CHAPTER 74

A uniform mild-slope model for waves over varying bottom

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Abstract

A time-dependent mild-slope equation is derived, based on the formal derivation of Smith & Sprinks (1975), from which the higher-order dispersion relation for waves over uneven bottom is obtained. If only linear dispersion is used for steady waves, the equation (2.12) of Chamberlain & Porter (1995), referred as the modified mild-slope equation (MMSE), is recovered from this equation. To the leading-order solution, it is found that the modified curvature terms of the MMSE have significant effects, while the slope square terms are negligible in accordance with mild-slope assumption. Therefore, retaining all the modified curvature terms neglected in the MSE, the uniform model is developed.

A numerical model using the finite element method(FEM) is developed to predict wave scattering by a varying bottom. In general, the uniform model can predict salient features of waves over various sea beds, such as sinusoidal beds and man-made bars. For sinusoidal beds, the results of the uniform model are in closer agreement with the experimental data than other established models. For man-made bars, the results of both the MMSE and uniform model are in closer agreement with the experimental data than the results of Kirby (1986). An important result of the FEM model is application to transformation of waves over arbitrarily-varying bathymetry.

Introduction

Surface wave scattering by rippled seabeds has been studied by many researchers during the past decade. Rippled seabed are used to represent both regular and irregular ocean bottoms. Davies & Heathershaw(1984) provided solutions to the linear problem and verified their solutions experimentally. Mei(1985) concentrated his studies on the process close to the Bragg resonance condition. Dalrymple & Kirby (1986) studied a single sinusoidal bed for both resonant and non-resonant cases, using a boundary integral equation method. The step-approximation model (Guazzelli et al. 1992), or the successive-application matrix (O'Hare & Davies 1993), in which the bed is divided into a series of very small horizontal shelves, was also developed to model the wave scattering by the rippled bed. Edge & Zhang (1996) first predicted random wave scattering by berms using P-M spectrum.

Alternative methods related to the mild-slope equation(MSE), such as the modified mild-slope equation (MMSE), have been investigated extensively since it was derived by Berkhoff (1972) and Smith & Sprinks (1975). Kirby(1986) developed the equations to predict wave propagation over rapidly varying to-

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pography, which is based on the concept of expanding the rapidly varying component on a very mild-slope bottom. Massel (1993) extended the mild-slope equation by using a Galerkin-eigenfunction method. Recently, Chamberlain & Porter (1995) presented several modified versions of the MSE (or MMSE) by using both variational and Galerkin methods.

Kirby's model (1986) is limited to a rapidly-varying topography. The deficiency of the Kirby's model in prediction of the higher- and sub-harmonic resonance peaks was found by Guazzelli et al. (1992) and O'Hare & Davies (1993). Two of Chamberlain & Porter's models, i.e. Eqs.(3.2) and (3.3) of Chamberlain & Porter (1995), were shown to be ineffective in predicting some resonance peaks (see Fig.3 of Chamberlain & Porter, 1995). In other words, these models break down at some frequencies, similar to the MSE's breaking down as described by Kirby (1986). It means that the approximations of these models are not uniform, but are frequency-dependent. Therefore, to overcome the deficiency of the models mentioned above, the uniform model is developed to predict wave scattering by both slowly-varying and rapidly-varying components of beds uniformly.

The first part of this paper presents the three-dimensional wave model based on Green's identity method for general topography. Secondly, the paper presents a numerical implementation for the finite element method (FEM). Thirdly, the uniform model is verified and the results for sinusoidal beds are given. Finally, numerical results of the uniform models for waves over the man-made bars are presented and compared with the experimental data and other numerical solutions.

Governing equations for general topography

For a small-amplitude wave with angular frequency ω, it is assumed that the flow is incompressible and irrotational and that the pressure is constant at the free surface. The rectilinear coordinates (x, y, z) are fixed in space and z=0 is located at the calm water level. The wave potential $\Phi(x, y, z)$ satisfies the equations:

$$\nabla^2\Phi + \Phi_{zz} = 0, \qquad (-h \le z \le 0), \tag{1}$$

$$\Phi_{tt} + g\Phi_z = 0, \qquad at \quad z = 0, \tag{2}$$

Since the sea bottom is fixed at $z = -h(x, y)$, the normal velocity vanishes; this implies

$$\Phi_z + \nabla h \cdot \nabla\Phi = 0, \qquad at \quad z = -h, \tag{3}$$

where ∇ denotes the horizontal gradient operator, i.e. $\nabla = \{\partial/\partial x, \partial/\partial y\}$.

