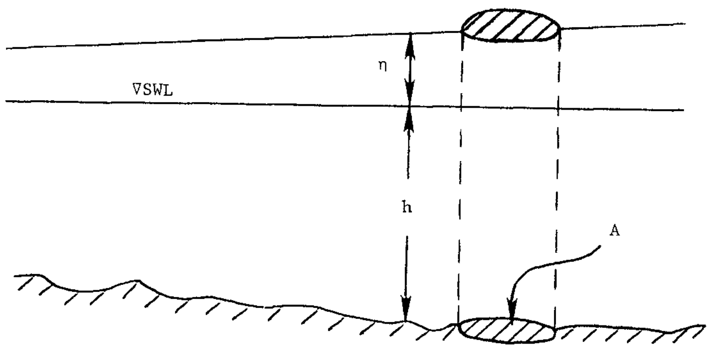


Figure 2. Cylinder of Fluid Extending from Free Surface η to the Bottom -h.

The position of the lower left hand corner of the element is designated s(a,b,t) and p(a,b,t).

2.3 Normal Strain

Normal Strain is defined as the change in length of a fluid element divided by its initial length. Accordingly, normal strain in the x direction is given by

$$\varepsilon_x\ (a,b,t) = \lim_{\substack{\Delta t \to o \\ \Delta a \to o}} \frac{\left.\frac{\partial s}{\partial a}\right|^{t+\Delta t} \Delta a - \Delta a}{\Delta a}$$

and thus

$$\varepsilon_x = \frac{\partial s}{\partial a} - 1 \tag{2.1}$$

similarly

$$\varepsilon_y = \frac{\partial p}{\partial b} - 1 \tag{2.2}$$

By convention an increase in length corresponds to a positive normal strain.

2.4 Shear Strain

Shear Strain is defined as the change in angle between two originally perpendicular lines as the element deforms. Referring to Fig. 1, for <u>small deformations</u>

$$\gamma_1 = \frac{\frac{\partial p}{\partial a}}{\frac{\partial s}{\partial a}} \simeq \frac{\partial p}{\partial a} \quad \text{since} \quad \frac{\partial s}{\partial a} = 1 + \varepsilon_x \simeq 1$$

similarly

$$\gamma_2 = \frac{\frac{\partial s}{\partial b}}{\frac{\partial p}{\partial b}} \simeq \frac{\partial s}{\partial b}$$

and thus the shear strain is

$$\gamma_{xy} = \gamma_1 + \gamma_2 = \frac{\partial p}{\partial a} + \frac{\partial s}{\partial b} \tag{2.3}$$

Shear strain is positive for a decrease in the angle between two originally perpendicular line elements.

2.5 Rotation

Rotation is defined as $\gamma_1 - \gamma_2$. Counter-clockwise rotation is taken as positive. Accordingly for small deformations the rotation in the x-y plane may be written as

$$\Theta_{xy} = \gamma_1 - \gamma_2 = \frac{\partial p}{\partial a} - \frac{\partial s}{\partial b} \tag{2.4}$$

2.6 Normal Stress

Water can be considered a Newtonian fluid where the stresses are linearly related to the time rate of strain, rather than to the strain itself as in elastic solids. Assuming an incompressible fluid, the normal stresses in the x direction are given by

$$\sigma_x = \bar{\sigma} + 2\mu \frac{\partial \epsilon_x}{\partial t} \tag{2.5}$$

where $\bar{\sigma}$ is the mean normal stress and μ is the dynamic viscosity coefficient. Based on experiments with incompressible fluids Daily and Harleman [1966], the mean normal stress $\bar{\sigma}$ is just the pressure as given below.

$$\bar{\sigma} = \frac{1}{3}(\sigma_x + \sigma_y + \sigma_z) = -p \tag{2.6}$$

Substituting for the mean normal stress in Eq. (2.5) and makinhg use of Eq. (2.1)

$$\sigma_x = -p + 2\mu \frac{\partial}{\partial a}\left(\frac{\partial s}{\partial t}\right) \tag{2.7}$$

similarly

$$\sigma_y = -p + 2\mu \frac{\partial}{\partial b}\left(\frac{\partial p}{\partial t}\right) \tag{2.8}$$

2.7 Shear Stress

The shear stress is linearly related to the time rate of change of the shear strain in the fluid.

$$\tau_{xz} = \tau_{yx} = \mu \frac{\partial}{\partial t}(\gamma_{xy}) \tag{2.9}$$

Substituting for the shear strain in Eq. (2.9) yields the shear stress

$$\tau_{xy} = \mu \frac{\partial}{\partial t}\left(\frac{\partial p}{\partial a} + \frac{\partial s}{\partial b}\right) \tag{2.10}$$

2.8 Vorticity

Vorticity is related to the time rate of change of rotation as follows.

$$\zeta = \frac{1}{2}\frac{\partial}{\partial t}(\Theta_{xy}) \qquad (2.11)$$

Substituting the expression for rotation, the vorticity is

$$\zeta = \frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial p}{\partial a} - \frac{\partial s}{\partial b}\right) \qquad (2.12)$$

which is one half times the curl of the Lagrangian velocity vector.

3. CONSERVATION EQUATIONS

3.1 General Mathematical Relations

When mapping a region A_2 in the sp-plane into a region A_1 in the ab-plane (see Fig. 1), the Jacobian operator is

$$J = \frac{\partial(s,p)}{\partial(a,b)} = \begin{vmatrix} \frac{\partial s}{\partial a} & \frac{\partial p}{\partial a} \\ \frac{\partial s}{\partial b} & \frac{\partial p}{\partial b} \end{vmatrix} \qquad (3.1)$$

The Jacobian operator is used when transforming double integrals from one coordinate system to another. Referring to Fig. 1,

$$\int_{A_2} F(s,p)dsdp = \int_{A_1} F[s(a,b,t),\ p(a,b,t)]\ J\ dadb \qquad (3.2)$$

where F is an arbitrary function. For a geometric interpretation of the Jacobian set F = 1. The left hand side of Eq. (3.2) represents the area A_2, which can be expressed as the magnitude of the cross product of the vectors representing two adjoining sides.

$$A_2 = \left| \left(\frac{\partial s}{\partial a}\Delta a\hat{i} + \frac{\partial p}{\partial a}\Delta a\hat{j}\right) \times \left(\frac{\partial s}{\partial b}\Delta b\hat{i} + \frac{\partial p}{\partial b}\Delta b\hat{j}\right) \right|$$

which reduces to

$$A_2 = J\ A_1 \qquad (3.3)$$

Other mathematical relations that will be used in the derivation of the conservation equations are the Lagrangina Del operator (∇_h) and the Lagrangian Laplacian (∇_h^2). The subscript h designates a 2-D horizontal operation. In the s-p system the Del operator is written as

$$\nabla_h = \frac{\partial}{\partial s}\hat{i} + \frac{\partial}{\partial p}\hat{j} \tag{3.4}$$

