On the Variation of Characteristics of Two Wave Trains Crossing in Intermediate Depth

by

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ABSTRACT

In this paper, the authors paid attention to the non-linear interaction of two free wave trains crossing in intermediate water depth. From theoretical approaches, the velocity potential and water surface elevation have been expanded to the second-order by perturbation method. And wave height, velocity of fluid particles, pressure distribution, wave thrust and energy density of the common wave are also investigated. In order to verify the theoretical results, elaborated and numerous experiments have been performed. Some of the remarkable coincidence are obtained and the conclusions have been presented in this paper.

1. *INTRODUCTION :*

Concerning the non-linear interaction of the shortcrested waves has been obtained by Fuchs (1952) for a second-order solution. Chappelear (1961) extended this to third-order in the same manner using a formal power expansion. Hsu & Silve-

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ster (1979) have also presented the third-order solution which encompass all angles of incidence and can be extended to the limits for both standing and stokes waves. Further , the second-order Eulerian water particle velocities throughout the bottom boundary layer , mass transport of the first approximation and the limiting cases of progressive and standing waves have been obtained by Hsu , Silvester and Tsuchiya (1980). A comparision is also made with their available experimental data.

The above comprehensive program of research on short-crested waves are only applicable for the same period and height of the two component waves. However , for the non-linear interaction between pairs of intersecting gravity wave trains of arbitrary wavelength and direction on the surface in deep water has been developed by O. M. Phillips (1960). And the conclusion was made that the second-order terms give rise to Fourier components with wave numbers and frequencies formed by the sums and differences of those of primary components , and the amplitudes of these secondary components is always bounded in time and small in magnitude. But the third-order terms can give rise to tertiary components whose amplitude grows linearly with time in a resonant manner as the interactions are satisfied to the resonance loop. Furthermore , Longuet-Higgins (1962) has found that the rate of growth of the tertiary wave with time is a maximum when $\theta \doteqdot 17^\circ$; *the rate of growth with horizontal distance is a maximum when* $\theta \doteqdot 24^\circ$, *where* θ *denotes the angle between the two primary wave components.*

Based on the approaching method of Longuet-Higgins (1962) , the paper presents the second-order solutions of *velocity potential and wave profile after two wave trains crossing in intermediate water depth. Then the water particle velocity , pressure distribution , wave thrust and energy desity are also investigated. Moreover , the verification of the experiments has been performed in the laboratory.*

2. *THEORETICAL APPROACH*

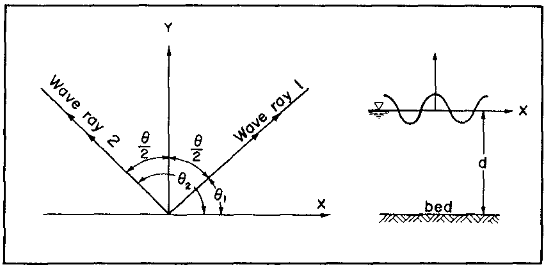

Fig. 1

Let us suppose that two free wave trains are crossing in the uniform , impervious intermediate water depth as shown in Fig. 1. *Since the presence of vorticity is very small , so that it is permissible to assume the existence of a potential function ϕ for the velocity U in an imcompressible non-viscous flow : thus*

$$U = \nabla \phi \quad , \quad \nabla^2 \phi = 0 \qquad \cdots\cdots\cdots(1)$$

And let Z be the vertical coordinate , then we have the dynamic condition at the free surface , as

$$g\eta + \phi_t + \frac{1}{2}U^2 = F(t) \qquad at\ Z = \eta \qquad \cdots\cdots\cdots(2)$$

where $F(t)$ *is Bernoulli's constant for linear case , but the nonlinear which can be expressed by Longuet-Higgins* (1953)

$$F(t) = (\frac{1}{2}\overline{\eta^2})_{tt} + \frac{1}{2}(\overline{(\nabla_h \phi)^2} - \overline{w^2}) \qquad \cdots\cdots\cdots(3)$$

Here " — " *denotes the average for wave period and*

$$\nabla_h = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$$

The kinematic condition at the free surface is

$$\eta_t - \phi_z + \phi_x \eta_x + \phi_y \eta_y = 0 \qquad at\ Z = \eta \qquad \cdots\cdots\cdots(4)$$

For notational simplicity that it is better to denote the partial differentiation by the subscript , $\phi_x = \partial\phi/\partial x$ $\phi_{tz} = \partial^2\phi/\partial t\,\partial z$ *etc. Then taking the material derivative* , D/Dt , *of eq*(2) *and subtracting* g *times eq*. (4) , *we obtain*

$$\phi_{tt} + g\phi_z + (U^2)_t + U \cdot \nabla(\frac{1}{2}U^2) = 0 \qquad at\ Z = \eta \qquad \cdots\cdots\cdots(5)$$

Now let eq. (2) *and eq*. (5) *be expanded in Taylor's series about* $Z = 0$ *to give*

$$g\eta + [\phi_t + \eta\phi_{tz} + \frac{1}{2}\eta^2\phi_{tzz} + \cdots] + [\frac{1}{2}U^2 + \eta(\frac{1}{2}U^2)_z + \cdots] = Q$$

$$at\ Z = 0 \qquad \cdots\cdots\cdots(6)$$

$$\{(\phi_{tt}+g\phi_z)+\eta(\phi_{tt}+g\phi_z)_z+\cdots\}+\{(U^2)_t+\eta(U^2)_{tz}+\cdots\}$$

$$+\{U\cdot\nabla(\frac{1}{2}U^2)+\cdots\}=0 \qquad at\ Z=0 \qquad \cdots\cdots\cdots(7)$$