By utilizing the Green's identity method, Equations (1)-(3) can be combined to obtain the time-dependent equation governing the velocity potential $\phi(x, y)$ and the wave number $k(x, y)$:

$$\phi_{tt} - \nabla \cdot (CC_g\nabla\phi) + (\omega^2 - k^2CC_g)\phi - gF\phi = 0 \tag{4}$$

where

$$\omega^2 = gk \tanh kh, \tag{5}$$

is leading-order dispersion relation, and

$$F = \alpha_1(\nabla h \cdot \nabla h)k + \alpha_2 \nabla^2 h + \alpha_3 \nabla k \cdot \nabla h/k + \alpha_4 \nabla^2 k/k^2 + \alpha_5(\nabla k \cdot \nabla k)/k^3. \quad (6)$$

$C(x,y) = \omega/k$ and $C_g(x,y) = \partial\omega/\partial k$ are the wave celerity and the group velocity, respectively. The dimensionless parameters of $\alpha_i \quad (i = 1,5)$ become

$$\alpha_1 = -\sigma(1-\sigma^2)(1-\sigma q) \quad (7)$$

$$\alpha_2 = -\sigma q(1-\sigma^2)/2 \quad (8)$$

$$\alpha_3 = q(1-\sigma^2)(2q\sigma^2 - 5\sigma/2 - q/2) \quad (9)$$

$$\alpha_4 = q(1-\sigma^2)(1-2\sigma q)/4 - \sigma/4 \quad (10)$$

$$\alpha_5 = q(1-\sigma^2)(4\sigma^2 q^2 - 4q^2/3 - 2\sigma q - 1)/4 + \sigma/4 \quad (11)$$

Throughout this paper, the notations $q = kh$, $Q = k(z+h)$, and $\sigma = \tanh kh$ are used for convenience. The detailed derivation of Eq.(4) is given in the Appendix.

Higher-order dispersion relation

The time-dependent MMSE can be further decomposed into the real and imaginary parts, which are corresponding to the dispersion relation and wave action conservation, respectively. Let

$$\phi = Ae^{i\theta}$$

where A is the potential amplitude and θ is the phase and is defined by

$$\nabla\theta = \hat{k} \quad \theta_t = -\hat{\omega}$$

Separating the time-dependent equation into real and imaginary parts, the higher-order dispersion relation is obtained:

$$\hat{\omega}^2 = \omega^2 + CC_g(\hat{k}^2 - k^2) + \frac{A_{tt}}{A} - CC_g\frac{\nabla^2 A}{A} - \frac{\nabla(CC_g)\cdot\nabla A}{A} - gF, \quad (12)$$

where ω is defined in Eq.(5). If the frequency is fixed and temporal variation of amplitude is not considered, the effective wave number for the MMSE is

$$\hat{k}^2 = k^2 + \frac{\nabla^2 A}{A} + \frac{\nabla(CC_g)\cdot\nabla A}{CC_g A} + \frac{gF}{CC_g}. \quad (13)$$

Discarding the last terms, the effective wave number for the MSE (Liu, 1990, p35)

$$\hat{k}^2 = k^2 + \frac{\nabla^2 A}{A} + \frac{\nabla(CC_g)\cdot\nabla A}{CC_g A} \quad (14)$$

is recovered. It is clear that the last terms in general MMSE will directly affect the wave phase prediction if they are comparable with the original terms in the MSE.