A direct transformation from the s-p space to the a-b space will be employed to find the Del operator in Lagrangian coordinates. The partial derivatives $\partial/\partial a$ and $\partial/\partial b$ are

$$\frac{\partial}{\partial a} = \frac{\partial}{\partial s} + \frac{\partial p}{\partial a}\frac{\partial}{\partial p} \tag{3.5}$$

$$\frac{\partial}{\partial b} = \frac{\partial s}{\partial b}\frac{\partial}{\partial s} + \frac{\partial p}{\partial b}\frac{\partial}{\partial p} \tag{3.6}$$

Equations (3.5) and (3.6) are solved for the partials $\partial/\partial s$ and $\partial/\partial p$

$$\frac{\partial}{\partial s} = \frac{\begin{vmatrix} \frac{\partial}{\partial a} & \frac{\partial p}{\partial a} \\ \frac{\partial}{\partial b} & \frac{\partial p}{\partial b} \end{vmatrix}}{\begin{vmatrix} \frac{\partial s}{\partial a} & \frac{\partial p}{\partial a} \\ \frac{\partial s}{\partial b} & \frac{\partial p}{\partial b} \end{vmatrix}} = \frac{\partial(\ ,p)}{\partial(a,b)}\frac{1}{J} \tag{3.7}$$

similarly

$$\frac{\partial}{\partial p} = \frac{\partial(s,\ )}{\partial(a,b)}\frac{1}{J} \tag{3.8}$$

Substituting Eqs. (3.7) and (3.8) into Eq. (3.4), the Lagrangian Del operator is obtained

$$\nabla_h = [\frac{\partial(\ ,p)}{\partial(a,b,)}\hat{i} + \frac{\partial(s,\ )}{\partial(a,b)}\hat{j}]\frac{1}{J} \tag{3.9}$$

The derivation of the Lagrangian Laplacian operator follows along the same lines as the Del operator and yields

$$\nabla_h^2 = \frac{\partial[\frac{\partial(\ ,p)}{\partial(a,b)}\frac{1}{J},p]}{\partial(a,b)} + \frac{\partial[s,\frac{\partial(s,\ )}{\partial(a,b)}\frac{1}{J}]}{\partial(a,b)}\frac{1}{J} \tag{3.10}$$

The Laplacian becomes highly nonlinear when converted to Lagrangian coordinates. A similar 3-D expression for the Lagrangian Laplacian operator is presented in Pierson [1962].

In deriving the conservation equations use will be made of the Reynolds' or Kinematic transport theorem. This theorem expresses the rate of change of a property moving with a body of fluid.

Let $s(a,b,c,t)$, $p(a,b,c,t)$ and $r(a,b,c,t)$ be the coordinates of a fluid particle, where a, b and c are the coordinates of the original position. V is the volume of the fluid body under consideration. Take $F(s,p,r,t)$ to be any function representing for example momentum or constituent concentration, and introduce the volume integral $G(s,p,r,t)$.

$$G(s,p,r,t) = \iiint_V F(s,p,r,t)\, dsdpdr \tag{3.11}$$

The Reynolds' transport theorem is used in order to find the change in G with time, that is the total derivative dG/dt.

$$\frac{dG}{dt} = \frac{d}{dt} \iiint_V F(s,p,r,t) dsdpdr \tag{3.12}$$

Since the volume is deforming with time, the order of integration and differentiation cannot be interchanged. Making use of the 3-D equivalent of Eq. (3.1), it follows from Eq. (3.12)

$$\frac{dG}{dt} = \frac{d}{dt} \iiint_{V_o} F[s(a,b,c,t),p(a,b,c,t),r(a,b,c,t),t]\, Jdadbdc \tag{3.13}$$

V_o is the volume at $t = 0$. Because V_o is constant, the order of integration and differentiation can be interchanged.

$$\frac{dG}{dt} = \iiint_{V_o} \frac{d}{dt}(FJ)dadbdc = \iiint_{V_o} \left(\frac{dF}{dt} J + F \frac{dJ}{dt}\right)dadbdc \tag{3.14}$$

It can be shown, Aris [1962] pp. 84, that

$$\frac{dJ}{dt} = [\nabla.(\frac{\partial s}{\partial t} \hat{i} + \frac{\partial p}{\partial t} \hat{j} + \frac{\partial r}{\partial t} \hat{k})]J \tag{3.15}$$

where the Del operator is the 3-D equivalent of ∇_h in Eq. (3.9). Substituting Eq. (3.15) in Eq. (3.14) yields the Reynolds' transport theorem.

$$\frac{dG}{dt} = \frac{d}{dt} \iiint_V Fdsdpdr = \iiint_{V_o} [\frac{dF}{dt} + F\nabla.(\frac{\partial s}{\partial t} \hat{i} + \frac{\partial p}{\partial t} \hat{j} + \frac{\partial r}{\partial t} \hat{k})]Jdadbdc \tag{3.16}$$

where $\partial s/\partial t$, $\partial p/\partial t$ and $\partial r/\partial t$ are the particle velocities in respectively the x, y and z directions. This result will be used repeatedly in the derivation of the conservation equations. For further reference on the Reynolds' transport theorem see Aris [1962].

3.2 Basic Assumptions

In deriving the Lagrangian form of the long wave equations the following basic assumptions will be employed.

1) Incompressible homogeneous fluid

2) Hydrostatic pressure, i.e. $\partial^2 r/\partial t^2 << g$

3) Vertical variations in horizontal velocity are negligible, i.e. planar motion

3.3 Conservation of Mass; Continuity

Consider a cylinder with volume V moving with the fluid and consisting of the same fluid particles. The surface and bottom of the cylinder are respectively $\eta(s,p,t)$ and $h(s,p)$. η and h are measured from the Still Water Level. From the assumption of planar motion it follows that the cylinder remains a cylinder even though the horizontal cross-section is allowed to deform; see Fig. 2.