Suppose that we define ϕ_{ij} , η_{ij} , U_{ij} *and* Q_{ij} *by the formal expressions* :

$$\left.\begin{aligned}\phi&=(\alpha\phi_{10}+\beta\phi_{01})+(\alpha^2\phi_{20}+\alpha\beta\phi_{11}+\beta^2\phi_{02})+\cdots\cdots\\ \eta&=(\alpha\eta_{10}+\beta\eta_{01})+(\alpha^2\eta_{20}+\alpha\beta\eta_{11}+\beta^2\eta_{02})+\cdots\cdots\\ U&=(\alpha U_{10}+\beta U_{01})+(\alpha^2U_{20}+\alpha\beta U_{11}+\beta^2U_{02})+\cdots\cdots\\ Q&=(\alpha Q_{10}+\beta Q_{01})+(\alpha^2Q_{20}+\alpha\beta Q_{11}+\beta^2Q_{02})+\cdots\cdots\end{aligned}\right\}\cdots\cdots(8)$$

where α , β *are to be small , independent and proportional to the surface slopes ,* $\alpha\phi_{10}$ *and* $\beta\phi_{01}$ *represent the first approximation of two crossing wave trains. The remaining terms represent wave interaction. Then substitute eq.* (8) *in eq.* (1) , (6) , (7) , *that we obtain* :

$0(\alpha)$:

$$\nabla^2\phi_{10}=0 \qquad \cdots\cdots\cdots(9)$$

$$g\eta_{10}+\phi_{10t}=Q_{10} \qquad at\ Z=0 \qquad \cdots\cdots\cdots(10)$$

$$\phi_{10tt}+g\phi_{10z}=0 \qquad at\ Z=0 \qquad \cdots\cdots\cdots(11)$$

$$\phi_{10z}=0 \qquad at\ Z=-d \qquad \cdots\cdots\cdots(12)$$

$0(\alpha^2)$:

$$\nabla^2 \phi_{20} = 0 \qquad \cdots\cdots\cdots(13)$$

$$g\eta_{20} + \phi_{20t} + \eta_{10}\phi_{10tz} + \frac{1}{2}U_{10}^2 = Q_{20} \quad at\ Z = 0 \qquad \cdots\cdots\cdots(14)$$

$$\phi_{20tt} + g\phi_{20z} + \eta_{10}(\phi_{10tt} + g\phi_{10z})_z + (U_{10}^2)_t = 0 \quad at\ Z = 0 \cdots\cdots(15)$$

$$\phi_{20z} = 0 \qquad at\ Z = -d \qquad \cdots\cdots\cdots(16)$$

$0(\alpha\beta)$:

$$\nabla^2 \phi_{11} = 0 \qquad \cdots\cdots\cdots(17)$$

$$g\eta_{11} + \phi_{11t} + \eta_{10}\phi_{01tz} + \eta_{01}\phi_{10tz} + U_{10}\cdot U_{01} = Q_{11} \quad at\ Z = 0 \qquad \cdots\cdots\cdots(18)$$

$$\phi_{11tt} + g\phi_{11z} + \eta_{10}(\phi_{01tt} + g\phi_{01z})_z + \eta_{01}(\phi_{10tt} + g\phi_{10z})_z$$
$$+ 2(U_{10}\cdot U_{01})_t = 0 \qquad at\ Z = 0 \qquad \cdots\cdots\cdots(19)$$

$$\phi_{11z} = 0 \qquad at\ Z = -d \qquad \cdots\cdots\cdots(20)$$

And the terms of $0(\beta)$, $0(\beta^2)$ *are the same expressions as* $0(\alpha)$, $0(\alpha^2)$ *but the subscript of* 10 , 20 *from eq.* (9) *through eq.* (16) *will be changed by* o1 , o2. *If we assume that the water is intermediate for the first-and second-order waves that the solutions will be obtained as* :

$$\phi_{10} = A_1 \cosh k_1(z+d) \sin\varphi_1 \qquad \cdots\cdots\cdots(21)$$

$$\eta_{10} = \frac{\sigma_1 A_1}{g} \cosh k_1 d \cos\varphi_1 \qquad \cdots\cdots\cdots(22)$$

$$\sigma_1^2 = g k_1 \ tanh \ k_1 d \qquad \cdots\cdots\cdots(23)$$

$$\phi_{20} = \frac{3}{8} \frac{k_1^2 A_1^2}{\sigma_1 sinh^2 k_1 d} cosh\, 2 k_1 (z+d) \, sin\, 2\varphi_1 \qquad \cdots\cdots\cdots(24)$$

$$\eta_{20} = \frac{k_1^2 A_1^2}{4g} \frac{cosh^2 k_1 d \,(cosh\, 2 k_1 d + 2)}{sinh^2 k_1 d} \ cos\, 2\varphi_1 \qquad \cdots\cdots\cdots(25)$$

and

$$\phi_{01} = B_1 \, cosh\, k_2 (z+d) \, sin\varphi_2 \qquad \cdots\cdots\cdots(26)$$

$$\eta_{01} = \frac{\sigma_2 B_1}{g} \ cosh\, k_2 d \ cos\, \varphi_2 \qquad \cdots\cdots\cdots(27)$$

$$\sigma_2^2 = g k_2 \ tanh \ k_2 d \qquad \cdots\cdots\cdots(28)$$

$$\phi_{02} = \frac{3}{8} \frac{k_2^2 B_1^2}{\sigma_2 sinh^2 k_2 d} cosh\, 2 k_2 (z+d) \, sin\, 2\varphi_2 \qquad \cdots\cdots\cdots(29)$$

$$\eta_{02} = \frac{k_1^2 B_1^2}{4g} \frac{cosh^2 k_2 d \,(cosh^2 k_2 d + 2)}{sinh^2 k_2 d} \ cos\, 2\varphi_2 \qquad \cdots\cdots\cdots(30)$$