Correspondence to previous models

For the monochromatic and steady waves, the general MMSE becomes:

$$\nabla \cdot (CC_g \nabla \phi) + k^2 CC_g \phi + gF\phi = 0 \tag{15}$$

by using

$$\phi_{tt} = -\phi\omega^2. \tag{16}$$

The relations of the wavenumber slope and curvature with those of bottom, can be obtained from $\nabla\omega = 0$, i.e.

$$\frac{\nabla k}{k} = \beta_1 \frac{\nabla h}{h}, \tag{17}$$

and

$$\frac{\nabla^2 k}{k} = \beta_1 \frac{\nabla^2 h}{h} + \beta_2 \frac{(\nabla h \cdot \nabla h)}{h^2}, \tag{18}$$

where

$$\beta_1 = -q(1-\sigma^2)/\gamma, \tag{19}$$

$$\beta_2 = 2q^2(1-\sigma^2)(\gamma - \alpha_1)/\gamma^3, \tag{20}$$

$$\gamma = \sigma + q(1-\sigma^2). \tag{21}$$

Thus the MMSE in specific case reads

$$\nabla \cdot (CC_g \nabla \phi) + k^2 CC_g \phi + g\left[f_c \nabla^2 h + k f_s (\nabla h \cdot \nabla h)\right]\phi = 0 \tag{22}$$

where

$$f_c(kh) = \alpha_2 + \alpha_4 \beta_1 / q, \tag{23}$$

$$f_s(kh) = \alpha_1 + \alpha_3 \beta_1 / q + \alpha_4 \beta_2 / q^2 + \alpha_5 \beta_1^2 / q^2. \tag{24}$$

Using the notations of Massel(1993) and Chamberlain & Porter(1995), the following relations are found:

$$R_{00}^1 = 2qf_s, \quad R_{00}^2 = 2q\sigma f_c, \tag{25}$$

$$u_1 = f_c, \quad u_2 = kf_s. \tag{26}$$

Therefore, both Equation (34) of Massel(1993) and Equation (2.12) of Chamberlain & Porter(1995) can be recovered from Eq.(4). Obviously, Eq.(4) for the MMSE is more general. The plots of R_{00}^1 and R_{00}^2 have been done for checking with Massel (1993).

However, if the water depth is constant but k is still arbitrary due to the variation of wave amplitude, the general equation is reduced to:

$$\phi_{tt} - \nabla \cdot (CC_g \nabla \phi) + (\omega^2 - k^2 CC_g)\phi - g\{\alpha_4 \nabla^2 k/k^2 + \alpha_5 (\nabla k \cdot \nabla k)/k^3\}\phi = 0 \tag{27}$$

In this case the equations of Massel(1993) and Chamberlain & Porter(1995) fail to consider the variation of wavenumber k. Obviously, the equation (4) in this paper is more general.

Uniform model equation

To exam errors induced by omitting the last terms, which consist of either curvature or slope terms, we compare the ratio of modified terms to the original term of the conventional MSE by using Eqs.(17), (18) and

$$CC_g = g\gamma/(2k), \tag{28}$$

i.e.

$$\frac{gF\phi}{k^2CC_g\phi} = R_c\frac{\nabla^2 h}{k} + R_s(\nabla h \cdot \nabla h) \tag{29}$$

where

$$R_c(kh) = 2(\alpha_2 + \alpha_4\beta_1/q)/\gamma, \tag{30}$$

and

$$R_s(kh) = 2(\alpha_1 + \alpha_3\beta_1/q + \alpha_4\beta_2/q^2 + \alpha_5\beta_1^2/q^2)/\gamma, \tag{31}$$

are functions of only kh. The plots of R_c and R_s are presented in Fig. 1. It is found that approximately $max|R_c, R_s| < 0.2$ in Fig. 1. Nevertheless, from Eq.(29) it is clear that the relative errors still depend on both

$$\frac{\nabla^2 h}{k}, \qquad and \qquad (\nabla h \cdot \nabla h).$$

It is well known that the MSE was derived by neglecting all the modified terms based on the assumption that

$$\frac{\nabla h}{kh} \ll 1.$$

In the following we will show that the slope square terms are negligible, but the mild-slope assumption does not mean that the terms proportional to bottom curvature (i. e. $\nabla^2 h$) can also be neglected. For example, Massel (1993, p109) mentioned an example that the curvature terms are quite significant even for small mean bottom slope for a sinusoidal bed:

$$|\nabla h| = 0.1\pi \approx 0.3 \quad |\nabla^2 h| = 0.2\pi^2 \approx 2.$$

Instead of using $|\nabla^2 h|$, dimensionless $|\nabla^2 h|/k$ is used in this paper. It can be proved that the slope square terms are higher order than the curvature terms, if the ratio of bed wave number to surface wave number is of unity order. For simplicity, let the sinusoidal bed be defined by

$$h(x) = h_0 - bsinKx, \tag{32}$$

where h_0 is mean depth, K is bed wave number, and b is bed amplitude. It is easily found that

$$\frac{|\nabla^2 h|}{k} = \frac{K}{k}|\nabla h|$$

for sinusoidal bed case. It indicates that curvature terms depend not only on bed slope but also the ratio of bed wave number to surface wave number. Since

the modified terms are omitted in the MSE, the errors induced by the MSE will be significant in the lower frequency ranges.