Continuity implies that when moving with the fluid the volume of the cylinder remains the same.

$$V = \iint_A (h+\eta)dsdp = \text{constant} \tag{3.17}$$

where A is the cross-sectional area of the cylinder. The change in volume with time is zero. Making use of the 2-dimensional form of the Reynolds' transport theorem, Eq. (3.16), with $F = h+\eta$.

$$\frac{dV}{dt} = \iint_{A_0} \left\{ \frac{d(h+\eta)}{dt} + (h+\eta)\left[\nabla_h \cdot \left(\frac{\partial s}{\partial t}\hat{i} + \frac{\partial p}{\partial p}\hat{j}\right)\right]\right\} Jdadb = 0 \tag{3.18}$$

Because A_0 is an arbitrary area the integrand must be equal to zero. From this and Eq. (3.15) it follows

$$\frac{d}{dt}\left[(h+\eta)J\right] = 0 \tag{3.19}$$

When integraing Eq.(3.19) with respect to time from t = 0 to some later time t, the condition of continuity can be written in the form.

$$[h(a,b,t) + \eta(a,b,t)]\ J = h(a,b,0) + \eta(a,b,0) \tag{3.20}$$

It is noted that for 3-D and 2-D vertical incompressible fluid motion the continuity equation is simply J = 1, Lamb [1932], Neuman and Pierson [1966], Defant [1961], Miche [1944], Goto [1979] and Shuto and Goto [1978], where J is respectively the 3-D and 2-D equivalent of Eq. (3.1).

3.4 Conservation of Momentum; Equations of Motion

The rate of change of momentum within a material volume (cylinder) moving with the fluid equals the sum of the external forces. For the momentum equation in the x-direction.

$$\frac{d}{dt} \iint_A \rho \frac{\partial s}{\partial t} (h+\eta) ds dp = \Sigma F_x \tag{3.21}$$

Using the 2-dimensional form of the Reynolds' transport theorem, Eq. (3.16), with F replaced by $\rho(h+\eta)\ \partial s/\partial t$.

$$\frac{d}{dt} \iint_A \rho \frac{\partial s}{\partial t} (h+\eta) ds dp = \iint_{A_0} \rho(h+\eta) \frac{\partial^2 s}{\partial t^2} J da db \tag{3.22}$$

The external forces acting on the cylinder consist of the surface forces (F_s) i.e. pressure and internal stresses, bottom shear stress (F_b) and Coriolis Force (F_C). Shear stress and horizontal gradients of the normal stress at the free surface, i.e. wind stress and atmospheric pressure respectively are assumed to be zero. In the following each of the external forcing terms will be discussed separately.

The total force associated with the pressure and internal stresses in the x-direction is given by the line integral

$$F_s = \oint [\sigma_x^* \, dp - \tau_{xy}^* \, ds] \tag{3.23}$$

in which

$$\sigma_x^* = \int_{-h}^{\eta} \sigma_x dr \text{ and } \tau_{xy}^* = \int_{-h}^{\eta} \tau_{xy} dr$$

The line integral is along the intersection of the cylinder and a horizontal plane. Green's theorem in the plane may be applied to Eq. (3.23) to transform the line integral into a double integral

$$F_s = \iint_A \left[\frac{\partial \sigma_x^*}{\partial s} + \frac{\partial \tau_{xy}^*}{\partial p}\right] ds dp \tag{3.24}$$

where as before A is the cross-sectional area of the cylinder. Substituting the expressions for σ_x^* and τ_{xy}^*

$$F_s = \iint_A \left[\frac{\partial \int_{-h}^{\eta} \sigma_x dr}{\partial s} + \frac{\partial \int_{-h}^{\eta} \tau_{xy} dr}{\partial p}\right] ds dp \tag{3.25}$$

The normal stress σ_x is composed of pressure and normal stresses associated with deformations. Considering the contribution of the pressure only, Eq. (3.25) reduces to

$$F_{s1} = \iint_A \frac{\partial \int_{-h}^{\eta} -p\, dr}{\partial s}\, ds dp \tag{3.26}$$

(note the difference between the notation p for pressure and p for particle displacement in the y direction). The subscript 1 refers to the contribution of pressure to the surface force term in the x-direction. Transforming to the a-b space and making use of Eq. (3.7)

$$F_{s1} = \iint_{A_o} \left[\frac{\partial \int_{-h}^{\eta} p\, dr}{\partial a} \frac{\partial p}{\partial b} - \frac{\partial \int_{-h}^{\eta} p\, dr}{\partial b} \frac{\partial p}{\partial a}\right] dadb \tag{3.27}$$

From the assumption of hydrostatic pressure $p = \rho g r + \rho g \eta$, the pressure may be vertically integrated to yield

$$\int_{-h}^{\eta} p\, dr = \frac{1}{2} \rho g\, (h+\eta)^2 \tag{3.28}$$

Substituting in the expression for F_{s1}, the pressure force is obtained

$$F_{s1} = - \iint_{A_o} \rho g (h+\eta) \left[\frac{\partial(h+\eta)}{\partial a}\frac{\partial p}{\partial b} - \frac{\partial(h+\eta)}{\partial b}\frac{\partial p}{\partial a}\right] dadb$$

$$= \iint_{A_o} \rho g (h+\eta) \frac{\partial(h+\eta), p)}{\partial(a,b)}\, dadb \tag{3.29}$$

An attempt was made to evaluate the contribution of the viscous stresses, F_{s2}, by substituting in Eq. (3.14) the Lagrangian formulation for the normal stress σ_* and shear stress τ_{xy} respectively Eqs. (2.7) and (2.10). This resulted in a long and cumbersome expression. In order to arrive at a simpler and manageable result a direct transformation from Eulerian to Lagrangian variables is applied. Noting that f(s,p,t) is equivalent to f(x,y,t), the Eulerian expressions for the normal and shear stresses (Daily and Harleman [1966] pp. 102-104) can be written as

$$\sigma_* = 2\mu \frac{\partial(\frac{\partial s}{\partial t})}{\partial s} - p$$

$$\tau_{xy} = \mu\left[\frac{\partial(\frac{\partial s}{\partial t})}{\partial p} + \frac{\partial(\frac{\partial p}{\partial t})}{\partial s}\right]$$

Neglecting the pressure part of the normal stress term since it has already been accounted for, and substituting the expressions for the stresses in Eq. (3.25) yields

$$F_{s2} = \iint_A \left\{ 2\mu \frac{\partial\left[(h+\eta)\frac{\partial(\frac{\partial s}{\partial t})}{\partial s}\right]}{\partial s} + \mu \frac{\partial\left[(h+\eta)\frac{\partial(\frac{\partial s}{\partial t})}{\partial p}\right]}{\partial p} + \mu \frac{\partial\left[(h+\eta)\frac{\partial(\frac{\partial p}{\partial t})}{\partial s}\right]}{\partial p} \right\} ds\,dp \tag{3.30}$$

or, when rearranging

$$F_{s2} = \iint_A \left\{ \mu(h+\eta)\left[\frac{\partial^2(\frac{\partial s}{\partial t})}{\partial s^2} + \frac{\partial^2(\frac{\partial s}{\partial t})}{\partial p^2}\right] + \underbrace{\cancel{2\mu \frac{\partial(h+\eta)}{\partial s}\frac{\partial(\frac{\partial s}{\partial t})}{\partial s}}}_{\text{small}} + \underbrace{\cancel{\mu \frac{\partial(h+\eta)}{\partial p}\frac{\partial(\frac{\partial s}{\partial t})}{\partial p}}}_{\text{small}} + \underbrace{\cancel{\mu \frac{\partial(h+\eta)}{\partial p}\frac{\partial(\frac{\partial p}{\partial t})}{\partial s}}}_{\text{small}} + \underbrace{\cancel{\mu(h+\mu)\frac{\partial}{\partial s}\left[\frac{\partial(\frac{\partial s}{\partial t})}{\partial s} + \frac{\partial(\frac{\partial p}{\partial t})}{\partial p}\right]}}_{= 0 \text{ (continuity)}} \right\} ds\,dp \tag{3.31}$$