for convenience , where

$$\varphi_1 = k_1 x - \sigma_1 t \ \ and \ \ \varphi_2 = k_2 x - \sigma_2 t$$

Combining the solutions of the first-order from eq. (17) *through eq.* (20) *, then* ϕ_{11} *and* η_{11} *of the nonlinear term will also be found out.*

$$\phi_{11} = A_1 B_1 k_1 k_2 \left[\frac{F_1}{F_3} \cosh k' (z+d) \sin(\varphi_1 + \varphi_2) \right.$$

$$\left. + \frac{F_2}{F_4} \cosh k'' (z+d) \sin(\varphi_1 - \varphi_2) \right] \qquad \cdots\cdots\cdots(31)$$

$$\eta_{11} = \frac{1}{g} A_1 B_1 k_1 k_2 \left\{ \left[(\sigma_1 + \sigma_2) \frac{F_1}{F_3} \cosh k' d + \frac{1}{4} G \right. \right.$$

$$\cdot \cosh(k_1 + k_2) d - \frac{1}{4} H \cosh(k_1 - k_2) d \Big] \cos(\varphi_1 + \varphi_2)$$

$$+ \Big[(\sigma_1 - \sigma_2) \frac{F_2}{F_4} \cosh k'' d - \frac{1}{4} H \cosh(k_1 - k_2) d$$

$$\left. \left. + \frac{1}{4} G \cosh(k_1 + k_2) d \right] \cos(\varphi_1 - \varphi_2) \right\} \qquad \cdots\cdots\cdots(32)$$

where

$$F_1 = \frac{1}{2} \left(\frac{\sigma_1^2}{\sigma_2} \frac{\sinh k_2 d}{\sinh k_1 d} + \frac{\sigma_2^2}{\sigma_1} \frac{\sinh k_1 d}{\sinh k_2 d} \right) + 2(\sigma_1 + \sigma_2)$$

$$\left(\frac{\cos\theta}{2} \cosh k_1 d \cosh k_2 d - \frac{1}{2} \sinh k_1 d \sinh k_2 d \right)$$

$$F_2 = \frac{1}{2} \left(\frac{\sigma_1^2}{\sigma_2} \frac{\sinh k_2 d}{\sinh k_1 d} - \frac{\sigma_2^2}{\sigma_1} \frac{\sinh k_1 d}{\sinh k_2 d} \right) + 2(\sigma_1 - \sigma_2)$$

$$\left(\frac{\cos\theta}{2} \cosh k_1 d \cosh k_2 d + \frac{1}{2} \sinh k_1 d \sinh k_2 d \right)$$

$$F_3 = (\sigma_1 + \sigma_2)^2 \cosh k' d - g k' \sinh k' d$$

$$F_4 = (\sigma_1 - \sigma_2)^2 \cosh k'' d - g k'' \sinh k'' d$$

$$G = \frac{(\sigma_1 + \sigma_2)^2}{\sigma_1 \sigma_2} - 2\cos^2\frac{\theta}{2}$$

$$H = \frac{(\sigma_1 + \sigma_2)^2}{\sigma_1 \sigma_2} - 2\sin^2\frac{\theta}{2}$$

$$k' = |K_1 + K_2| \;,\; k'' = |K_1 - K_2|$$

$$\theta = |\theta_1 - \theta_2|$$

If a_1 , a_2 *denotes the amplitude of two primary wave components respectively. Then both of the velocity potential and water surface elevation function of the second-order solution of the common wave will be given by*

$$\phi = \frac{a_1 g}{\sigma_1}\frac{\cosh k_1(z+d)}{\cosh k_1 d}\sin\varphi_1 + \frac{a_2 g}{\sigma_2}\frac{\cosh k_2(z+d)}{\cosh k_2 d}\sin\varphi_2$$

$$+\frac{3}{8}a_1^2\sigma_1\frac{\cosh 2k_1(z+d)}{\sinh^4 k_1 d}\sin 2\varphi_1 +$$

$$+\frac{3}{8}a_2^2\sigma_2\frac{\cosh 2k_2(z+d)}{\sinh^4 k_2 d}\sin 2\varphi_2 + \frac{a_1 a_2 \sigma_1 \sigma_2}{\sinh k_1 d \sinh k_2 d}$$

$$\cdot\left[\frac{F_1}{F_3}\cosh k'(z+d)\cdot\sin(\varphi_1+\varphi_2)\right.$$

$$\left.+\frac{F_2}{F_4}\cosh k''(z+d)\sin(\varphi_1-\varphi_2)\right] \qquad \cdots\cdots\cdots(33)$$

$$\eta = a_1 \cos\varphi_1 + a_2 \cos\varphi_2 + \frac{a_1^2 k_1}{4} \frac{\cosh k_1 d}{\sinh^3 k_1 d} (\cosh 2 k_1 d + 2)$$

$$\cos 2\varphi_1 + \frac{a_2^2 k_2}{4} \frac{\cosh k_2 d}{\sinh^3 k_2 d} (\cosh 2 k_2 d + 2) \cos 2\varphi_2$$

$$+ \frac{a_1 a_2 \sigma_1 \sigma_2}{\sinh k_1 d \sinh k_2 d} \cdot \{ [(\sigma_1 + \sigma_2) \frac{F_1}{F_3} \cosh k' d +$$

$$+ \frac{1}{4} G \cosh (k_1 + k_2) d - \frac{1}{4} H \cosh (k_1 - k_2) d]$$

$$\cdot \cos (\varphi_1 + \varphi_2) + [(\sigma_1 - \sigma_2) \frac{F_2}{F_4} \cosh k'' d$$

$$- \frac{1}{4} H \cosh (k_1 - k_2) d + \frac{1}{4} G \cosh (k_1 + k_2) d] \cos (\varphi_1 - \varphi_2) \} \qquad \cdots\cdots\cdots (34)$$

From above two equations, examples of water surface elevation after two wave trains crossing are graphed in Fig. 2. and Fig. 3. In Fig. 2, curve A is the first-order solution of common wave, curve B is the solution of Stokes wave which obtained from the front of four terms on the left side of eq. (34), and curve C is the solution of eq. (34), also representing the condition of short-crested wave, as noted by Hsu et al (1979). In Fig. 3, the dash-line and solid-line denotes the solution of first-order and second-order respectively, and it is evident that the period of the common wave is equal to the least common multiple of the two primary wave components.