Moreover, the ratio of slope square terms to curvature terms in Eq.(22) can be found from Eq.(32):

$$\frac{f_s}{f_c}\frac{k(\nabla h \cdot \nabla h)}{\nabla^2 h} = \frac{f_s}{f_c}(kb)\cot Kx = O(kb) \tag{33}$$

From Eqs.(23) and (24) it is seen that the ratio of f_s to f_c is of $O(1)$. It means that the magnitude of Eq.(33) depends on surface wave number k and bottom amplitude b. Generally, kb is small value, except for very high frequency waves. Thus based on this ratio of Eq.(33), it is clear that slope square terms are only comparable to curvature terms for very large k, i.e. in the very high frequency ranges. According to Guazzelli et al. (1992), the high-frequency waves are hardly scattered by bottom undulation. Therefore, if the mild-slope assumption

$$\frac{\nabla h}{kh} \ll 1,$$

is applied, only slope square terms can be omitted, i.e.

$$\nabla \cdot (CC_g \nabla \phi) + k^2 CC_g \phi + g f_c \nabla^2 h \phi = 0 \tag{34}$$

The conventional mild-slope equation (MSE)

$$\nabla \cdot (CC_g \nabla \phi) + k^2 CC_g \phi = 0 \tag{35}$$

is recovered, if the modified term (last term) in Eq.(34) is neglected. Since the magnitude of neglected terms (curvature terms) is depend on the ratio of bed wave number to surface wave number (K/k), the error induced by the MSE is frequency-dependent.

On the other hand, Kirby's equation (1986) is originally derived for a rapidly-varying bottom. For the fixed bottom, the surface wave number k should be smaller than bed wave number K. It means that the solutions of Kirby's model are also frequency-dependent, which have been confirmed by other numerical methods and experimental data. The error of the model based on Eq.(34) is more uniform over a large frequency range than other models, thus it is called uniform mild-slope model.

Finite element method based on a weak form

For simplicity, we limit the remaining parts to the problem of two dimensional wave problems.For steady waves, the MMSE reads:

$$(CC_g \phi_x)_x + p\phi = 0 \tag{36}$$

where

$$p = k^2 CC_g + gF \tag{37}$$

The boundary conditions for a patch of rippled beds are

$$\phi_x = -ik(\phi - 2\phi_I) \quad (x_1 \leq 0) \tag{38}$$

$$\phi_x = ik\phi \quad (x_2 \geq L) \tag{39}$$

where $\phi_I = e^{ikx}$ is the incident wave of unit amplitude, x_1 and x_2 represent the upwave and downwave limits of the computational grid, and L is the length of computation domain. These boundary conditions have been given previously by Kirby (1986).

Considering potential applications in the three dimensional wave, a FEM model is developed for general topography. It seems that the numerical model in Chamberlain & Porter (1995) is suitable for periodic topography. The FEM model has advantages in the three dimensional problems with complex geometries, where it is desirable to use irregular meshes.

Multiplying the entire left hand side of equation (36) with a weight function w, and integrating over the domain (0,L) gives the weighted-residual statement:

$$\int_0^L \{(CC_g\phi_x)_x + p\phi\} w dx = 0 \tag{40}$$

Mathematically, the above equation is a statement that the numerical error is needed to be zero in the weighted-integral sense. The trading of differentiability from ϕ to w provides the weak form

$$\int_0^L (CC_g\phi_x w_x - p\phi w) dx - [CC_g w\phi_x]\,|_0^L = 0 \tag{41}$$

The trading of differentiability from ϕ to w can only be performed if it leads to boundary terms that are physically meaningful. The choice of the approximation ϕ for weight function gives the boundary terms $CC_g(\phi)_x\phi$, which has physical meaning of energy flux through a section. It is easy to find that the primary variable and the secondary variable are ϕ and $(\phi)_x$ respectively. Thus $[\phi_x]\,|_0^L$ is the natural boundary condition. Using the notations of Reddy (1993), we have