If typical values are taken for the particle velocities in long waves $[\partial s/\partial t = \eta\sqrt{g/h}\,\sin(k_x s + k_y p - \omega t)]$, it can be shown that the first term on the righthand side of Eq. (3.31) is the dominant term.

Thus,

$$F_{s2} = \iint_A \mu(h+\eta)\,\nabla_h^2\left(\frac{\partial s}{\partial t}\right) ds\,dp \tag{3.32}$$

Transforming to the a-b space

$$F_{s2} = \iint_{A_o} \mu(h+\eta)\nabla_h^2 \left(\frac{\partial s}{\partial t}\right) Jdadb \tag{3.33}$$

In the case of turbulent motion the coefficient of dynamic viscosity, μ, is replaced by the eddy viscosity or momentum transfer coefficient A_h.

In addition to stresses on the cylinder wall, the bottom of the cylinder is subjected to a shear stress. In general, the horizontal components of this stress are taken proportional to the square of the velocity. E.g. in the x-direction

$$\tau_{xz} = \rho F \frac{\partial s}{\partial t} \sqrt{\left(\frac{\partial s}{\partial t}\right)^2 + \left(\frac{\partial p}{\partial t}\right)^2}$$

where F is the friction coefficient. Integrating over the cross-section of the control volume, the bottom frictional force is

$$F_b = - \iint_{A_o} \rho F \frac{\partial s}{\partial t} \sqrt{\left(\frac{\partial s}{\partial t}\right)^2 + \left(\frac{\partial p}{\partial t}\right)^2}\, Jdadb \tag{3.34}$$

Sometimes the bottom stress is taken proportional to the velocity. In that case, the expression corresponding to Eq. (3.34) is

$$F_b = - \iint_{A_o} \rho F_\ell \frac{\partial s}{\partial t} Jdadb \tag{3.35}$$

where F_ℓ is the linear bottom friction coefficient.

Assuming the vertical velocities to be small compared to horizontal velocities the Coriolis force per unit volume can be written as

$\rho f \frac{\partial p}{\partial t}$ x-direction

$- \rho f \frac{\partial s}{\partial t}$ y-direction

where $f = 2\Omega \sin \phi$

Ω = angular velocity of earth

ϕ = latitude

Integrating the Coriolis force over the volume of the cylinder, the force acting on the control volume in the x-direction is

$$F_{cx} = \iint_{A_o} \rho(h+\eta)\, f \frac{\partial p}{\partial t} Jdadb \tag{3.36}$$

Similarly for the Coriolis force in the y-direction

$$F_{cy} = - \iint_{A_o} \rho(h+\eta)\, f \frac{\partial s}{\partial t} J da db \tag{3.37}$$

Equating the rate of change of momentum and the external forces for the x-direction, omitting the integral signs (this is justified because the integration is over an arbitrary area) and dividing by $\rho(h+\eta)$ yields

$$\left[\frac{\partial^2 s}{\partial t^2} + \frac{F}{h+\eta} \frac{\partial s}{\partial t} \sqrt{\left(\frac{\partial s}{\partial t}\right)^2 + \left(\frac{\partial p}{\partial t}\right)^2} - f \frac{\partial p}{\partial t} - \frac{A_h}{\rho} \nabla_h^2 \left(\frac{\partial s}{\partial t}\right)\right] J = - g \frac{\partial(h+\eta,\ p)}{\partial(a,b)} \tag{3.38}$$

This is the Lagrangian conservation of momentum equation for long waves in the x-direction. Similarly for the y-direction

$$\left[\frac{\partial^2 p}{\partial t^2} + \frac{F}{h+\eta} \frac{\partial p}{\partial t} \sqrt{\left(\frac{\partial s}{\partial t}\right)^2 + \left(\frac{\partial p}{\partial t}\right)^2} + f \frac{\partial s}{\partial t} - \frac{A_h}{\rho} \nabla_h^2 \left(\frac{\partial p}{\partial t}\right)\right] J = - g \frac{\partial(s,\ h+\eta)}{\partial(a,b)} \tag{3.39}$$

An alternate form of the Lagrangian equation of motion is obtained by multiplying Eq. (3.38) by $\partial s/\partial a$ and Eq. (3.39) by $\partial p/\partial a$. This yields

$$\left[\frac{\partial^2 s}{\partial t^2} + \frac{F}{h+\eta} \frac{\partial s}{\partial t} \sqrt{\left(\frac{\partial s}{\partial t}\right)^2 + \left(\frac{\partial p}{\partial t}\right)^2} - f \frac{\partial p}{\partial t} - \frac{A_h}{\rho} \nabla_h^2 \left(\frac{\partial s}{\partial t}\right)\right] \frac{\partial s}{\partial a} \tag{3.40}$$

$$+ \left[\frac{\partial^2 p}{\partial t^2} + \frac{F}{h+\eta} \frac{\partial p}{\partial t} \sqrt{\left(\frac{\partial s}{\partial t}\right)^2 + \left(\frac{\partial p}{\partial t}\right)^2} + f \frac{\partial s}{\partial t} - \frac{A_h}{\rho} \nabla_h^2 \left(\frac{\partial p}{\partial t}\right)\right] \frac{\partial p}{\partial a} = - g \frac{\partial(h+\eta)}{\partial a}$$