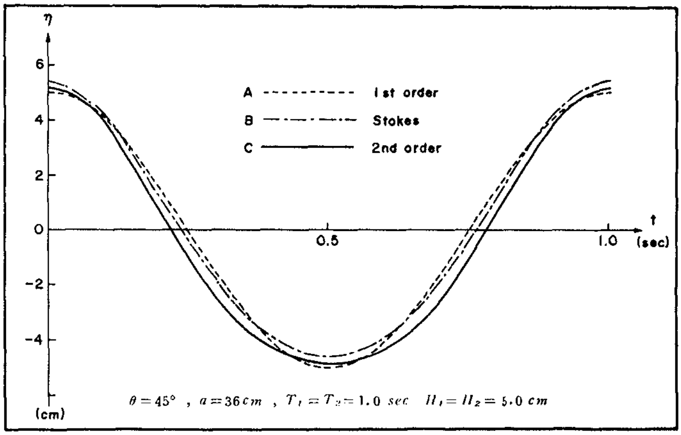
η
6
4
2
0
-2
-4
(cm)
A 1st order
B Stokes
C 2nd order
0.5
1.0
t
(sec)
θ = 45° , a = 36 cm , T₁ = T₂ = 1.0 sec H₁ = H₂ = 5.0 cm

Fig. 2

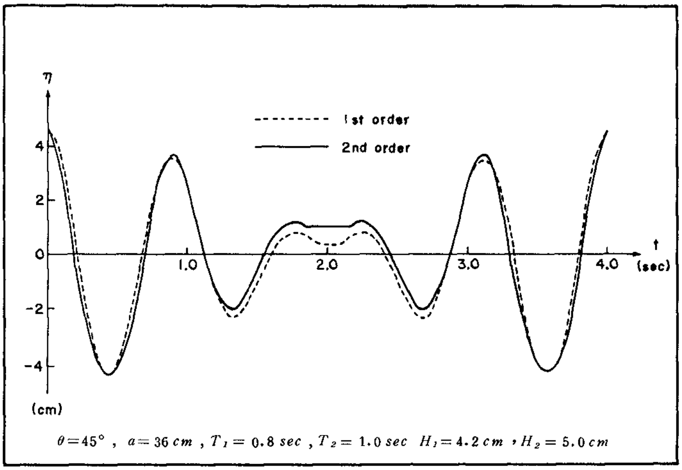
η
4
2
0
-2
-4
(cm)
1st order
2nd order
1.0
2.0
3.0
4.0
t
(sec)
θ = 45° , a = 36 cm , T₁ = 0.8 sec , T₂ = 1.0 sec H₁ = 4.2 cm , H₂ = 5.0 cm

Fig. 3

Based on the velocity potential and water surface elevation funtion, the water particle velocity, pressure distribution, wave thrust and energy density can also be investigated as the following form:

(1) *water particle velocity:*

After two wave trains crossing, the components of water particle velocity (u , v , w) for the second-order can be expressed as

$$u = u_1 \cos\varphi_1 + u_2 \cos\varphi_2 + u_3 \cos 2\varphi_1 + u_4 \cos 2\varphi_2 + u_5 \cos(\varphi_1+\varphi_2) + u_6 \cos(\varphi_1-\varphi_2) \quad \cdots\cdots\cdots(35)$$

$$v = v_1 \cos\varphi_1 + v_2 \cos\varphi_2 + v_3 \cos 2\varphi_1 + v_4 \cos 2\varphi_2 + v_5 \cos(\varphi_1+\varphi_2) + v_6 \cos(\varphi_1-\varphi_2) \quad \cdots\cdots\cdots(36)$$

$$w = w_1 \cos\varphi_1 + w_2 \cos\varphi_2 + w_3 \cos 2\varphi_1 + w_4 \cos 2\varphi_2 + w_5 \cos(\varphi_1+\varphi_2) + w_6 \cos(\varphi_1-\varphi_2) \quad \cdots\cdots\cdots(37)$$

from the above three equations, we have

$$(u_1, v_1, w_1) = \frac{a_1\sigma_1}{\sinh k_1 d}\left(\cos\theta_1 \cosh k_1(z+d),\ \sin\theta_1 \cosh k_1(z+d),\ \sinh k_1(z+d)\right)$$

$$(u_2, v_2, w_2) = \frac{a_2\sigma_2}{\sinh k_2 d}\left(\cos\theta_2 \cosh k_2(z+d),\ \sin\theta_2 \cosh k_2(z+d),\ \sinh k_2(z+d)\right)$$

$$(u_3, v_3, w_3) = \frac{3a_1^2\sigma_1 k_1}{4\sinh k_1 d}(\cos\theta_1 \cosh 2k_1(z+d), \sin\theta_1$$

$$\cosh 2k_1(z+d), \sinh 2k_1(z+d))$$

$$(u_4, v_4, w_4) = \frac{3a_2^2\sigma_2 k_2}{4\sinh^4 k_2 d}\{\cos\theta_2 \cosh 2k_2(z+d),$$