$$B(w,\phi) - l(w) = 0$$

where

$$B(w,\phi) = \int_0^L (\phi_x w_x - p\phi w) dx - [ikCC_g w\phi_x]\,|_0^L,$$

and

$$l(w) = -[2ikCC_g w\phi_I]\,|_{x=0}$$

are bilinear and linear forms, respectively. For a typical element, ϕ is approximated by

$$\phi = \sum_{j=1}^{4} \phi_j N_j$$

where N_j are cubic shape functions and ϕ_j are unknowns at the nodes. The water depth $h(x)$, slope and curvature of both h and k at each element in the FEM scheme can readily be evaluated as

$$h = \sum_{j=1}^{4} h_j N_j, \quad k = \sum_{j=1}^{4} k_j N_j$$

$$h_x = \sum_{j=1}^{4} h_j \frac{\partial N_j}{\partial x}, \quad k_x = \sum_{j=1}^{4} k_j \frac{\partial N_j}{\partial x}$$

$$h_{xx} = \sum_{j=1}^{4} h_j \frac{\partial^2 N_j}{\partial^2 x}, \quad and \quad k_{xx} = \sum_{j=1}^{4} k_j \frac{\partial^2 N_j}{\partial^2 x}.$$

The FEM schemes based on weak forms for the MSE, EMSE and UMSE are similar but much more simple. Bubnov-Galerkin method is adopted, thus the solution shape functions are used as weighting function.

It is worthy to note that most FEM models for water wave problems are established based on the functional formulation. For example, the hybrid element method of Mei (1983) and the modified hybrid element method of Zhang (1991,1996) are all based on variation of a functional. As noted in Reddy (1993), not all differential equations admit the functional formulation, and in order for the functional to exist, the bilinear form must be symmetric in its arguments. Since the weak form statement is equivalent to the differential equation and the specified natural boundary condition of the problem, the weak form FEM is used in this paper. Substituting the assumed approximate solution into the weak form (42), and following the procedure in Reddy (1993), the FEM is developed. Considering the fact that the coefficients of the FEM are high order functions of space, cubic shape functions are used here.

Verification of numerical model

In this section, the numerical solutions for sinusoidal beds and man-made bars are presented in $Figs.2 \sim 4$, and compared with experimental data.

For the case of the bottom having only one sinusoidal component, there are two resonant peaks of reflection coefficients, which were shown numerically by Dalrymple & Kirby (1986) by solving the 2-D Laplace equation. Later it was confirmed by Davies et al. (1989) and O'Hare & Davies (1993). Consider the case of sinusoidal bed $n = 2$, $b/h_0 = 0.32$, where h_0 is mean water depth, b is amplitude of bed undulation, and n is the number of sinusoidal bars. The numerical results of the MMSE, uniform and Kirby's model are presented in $Fig.2$, respectively, to compare with experimental data of Davies & Heathershaw (1984).

In $Fig.2$, it is seen that Kirby's model does not predict the second resonance peaks, while results of MMSE and uniform models are in good agreement with experimental data. Also it noted that the uniform model predicts the second peak better than the MMSE model.

Consider now the case of a doubly-sinusoidal bed (chosen here to be of equal amplitude b). The inclusion of a more rapid component makes the

bottom slope relatively larger, so the difference between the uniform model and the model of Chamberlain & Porter (1995) becomes significant. As indicated in *Fig.*3, Chamberlain & Porter (1995) overpredict reflection coefficients at higher-harmonic resonant peaks, if compared with experimental data (Guazzelli et al. 1992).

Numerical solutions and experimental results for the artificial bar field (Kirby & Anton 1990) are also studied. The bar field is periodic over intervals of width L, and can be conveniently represented by the Fourier series. The man-made bars have many sinusoidal components(Kirby & Anton 1990), some of which have shorter wavelengths than the surface wavelength, not belonging to rapidly-varying case. Since Kirby's model is valid only for rapidly-varying components, the slowly-varying components are not properly modeled. Therefore the overall reflections predicted by Kirby's model have larger discrepancy with experimental data than these predicted by the uniform model and MMSE as indicated in *Fig.*4.