Similarly multiplying Eq. (3.38) and (3.39) by $\partial s/\partial b$ and $\partial p/\partial b$ respectively and adding

$$\left[\frac{\partial^2 s}{\partial t^2} + \frac{F}{h+\eta} \frac{\partial s}{\partial t} \sqrt{\left(\frac{\partial s}{\partial t}\right)^2 + \left(\frac{\partial p}{\partial t}\right)^2} - f \frac{\partial p}{\partial t} - \frac{A_h}{\rho} \nabla_h^2 \left(\frac{\partial s}{\partial t}\right)\right] \frac{\partial s}{\partial b} \tag{3.41}$$

$$+ \left[\frac{\partial^2 p}{\partial t^2} + \frac{F}{h+\eta}\frac{\partial p}{\partial t}\sqrt{\left(\frac{\partial s}{\partial t}\right)^2 + \left(\frac{\partial p}{\partial t}\right)^2} + f\frac{\partial s}{\partial t} - \frac{A_h}{\rho}\nabla_h^2\left(\frac{\partial p}{\partial t}\right)\right]\frac{\partial p}{\partial b} = - g\frac{\partial(h+\eta)}{\partial b}$$

3.5 Boundary and Initial Conditions

In order to solve the Lagrangian long wave equations, boundary conditions have to be specified. The type and number of boundary conditions that are necessary are given below.

for inviscid flow ($A_h = 0$)

at closed boundaries:	normal particle displacement is zero
at open boundaries:	particle path or water level along path is prescribed.

for viscous flow ($A_h \neq 0$)

at closed boundaries:	normal and parallel particle displacement's are zero
at open boundaries:	particle path or water level along path is prescribed

Initially the system is assumed to be at rest, i.e. the water level is at the Still Water Level and all particle displacements are taken to be zero.

4. NUMERICAL SOLUTION OF THE 2-D LONG WAVE EQUATIONS

For small deformations the non-linear terms in Eqs. (3.20), (3.40) and (3.41) can be neglected. Introducing the variables s' and p' (see section 2.1). Equations (3.20), (3.40) and (3.41) reduce to respectively

$$h\frac{\partial s'}{\partial a} + h\frac{\partial h'}{\partial b} = \eta + [h-h(a,b,0)] = 0 \qquad (4.1)$$

$$\frac{\partial^2 s'}{\partial t^2} - f\frac{\partial p'}{\partial t} + g\frac{\partial(h+\eta)}{\partial a} = -\frac{F}{h+\eta}\frac{\partial s'}{\partial t}\sqrt{\left(\frac{\partial s'}{\partial t}\right)^2 + \left(\frac{\partial p'}{\partial t}\right)^2} \qquad (4.2)$$

$$\frac{\partial^2 p'}{\partial t^2} + f\frac{\partial s'}{\partial t} + g\frac{\partial(h+\eta)}{\partial b} = -\frac{F}{h+\eta}\frac{\partial p'}{\partial t}\sqrt{\left(\frac{\partial s'}{\partial t}\right)^2 + \left(\frac{\partial p'}{\partial t}\right)^2} \qquad (4.3)$$

When assuming linear bottom friction the righthand sides of Eqs. (4.2) and (4.3) are respectively $- F_\ell\ \partial s'/\partial t/h+\eta$ and $- F_\ell\ \partial p'/\partial t/h+\eta$

The employed difference equations are

$$\eta^n_{i,j}+h^n_{i,j}[(s^n_{i,j}-s^n_{i-1,j})/\Delta a + (p^n_{i,j}-p^n_{i,j-1})/\Delta b]+h^n_{ij} = h^0_{i,j} \qquad (4.4)$$

$$\frac{1}{\Delta t^2}(s^{n+1}_{i,j} - 2s^n_{i,j}+s^{n-1}_{i,j}) - \frac{f}{4\Delta t}(p^n_{i,j}+p^n_{i+1,j}+p^n_{i,j}+p^n_{i+1,j-1}-p^{n-1}_{i,j}$$
$$- p^{n-1}_{i+1,j}-p^{n-1}_{i,j-1}-p^{n-1}_{i+1,j-1}) + \frac{F_\ell}{2^x\overline{(h+\eta)}\Delta t}(s^{n+1}_{i,j}-s^{n-1}_{i,j})$$
$$= -\frac{g}{\Delta a}[(h+\eta)^n_{i+1,j} - (h+\eta)^n_{i,j}] \qquad (4.5)$$

$$\frac{1}{\Delta t^2}(p^{n+1}_{i,j} - 2p^n_{i,j}+p^{n-1}_{i,j}) + \frac{f}{4\Delta t}(s^n_{i,j}+s^n_{i-1,j}+s^n_{i,j+1}+s^n_{i-1,j+1}-s^{n-1}_{i,j}$$
$$-s^{n-1}_{i-1,j}-s^{n-1}_{i,j+1}-s^{n-1}_{i-1,j+1}) + \frac{F_\ell}{2^y\overline{(h+\eta)}\Delta t}(p^n_{i,j}-p^{n-1}_{i,j})$$
$$= -\frac{g}{\Delta b}[(h+\eta)^n_{i,j+1}-(h+\eta)^n_{i,j}]$$

with

$$^x\overline{(h+\eta)} = \frac{(h+\eta)^n_{i,j} + (h+\eta)^n_{i+1,j}}{2}$$

$$^x\overline{(h+\eta)} = \frac{(h+\eta)^n_{i,j} + (h+\eta)^n_{i,j+1}}{2}$$

$$h^n_{i,j} = \text{function}\left[\frac{s^n_{i,j} + s^n_{i-1,j}}{2}, \frac{p^n_{i,j} + p^n_{i,j-1}}{2}\right]$$

These equations operate over a finite number of points on a spatial and temporal grid. For a nonlinear bottom stress, the difference form of the friction term in the x momentum equation is written as

$$\frac{F}{h+\eta}\frac{\partial s'}{\partial t}\sqrt{\left(\frac{\partial s'}{\partial t}\right)^2 + \left(\frac{\partial p'}{\partial t}\right)^2} = \frac{F}{2^x\overline{(h+\eta)}\Delta t}(s^{n+1}_{i,j}-s^{n-1}_{i,j})\left[\frac{1}{\Delta t^2}(s^n_{i,j}-s^{n-1}_{i,j})^2\right.$$
$$\left.+\frac{1}{16\Delta t^2}(p^n_{i,j}+p^n_{i+1,j}+p^n_{i,j-1}+p^n_{i+1,j-1}-p^{n-1}_{i,j}-p^{n-1}_{i+1,j}-p^{n-1}_{i,j-1}-p^{n-1}_{i+1,j-1})^2\right]^{1/2}$$

Given the bottom topography the depth can be found as a function of the particle position.

Computations are started from rest. First h_{ij}^{n} is computed using the known bathymetry and particle positions at $n\Delta t$. Then η_{ij}^{n} values are computed from the continuity equation, Eq. (4.4). Last the particle positions, $s_{i,j}^{n+1}$ and $p_{i,j}^{n+1}$ at the next time step are computed from the momentum equations, Eqs. (4.5) and (4.6). For an analysis of the stability of the scheme the reader is referred to Savoie and van de Kreeke [1981].