$$\sin\theta_2 \cosh 2k_2(z+d), \sinh 2k_2(z+d)\}$$

$$(u_5, v_5, w_5) = \frac{a_1 a_2 \sigma_1 \sigma_2}{\sinh k_1 d \sinh k_2 d}\frac{F_1}{F_3}k'\{\cos\alpha_1 \cosh k_1(z+d),$$

$$\sin\alpha_1 \cosh k'(z+d), \sin k'(z+d)\}$$

$$(u_6, v_6, w_6) = \frac{a_1 a_2 \sigma_1 \sigma_2}{\sinh k_1 d \sinh k_2 d}\frac{F_2}{F_4}k''\{\cos\alpha_2 \cosh k''(z+d),$$

$$\sin\alpha_2 \cosh k''(z+d), \sinh k''(z+d)\}$$

where, α_1 *is the angle between the vector of* $(k_1 + k_2)$ *and* x *axis*, α_2 *is the angle between* $(k_1 - k_2)$ *and* x *axis.*

$$\alpha_1 = arc\ tan\left(\frac{k_1 \sin\theta_1 + k_2 \sin\theta_2}{k_1 \cos\theta_1 + k_2 \sin\theta_2}\right)$$

$$\alpha_2 = arc\ tan\left(\frac{k_1 \sin\theta_1 - k_2 \sin\theta_2}{k_1 \cos\theta_1 - k_2 \cos\theta_2}\right)$$

(2) *pressure distribution* :

From Bernoulli's equation, the pressure can be expressured by the following form :

$$\frac{P}{\rho} = -gz - \phi_t - \frac{1}{2}u^2 + F(t) \qquad \cdots\cdots\cdots(38)$$

from the perturbation method , and assume that

$$\frac{P}{\rho} + gz = (\alpha P_{10} + \beta P_{01}) + (\alpha^2 P_{20} + \alpha\beta P_{11} + \beta^2 P_{02} +) \cdots \qquad \cdots\cdots\cdots(39)$$

$$F(t) = (\alpha F_{10} + \beta F_{01}) + (\alpha^2 F_{20} + \alpha\beta F_{11} + \beta^2 F_{02}) + \cdots \qquad \cdots\cdots\cdots(40)$$

then substitute eq (39) , (40) *and eq* (8) *in eq* (38) , *that we have* :

$$O(\alpha) \ : P_{10} = -\phi_{10t} + F_{10} \qquad \cdots\cdots\cdots(41)$$

$$O(\alpha^2) \ : P_{20} = -\phi_{20t} - \frac{1}{2}\boldsymbol{u}_{10}{}^2 + F_{20} \qquad \cdots\cdots\cdots(42)$$

$$O(\alpha\beta) \ : P_{11} = -\phi_{11t} - \boldsymbol{u}_{10} \cdot \boldsymbol{u}_{01} + F_{11} \qquad \cdots\cdots\cdots(43)$$

As for $O(\beta)$, $O(\beta^2)$ *which corresponds to* $O(\alpha)$, $O(\alpha^2)$ *but the subcripts* 10 , 20 , *will be changed into* 01 , 02 *respectively. Since* $F_{10} = F_{01} = 0$, $F_{20} = \frac{1}{4}A_1^2 k_1^2$, $F_{02} = \frac{1}{4}B_1^2 k_1^2$ *and* $F_{11} = 0$, *that the pressure after two wave trains crossing will be given by* :

$$\frac{P}{\rho} = -gz + a_1 g \frac{\cosh k_1(z+d)}{\cosh k_1 d} \cos\varphi_1 + a_2 g \frac{\cosh k_2(z+d)}{\cosh k_2 d} \cos\varphi_2$$

$$+ \frac{a_1^2 g k_1}{2\sinh 2k_1 d} \{ [\frac{3\cosh 2k_1(z+d)}{\sinh^2 k_1 d} - 1] \cos 2\varphi_1 +$$

$$+\left[1-\cosh 2k_1(z+d)\right]\}$$

$$+\frac{a_2^2 g k_2}{2\sinh 2k_2 d}\left\{\left[\frac{3\cosh 2k_2(z+d)}{\sinh^2 k_2 d}-1\right]\cos 2\varphi_2\right.$$

$$+\left[1-\cosh 2k_2(z+d)\right]\}$$

$$+\frac{a_1 a_2 \sigma_1 \sigma_2}{\sinh k_1 d \sinh k_2 d}\{\left[(\sigma_1+\sigma_2)\frac{F_1}{F_3}\cosh k'(z+d)\right.$$

$$-\frac{\cos\theta}{2}\cosh k_1(z+d)\cosh k_2(z+d)$$

$$\left.+\frac{1}{2}\sinh k_1(z+d)\sinh k_2(z+d)\right]\cos(\varphi_1+\varphi_2)$$

$$+\left[(\sigma_1-\sigma_2)\frac{F_2}{F_4}\cosh k''(z+d)-\frac{\cos\theta}{2}\cosh k_1(z+d)\right.$$

$$\left.\cdot\cosh k_2(z+d)-\frac{1}{2}\sinh k_1(z+d)\sinh k_2(z+d)\right]$$

$$\cdot\cos(\varphi_1-\varphi_2)\} \qquad \cdots\cdots\cdots(44)$$

(3) *wave thrust*

From Longuet-Higgins & Stewart (1964), *wave thrust will be written as the following form*

$$S_{ij}=\overline{\int_{-d}^{\eta}(P\delta_{ij}+\rho\mu_i\mu_j)\,dz}+\int_{-d}^{0}\rho g z\,dz \qquad \cdots\cdots\cdots(45)$$