The convergence of the numerical model is confirmed by checking the energy relation $R^2 + T^2 = 1$, where R and T are reflection and transmission coefficients, respectively. The maximum error is less than 10^{-5}.

Conclusions

To overcome the limitations of Kirby's model, the uniform model for wave transformation over an arbitrary-varying topography has been developed. Several cases of sinusoidal bed configuration have been studied and presented to confirm the numerical models. Based on the comparison between the numerical results and experimental data, it is clear that Kirby's method is frequency-dependent.

For sinusoidal beds, the numerical results of the MMSE, uniform and Kirby's model are all in good agreement with the experimental data of Davies & Heathershaw (1984), except at the second resonance peak ($2k/K \sim 2$), the uniform model predicts a reflection coefficient closer to the laboratory data than the other numerical results. For man-made bar case, it is shown that both the MMSE and uniform model give excellent comparison with experiments. Since Kirby's equation is originally derived for a rapidly-varying bottom, it becomes less accurate as ripple-bed undulations become slower (i.e. $2k/K$ becomes larger). The uniform model is capable of modeling not only the slowly-varying components but also rapidly-varying of sea beds.

The success of the uniform model in model tests and better agreement with experimental data than other models, suggest that the uniform model is more efficient and accurate tool for the prediction of wave transformation over general sea bed. An important result of the uniform model is application to reflection of wave energy from offshore bars and dredged material dump sites. In addition, the uniform wave model can be extended to further the work of Kirby (1984).

Appendix: Derivation of the time-dependent wave equation

The depth-integrated wave equation for monochromatic, linear waves propagating over ripple beds may be formulated following the Green's identity method of Smith & Sprinks (1975). The solution to Equation (1)-(3) may be expressed as

$$\Phi(x,y,z,t) = f(q,Q)\phi(x,y,t) + (non-propagating \quad modes) \tag{A1}$$

where

$$f = \cosh Q / \cosh q \tag{A2}$$

is a function of z, k, and h. The propagation of waves is associated with only the propagating mode, thus extracting this mode component and applying Green's identity to f and Φ:

$$\int_{-h}^{0} (f\Phi_{zz} - \Phi f_{zz})dz = [f\Phi_z - \Phi f_z]_{-h}^{0}$$

or

$$\int_{-h}^{0} (f\nabla^2\Phi + \Phi f_{zz})dz = -[f\Phi_z - \Phi f_z]_{-h}^{0} \tag{A3}$$

Using (A1) and (A2)

$$\begin{aligned} f_{zz} &= k^2 f \\ \nabla\Phi &= f\nabla\phi + \phi\nabla f \\ \nabla^2\Phi &= f\nabla^2\phi + 2\nabla\phi\cdot\nabla f + \phi\nabla^2 f \\ \Phi_z|_{z=-h} &= -\nabla h\cdot(f\nabla\phi + \phi\nabla f) \end{aligned} \tag{A4}$$

Inserting (A4) into (A3)

$$\int_{-h}^{0} (\phi k^2 f^2 + \nabla^2\phi f^2 + 2f\nabla\phi\cdot\nabla f + \phi f\nabla^2 f)dz = (\phi_{tt} + \omega^2\phi)/g|_{z=0} + \Phi_z f|_{z=-h} \tag{A5}$$

Based on (A2), every term in (A5) is evaluated using the following

$$\nabla f = f_h\nabla h + f_k\nabla k$$

$$\nabla^2 f = f_{hh}(\nabla h\cdot\nabla h) + f_h\nabla^2 h + 2f_{hk}\nabla h\cdot\nabla k + f_k\nabla^2 k + f_{kk}(\nabla k\cdot\nabla k)$$

where $f_h = \frac{\partial f}{\partial h}$, $f_k = \frac{\partial f}{\partial k}$, $f_{hh} = \frac{\partial^2 f}{\partial h^2}$, $f_{hk} = \frac{\partial^2 f}{\partial h\partial k}$, $f_{kk} = \frac{\partial^2 f}{\partial k^2}$, and derived in the following:

$$\begin{aligned} f_h &= k(\sinh Q - \sigma\cosh Q)/\cosh q \\ f_k &= (Q\sinh Q - q\sigma\cosh Q)/(k\cosh q) \\ f_{hh} &= 2\sigma k^2(\sigma\cosh Q - \sinh Q)/\cosh q \\ f_{kk} &= \{Q^2\cosh Q - 2\sigma qQ\sinh Q - q^2(1-2\sigma^2)\cosh Q\}/(k^2\cosh q) \\ f_{hk} &= \{(2q\sigma^2 - \sigma - q)\cosh Q + (1-q\sigma)\sinh Q + Q\cosh Q - Q\sigma\sinh Q\}/\cosh q \end{aligned} \tag{A6}$$