5. NUMERICALLY COMPUTED WATER PARTICLE TRAJECTORIES IN A SEMI-ENCLOSED BASIN

Considered is a semi-enclosed basin of constant depth connected to the ocean by an inlet; see Fig. 3. The values used in the computations for the length of the basin, depth of the basin, and the tidal period are respectively L = 6,400 m, h = 3 m, T = 43,300 sec. The length and width of the inlet are equal to L/2.

Assuming nonlinear bottom friction, the water motion is described by Eqs. (4.1) - (4.3). For linear bottom friction the appropriate expressions for the friction terms are substituted. The friction terms are further simplified by assuming $h+\eta \approx =$ constant.

The boundary conditions at the open boundary are given by the particle trajectory $p = 2000 \sin (2\pi t/T)$ m. This results in a tidal range of approximately 0.7 m when assuming the water level in the basin to fluctuate uniformly. The boundary conditions at the closed boundary require that a particle initially at the wall stays at the wall, that is the particle is allowed to slip but no flow is permitted through the wall.

The trajectories of the particles originally located at the position of the water levels (+ in Fig. 3), are computed for different values of the linear friction factor, the nonlinear friction factor, and the Coriolis coefficient.

The equations are solved using the explicit finite difference scheme presented in section 4, where the time step $\Delta t = 90$ sec and the space step $\Delta a = \Delta b = 800$ m, see also Fig. 3.

Examples of computed particle paths are presented in Figs. 4, 5 and 6. In these figures the particle paths starting from t = 0 (designated by the symbol x) to t = 2T are shown. For linear friction and zero Coriolis acceleration the computed particle paths are presented in Fig. 4. The particle paths are virtually straight lines.

The straight line paths can be explained as follows. When neglecting the Coriolis term and assuming linear bottom friction and $h+\eta = h =$ constant in Eqs. (4.2) and (4.3), the corresponding vorticity equation is

$$\frac{\partial \zeta}{\partial t} + \frac{F_{\ell}\zeta}{h} = 0 \tag{5.1}$$

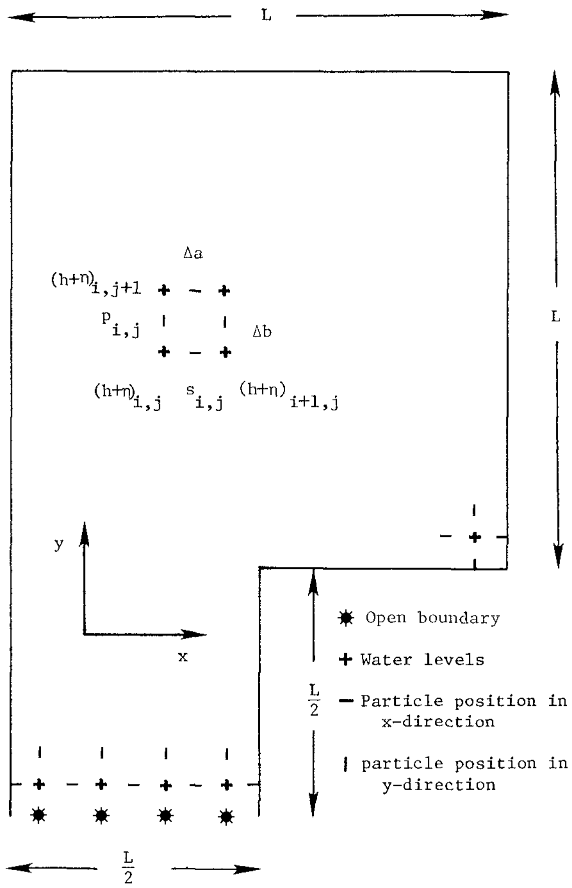
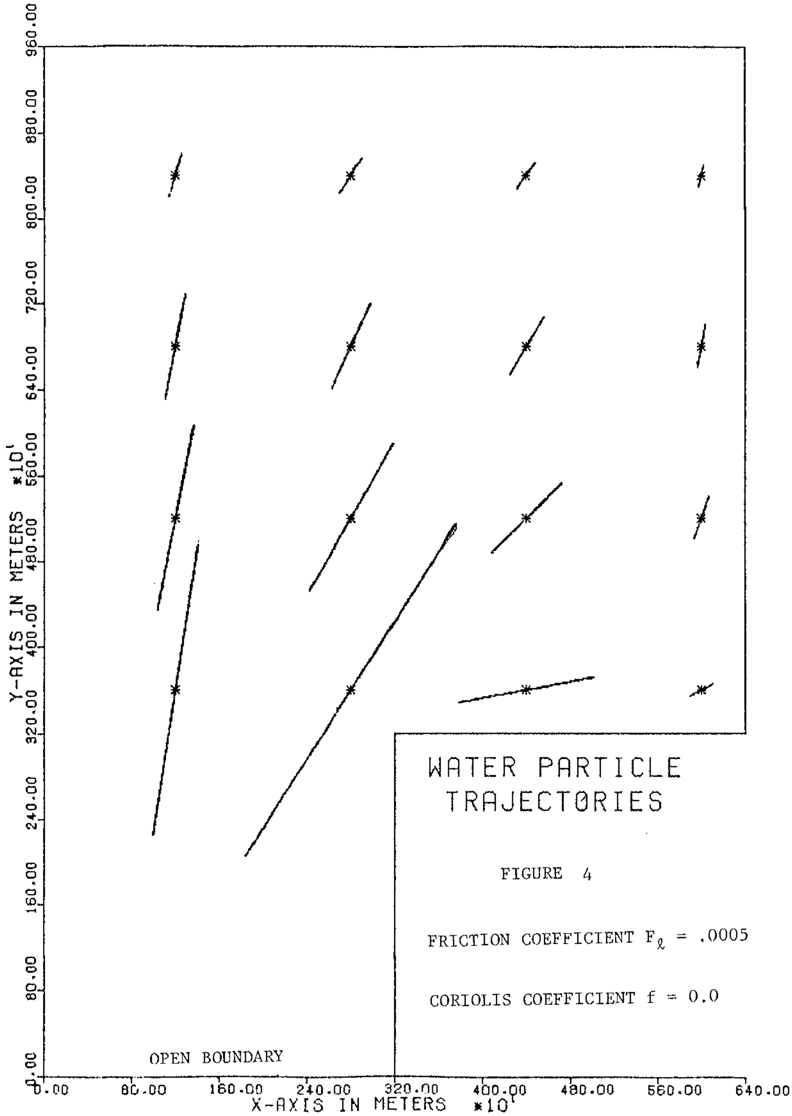