where " $-$ " *denotes the average of common wave period, and* δ_{ij} *is Kronecker's delta. For the second-order approximation, then the above equation can be simplified as*

$$S_{ij} = \frac{1}{2}\rho g \overline{\eta^2}\delta_{ij} + \rho\int_{-d}^{0} (\overline{\mu_i \mu_j} - \overline{w^2}\delta_{ij}) \, dz \qquad \cdots\cdots\cdots (46)$$

thus :

$$S_{xx} = \frac{1}{2}\rho g \overline{\eta^2} + \rho\int_{-d}^{0} (\overline{u^2} - \overline{w^2}) \, dz \qquad \cdots\cdots\cdots (47)$$

$$S_{yy} = \frac{1}{2}\rho g \overline{\eta^2} + \rho\int_{-d}^{0} (\overline{v^2} - \overline{w^2}) \, dz \qquad \cdots\cdots\cdots (48)$$

$$S_{xy} = S_{yx} = \rho\int_{-d}^{0} \overline{uv} \, dz \qquad \cdots\cdots\cdots (49)$$

where S_{xx} , S_{yy} *represents the flux of horizontal momentum parallel to the* x , y *axis and* S_{xy} , S_{yx} *represents the flux of* $x-$, $y-$ *momentum across the plane* $y =$ *constant and* $x =$ *constant respectively. Then suppose* $E_1 = \frac{1}{2}\rho g a_1^2$, $E_2 =$ $E_2 = \frac{1}{2}\rho g a_2^2$, *that the components of wave thrust of the common wave will be given by*

$$S_{xx} = E_1 \{ \cos^2\theta_1 (\frac{1}{2} + \frac{k_1 d}{\sinh 2 k_1 d}) + \frac{k_1 d}{\sinh 2 k_1 d} \}$$

$$+ E_2 \{ \cos^2\theta_2 (\frac{1}{2} + \frac{k_2 d}{\sinh 2 k_2 d}) + \frac{k_2 d}{\sinh 2 k_2 d} \}$$

$$+ \frac{8 E_1 E_2 k_1 k_2}{\rho \sinh 2k_1 d \sinh 2 k_2 d} \{ \frac{F_1^2}{F_3^2} k'^2 [\cos^2\alpha_1 (\frac{\sinh 2k'd}{4 k'} + \frac{d}{2})$$

$$-\left(\frac{\sinh 2k'd}{4k'}-\frac{d}{2}\right)\Big]+\frac{F_2^2}{F_4^2}k''^2\Big[\cos^2\alpha_2\left(\frac{\sinh 2k''d}{4k''}+\frac{d}{2}\right)$$

$$-\left(\frac{\sinh 2k''d}{4k''}-\frac{d}{2}\right)\Big]+\frac{1}{2g}\Big[(\sigma_1+\sigma_2)\frac{F_1}{F_3}\cos k'd$$

$$+\frac{1}{4}G\cosh(k_1+k_2)d-\frac{1}{4}H\cosh(k_1-k_2)d\Big]^2$$

$$+\frac{1}{2g}\Big[(\sigma_1-\sigma_2)\frac{F_2}{F_4}\cosh k''d-\frac{1}{4}H\cosh(k_1-k_2)d$$

$$+\frac{1}{4}G\cosh(k_1+k_2)d\Big]^2\Big\} \qquad \cdots\cdots\cdots(50)$$

$$S_{yy}=E_1\left\{\sin^2\theta_1\left(\frac{1}{2}+\frac{k_1d}{\sinh 2k_1d}\right)+\frac{k_1d}{\sinh 2k_1d}\right\}$$

$$+E_2\left\{\sin^2\theta_2\left(\frac{1}{2}+\frac{k_2d}{\sinh 2k_2d}\right)+\frac{k_2d}{\sinh 2k_2d}\right\}$$

$$+\frac{8E_1E_2k_1k_2}{\rho\sinh 2k_1d\sinh 2k_2d}\Big\{\frac{F_1^2}{F_3^2}k'^2\Big[\sin^2\alpha_1\left(\frac{\sinh 2k'd}{4k'}+\frac{d}{2}\right)$$

$$-\left(\frac{\sinh 2k'd}{4k'}-\frac{d}{2}\right)\Big]+\frac{F_2^2}{F_4^2}k''^2\Big[\sin^2\alpha_2\left(\frac{\sinh 2k''d}{4k''}+\frac{d}{2}\right)$$

$$-\left(\frac{\sinh 2k''d}{4k''}-\frac{d}{2}\right)\Big]+\frac{1}{2g}\Big[(\sigma_1+\sigma_2)\frac{F_1}{F_3}\cosh k'd$$

$$+\frac{1}{4}G\cosh(k_1+k_2)d-\frac{1}{4}H\cosh(k_1-k_2)d\Big]^2$$

$$+\frac{1}{2g}\Big[(\sigma_1-\sigma_2)\frac{F_2}{F_4}\cosh k''d-\frac{1}{4}H\cosh(k_1-k_2)d$$

$$+\frac{1}{4}G\cosh(k_1+k_2)d\Big]^2\Big\} \qquad \cdots\cdots\cdots(51)$$

$$S_{xy} = S_{yx} = E_1 \{ \sin 2\theta_1 (\frac{k_1 d}{2\sinh 2k_1 d} + \frac{1}{4}) \}$$

$$+E_2 \{ \sin 2\theta_2 (\frac{k_2 d}{2\sinh 2k_2 d} + \frac{1}{4}) \} + \frac{8E_1 E_2 k_1 k_2}{\rho \sinh 2k_1 d \sinh 2k_2 d}$$

$$\cdot \{ \frac{F_1^2}{F_3^2} k'^2 [\sin 2\alpha_1 (\frac{d}{2} + \frac{\sinh 2k'd}{4k'})]$$

$$+ \frac{F_2^2}{F_4^2} k''^2 [\sin 2\alpha_2 (\frac{d}{2} + \frac{\sinh 2k''d}{4k''})] \} \quad \cdots\cdots\cdots (52)$$

(4) *average energy density* :

The total average energy density is the sum of the kinetic and potential energy density , thus

$$\overline{E} = \overline{E}_p + \overline{E}_k \quad \cdots\cdots\cdots (53)$$

according to the definition that

$$\overline{E}_p = \overline{\int_{-d}^{\eta} \rho g z \, dz} - \int_{-d}^{0} \rho g z \, dz = \frac{1}{2} \rho g \overline{\eta^2} \quad \cdots\cdots\cdots (54)$$

$$\overline{E}_k = \frac{1}{2} \rho \overline{\int_{-d}^{\eta} u^2 dz} \quad \cdots\cdots\cdots (55)$$

Then substitute the solutions of η *and* U *in the eq.* (54) , (55) *, the second-order solution of average energy density will be given by*

$$\overline{E} = (E_1 + E_2) + \frac{2E_1 E_2 k_1 k_2}{\rho \sinh 2k_1 d \sinh 2k_2 d} \{ \frac{F_1^2}{F_3^2} k'^2 \sinh 2k'd$$

$$+\frac{F_2^2}{F_4^2}k''\sinh 2k''d+\frac{2}{g}[(\sigma_1+\sigma_2)\frac{F_1}{F_3}\cosh k'd$$

$$+\frac{1}{4}G\cosh(k_1+k_2)d-\frac{1}{4}H\cosh(k_1-k_2)d]^2$$

$$+\frac{2}{g}[(\sigma_1-\sigma_2)\frac{F_2}{F_4}\cosh k''d+\frac{1}{4}G\cosh(k_1+k_2)d$$

$$-\frac{1}{4}H\cosh(k_1-k_2)d]^2\} \qquad \cdots\cdots\cdots(56)$$

3. *EXPERIMENTAL RESULTS*

The experiments of two wave trains crossing in intermediate depth have been performed at Tainan Hydraulics Laboratory. A number of different testing runs are listed in Table. 1 *, and the comparison of the theoretical and experimental results are shown through Fig.* 4 *to Fig.* 10.

Table. 1

Run	θ	d (*cm*)	T_1 (*sec*)	T_2 (*sec*)	H_1 (*cm*)	H_2 (*cm*)	H_1/L_1	H_2/L_2	d/L_1	d/L_2
1	45°	36	1.0	1.0	4.0	4.0	0.028	0.028	0.252	0.252
2	45°	36	0.8	0.8	4.8	4.8	0.049	0.049	0.367	0.367
3	45°	36	0.8	0.8	3.2	6.4	0.032	0.064	0.367	0.367
4	45°	32	0.9	0.9	4.0	4.0	0.034	0.034	0.271	0.271
5	45°	36	0.8	1.0	4.8	4.0	0.049	0.028	0.367	0.252
6	45°	30	0.7	0.8	5.2	4.0	0.069	0.047	0.400	0.313
7	45°	32	0.7	0.9	4.4	4.0	0.058	0.034	0.421	0.271

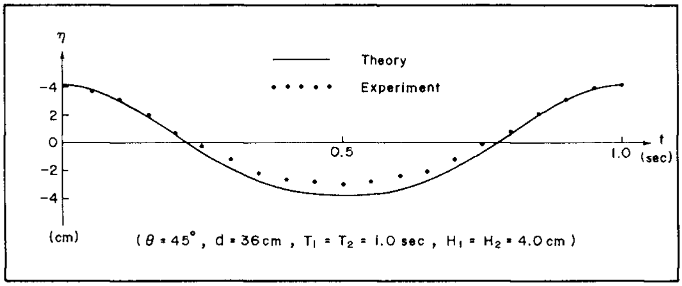

Fig. 4

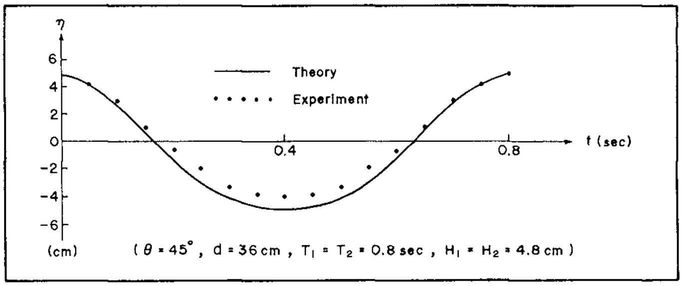

Fig. 5

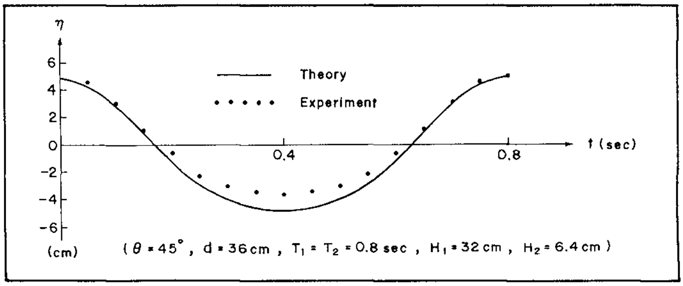

Fig. 6

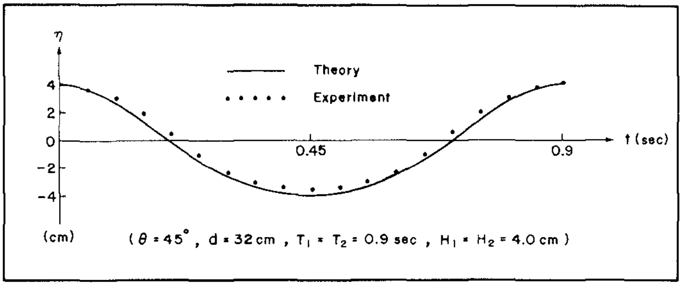

Fig. 7

The variation of water surface elevation for the same wave period and height or the same period but different wave height of two primary wave components crossing are illustrated from Fig. 4 *to Fig.* 7. *It is obvious from these figures that the profiles of common wave are sinusoidal curves and the theoretical values of common wave height are just equal to the sum of two primary waves. Although , those of the experimental values are less than theoretical values but they are coincident very well. The above mentioned results are in agreement with Hsu* (1979).