Figure 3. Configuration of Semi-enclosed Basin



WATER PARTICLE TRAJECTORIES

FIGURE 4

FRICTION COEFFICIENT F_ℓ = .0005

CORIOLIS COEFFICIENT f = 0.0

Because the basin is initially at rest, it follows from Eq. (5.1) $\zeta(t) = 0$. Thus the fluid motion is irrotational which allows introducing a (Lagrangian) velocity potential $\phi(a,b,t)$. Substituting in Eq. (4.1) yields

$$\nabla^2\phi = -\frac{1}{h}\frac{\partial\eta}{\partial t} \tag{5.2}$$

For a basin with dimensions much smaller than the wavelength

$$\frac{\partial\eta}{\partial t} \approx \frac{\omega A_c}{hA_b}\ \hat{\ell}\ \cos\omega t \tag{5.3}$$

where A_c is the cross-sectional area of the inlet, A_b is the surface area of the basin and $\hat{\ell}\ \sin\omega t$ represents the motion of the plunger. Combining Eqs. (5.2) and (5.3)

$$\nabla^2\phi = -\frac{\omega A_c}{hA_b}\ \hat{\ell}\ \cos\omega t \tag{5.4}$$

The solution to Eq. (5.4) is of the form

$$\phi(a,b,t) = \phi(a,b)\cos\omega t \tag{5.5}$$

It follows that the particle velocities

$$\frac{\partial s}{\partial t} = \frac{\partial\phi}{\partial a} \text{ and } \frac{\partial p}{\partial t} = \frac{\partial\phi}{\partial b}$$

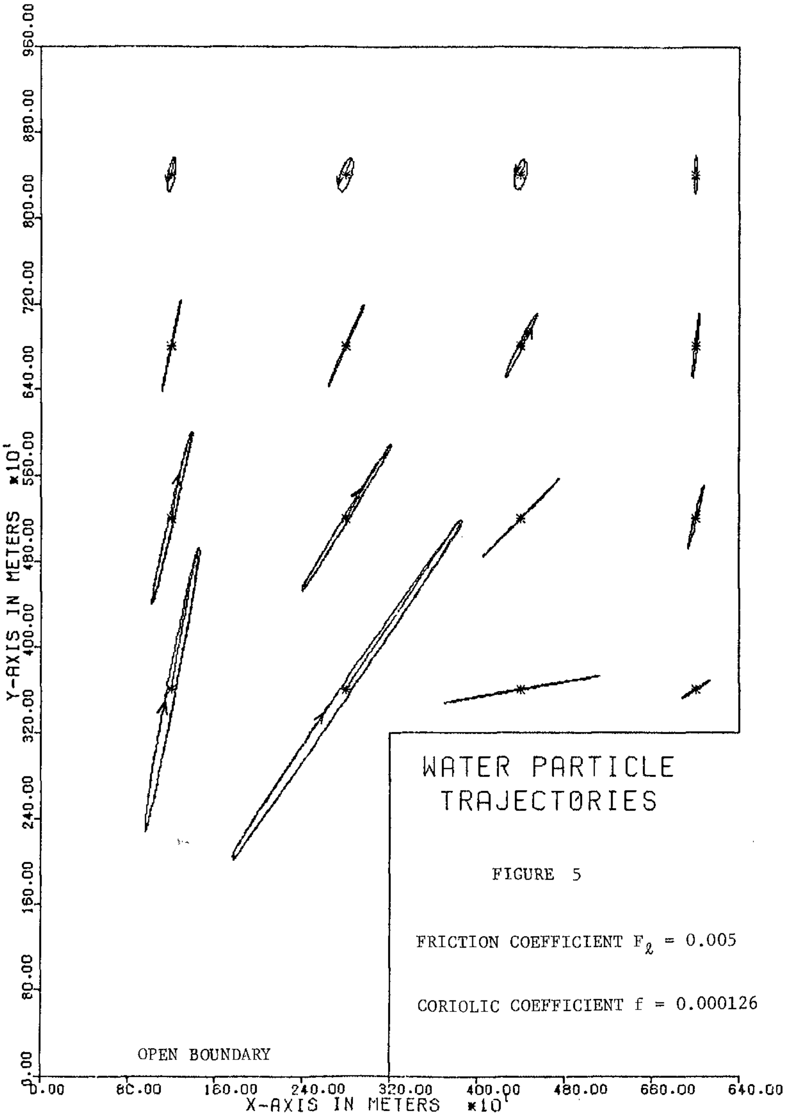
are in phase and thus the path of a particle is a straight line.

The results when the Coriolis force is added to the linear equations is presented in Fig. 5. The Coriolis acceleration causes moving particles to follow elliptical trajectories in the direction of the indicated arrows.

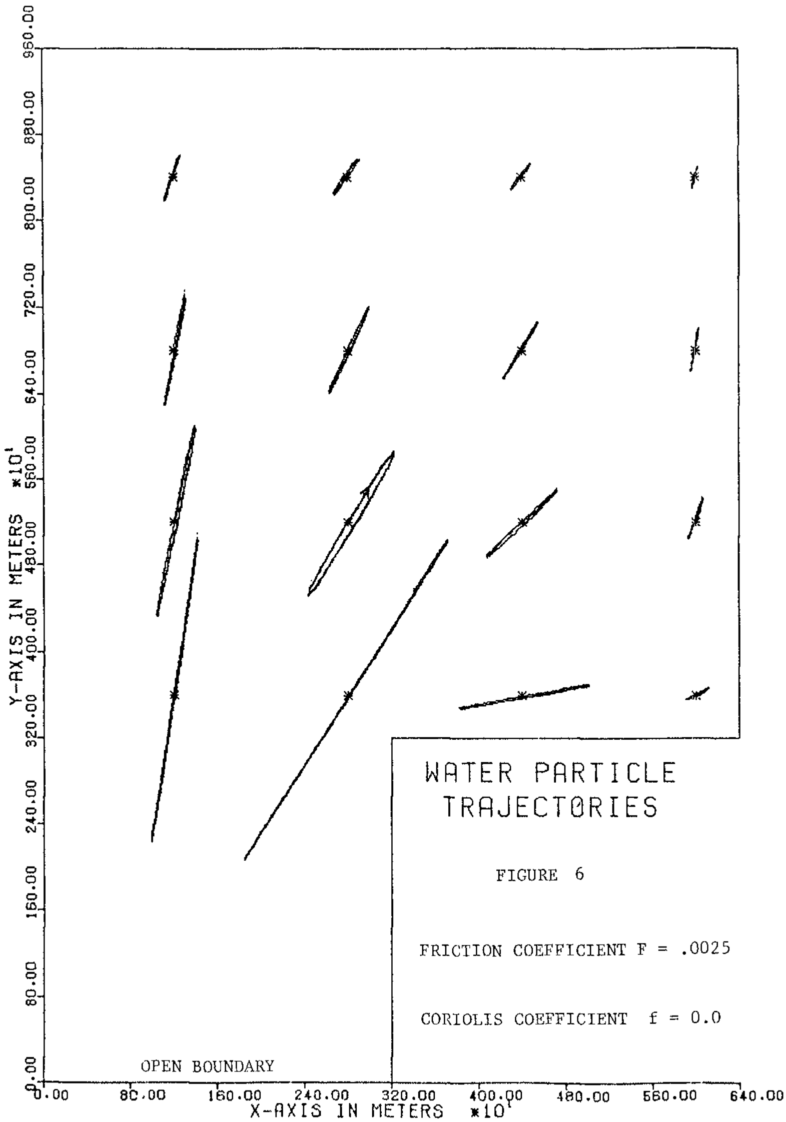
Nonlinear bottom friction is employed to compute the particle paths presented in Fig. 6. Particle paths are elliptical as opposed to the straight lines in the case of linear friction. When using linear friction particles in the basin are found to oscillate about their initial locations. For nonlinear friction particles encounter an initial displacement when the computation is first started and then oscillate about the new equilibrium position. The initial displacement can be explained by means of the equations for the mean particle displacement and mean water level.