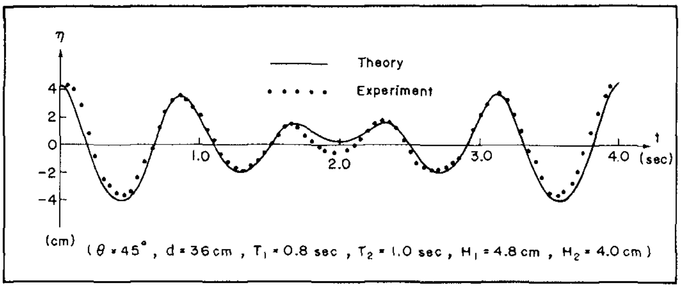

Fig. 8

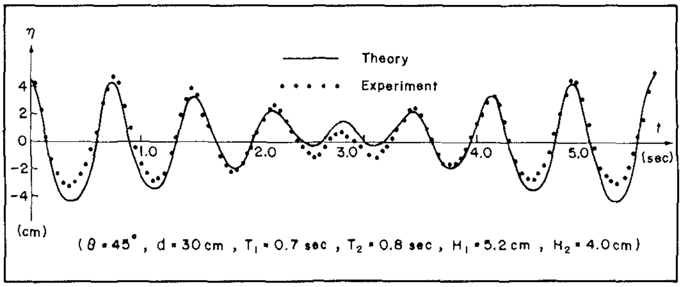

Fig. 9

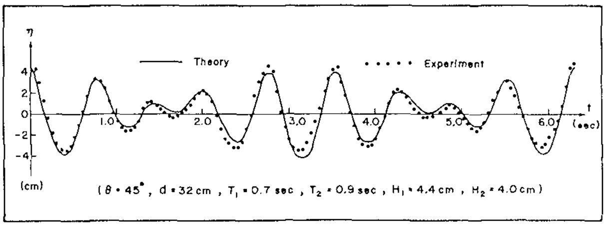

Fig. 10

Moreover, for the cases of different wave period and height, the variation of water surface elevation are shown from Fig. 8 *to Fig*. 10. *It is interested to find that the period of common wave are equal to the least common multiple of two primary wave components exactly and wave profiles of common wave are more complicated but the available experimental data are also coincident with theoretical curves. The comparative values of the theoretical and experimental results are presented in Table*. 2 *.*

Table. 2

Run	T_1 (sec)	T_2 (sec)	H_1 (cm)	H_2 (cm)	H_t^* (cm)	H_p^* (cm)	H_1+H_2 (cm)	$\sqrt{H_1^2+H_2^2}$ (cm)	$\frac{H_t}{\sqrt{H_1^2+H_2^2}}$	$\frac{H_p}{\sqrt{H_1^2+H_2^2}}$
1	1.0	1.0	4.0	4.0	8.0	7.2	8.0	5.66	1.414	1.272
2	0.8	0.8	4.8	4.8	9.6	8.8	9.6	6.79	1.414	1.296
3	0.8	0.8	3.2	6.4	9.6	8.7	9.6	7.16	1.341	1.215
4	0.9	0.9	4.0	4.0	8.0	7.5	8.0	5.66	1.414	1.325
5	0.8	1.0	4.8	4.0	8.6	8.2	8.8	6.25	1.376	1.312
6	0.7	0.8	5.2	4.0	9.0	9.0	9.2	6.56	1.372	1.372
7	0.7	0.9	4.4	4.0	8.0	8.2	8.4	5.95	1.345	1.378

$*H_t$ is the theoretical value , H_p is the experimental value.

4. *CONCLUSIONS*

In this paper , the authors pay attention to the non-linear interaction of two free wave trains crossing in intermediate water depth. From theoretical approaches , the velocity potential and water surface elevation have been expanded to the second order by perturbation method. And wave height , velocity of fluid particles , pressure distribution ,wave thrust and energy density of the common wave are also investigated. In order to verify the theoretical results , elaborated and numerous experiments have also been performed in our laboratory. Some of the remarkable conclusions can be submitted as follows :

(1) *After the interaction of two free wave trains , the common wave period is just equal to the least common multiple (L.C.M.) of the period of two primary component waves.*

(2) *From the theoretical calculation and experimental verification of the non-linear wave interactions, the maximum common waves height are higher about* 20 % *to* 41.4 % *than those derived from energy superposition.*

(3) *It is well simplified from the theoretical results developed by the authors : as* $k_1 = k_2$, $a_1 = a_2$, *that it becomes short-crest waves ; as* $k_1 = -k_2$, $a_1 = a_2$, *it becomes standing wave and as* $k_1 = k_2$, $a_1 = a_2$ *it becomes stokes progressive wave. Here* $k_1 = |K_1|$, $k_2 = |K_2|$. K_1 , K_2 *and* a_1 , a_2 *are wave numbers vector and wave amplitude of the two primary wave components respectively.*

The tertiary interactions are proceeding continuously in our series researches and will be presented later.

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