To derive these equations the Lagrangian long wave equations, Eqs. (3.38) and (3.39) are written in the form

$$\frac{\partial^2 s'}{\partial t^2} - f\frac{\partial p'}{\partial t} + g\frac{\partial\eta}{\partial a} + \frac{F_\ell}{h}\frac{\partial s'}{\partial t} = NL_1 \tag{5.6}$$



WATER PARTICLE TRAJECTORIES
FIGURE 5
FRICTION COEFFICIENT F_ℓ = 0.005
CORIOLIC COEFFICIENT f = 0.000126
OPEN BOUNDARY
Y-AXIS IN METERS ×10¹
X-AXIS IN METERS ×10¹
0.00
80.00
160.00
240.00
320.00
400.00
480.00
560.00
640.00
720.00
800.00
880.00
960.00


WATER PARTICLE
TRAJECTORIES
FIGURE 6
FRICTION COEFFICIENT F = .0025
CORIOLIS COEFFICIENT f = 0.0
OPEN BOUNDARY
X-AXIS IN METERS *10[1]
Y-AXIS IN METERS *10[1]

$$\frac{\partial^2 p'}{\partial t^2} - f\frac{\partial s'}{\partial t} + g\frac{\partial \eta}{\partial a} + \frac{F_\ell}{h}\frac{\partial p'}{\partial t} = NL_2 \tag{5.7}$$

$$h\left[\frac{\partial s'}{\partial a} + \frac{\partial p'}{\partial b}\right] + \eta = NL_3 \tag{5.8}$$

NL_1, NL_2 and NL_3, constitute the (higher order) nonlinear terms. It is assumed that the bay bottom is horizontal, i.e. h(a,b,y) = h. The nonlinear friction is approximated by a linear friction; the difference is contained in the terms NL_1 and NL_2.

Assuming a periodic forcing at the open boundaries in terms of s', p' or η, the solution to the Eqs. (5.6) - (5.8) can be written as

$$s'(a,b,t) = \langle s'\rangle(a,b) + s_p'(a,b,t) \tag{5.9}$$

$$p'(a,b,t) = \langle p'\rangle(a,b) + p_p'(a,b,t) \tag{5.10}$$

$$\eta(a,b,t) = \langle \eta\rangle(a,b) + \eta_p(a.b,t) \tag{5.11}$$

s_p', p_p' and η_p are periodic in t with a period equal to the tidal period. $\langle\ \rangle$ denotes averaging over the tidal period. Substituting in Eqs. (5.6) - (5.8) and averaging the equations over the tidal period yields the equations for the mean particle displacement $\langle s'\rangle$, $\langle p'\rangle$ and the mean water level $\langle\eta\rangle$.

$$g\frac{\partial\langle\eta\rangle}{\partial a} = \langle NL_1\rangle \tag{5.12}$$

$$g\frac{\partial\langle\eta\rangle}{\partial b} = \langle NL_2\rangle \tag{5.13}$$

$$h\left[\frac{\partial\langle s'\rangle}{\partial a} + \frac{\partial\langle p'\rangle}{\partial b}\right] + \langle\eta\rangle = \langle NL_3\rangle \tag{5.14}$$

For the semi-enclosed basin, when using the linear equations, i.e., NL_1, NL_2 = NL_3 = 0, it follows from Eqs. (5.12) - (5.14) and the boundary condition that $\langle s\rangle = \langle p\rangle = \langle\eta\rangle = 0$ everywhere. Particles thus oscillate about their original position. When the values of NL_1, NL_2 abd NL_3 are not all zero, e.g., when using nonlinear friction, particles encounter a net displacement $\langle s'\rangle$ (a,b), $\langle p'\rangle$ (a,b) and will oscillate about this position.

6. SUMMARY AND CONCLUSIONS

A rigorous derivation of the Lagrangian long wave equations, and expressions for the deformation of a fluid parcel is presented. In the derivation no use is made of the corresponding Eulerian form of these equations and expressions. The purpose of this is to preserve a true Lagrangian approach to the problem even though in some instances it would have been easier to transform directly from the Eulerian equations.

The equations of motion and continuity are highly nonlinear in Lagrangian form. The nonlinearities are associated with the deformation of the horizontal cross-section of a traveling water column. The Lagrangian expressions for stress, strain, rotation, and vorticity remain relatively simple.

An explicit finite difference solution is presented for the simplified set of equations i.e. neglecting the nonlinear terms except for the bottom friction. The technique is applied to a square bay of constant depth connected to the ocean by a single inlet. Particle trajectories in the basin are computed for different values of the friction coefficient and Coriolis parameter. The boundary condition i.e. the particle displacement in the inlet is described by a simple harmonic function of time. For nonlinear friction the particles tend to follow clockwise elliptical trajectories, whereas for linear friction the trajectories become straight lines. The Coriolis acceleration also induces an elliptical motion. In the case of linear friction the particles in the basin oscillate about their original locations. For nonlinear friction there is an initial displacement when the computations are started, but after the first tidal cycle the particle trajectories become virtually identical.

In summary it can be stated that

1) The numerical solution to the Lagrangian long wave equations when neglecting the nonlinear terms is not more involved than the numerical solution to the Eulerian equations.

2) Aside from the difficulties in prescribing the open boundary conditions, the method of computing particle trajectories using the Lagrangian equations rather than the Eulerian equations has a clear advantage in that it bypasses the computation of the velocity field.

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