Applying Leibniz's rule,

$$\int_{-h}^{0}(\nabla^2\phi f^2 + 2f\nabla\phi\cdot\nabla f)dz + f^2\nabla h\cdot\nabla\phi|_{z=-h} = \nabla\cdot(CC_g\nabla\phi)/g \qquad (A7)$$

Using the following integrations

$$g\int_{-h}^{0} k^2\phi f^2 dz = k^2 CC_g\phi$$

$$2k\int_{-h}^{0}\sinh Q\cosh Q dz = \sigma^2/(1-\sigma^2)$$

$$4k\int_{-h}^{0} Q\sinh Q\cosh Q dz = \{q(1+\sigma^2)-\sigma\}/(1-\sigma^2)$$

$$4k\int_{-h}^{0} Q\cosh Q\cosh Q dz = \{q^2(1-\sigma^2)+2q\sigma-\sigma^2\}/(1-\sigma^2)$$

$$4k\int_{-h}^{0} Q^2\cosh Q\cosh Q dz = \{2q^3(1-\sigma^2)+2q^2\sigma-q(1+\sigma^2)+\sigma\}/(1-\sigma^2)$$

and substituting (A6) and (A7) into (A5), finally the time-dependent Eq.(4) is obtained.

It may be worth remarking that the terms in (A5) involving

$$f_k = \frac{\partial f}{\partial k}, \quad f_{hk} = \frac{\partial^2 f}{\partial h\partial k}, \quad f_{kk} = \frac{\partial^2 f}{\partial k^2},$$

were mistakenly omitted by Smith & Sprinks (1975). Furthermore, Smith & Sprinks (1975) obtained the conversional mild-slope equation (MSE) by neglecting all the so-called 'forcing' terms, which are assumed to be of higher order. The validity of this assumption of higher order about the forcing terms must, however, be questioned. In the absence of current, similar treatment was made by Kirby (1984) (see Eq.17 of Kirby, 1984).

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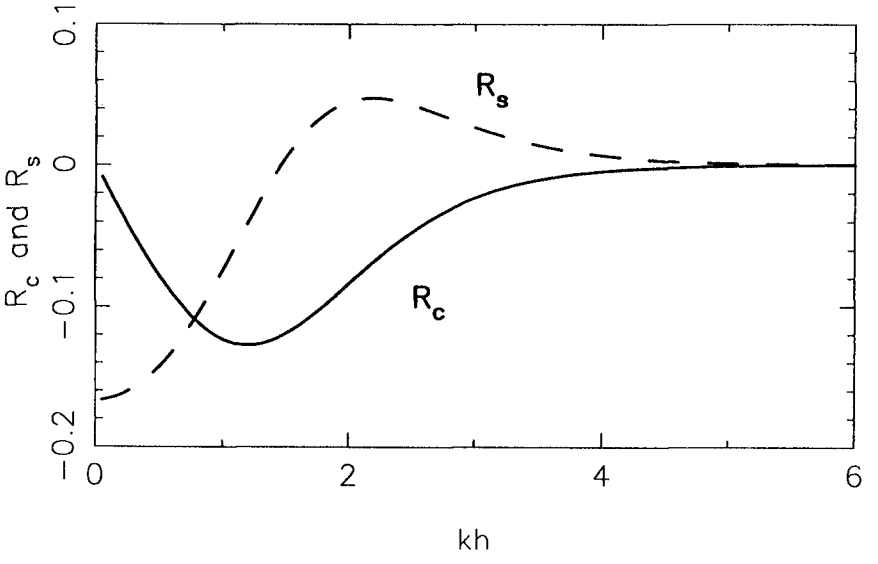

Fig.1 Curvature and Slope Functions(R_c, R_s)

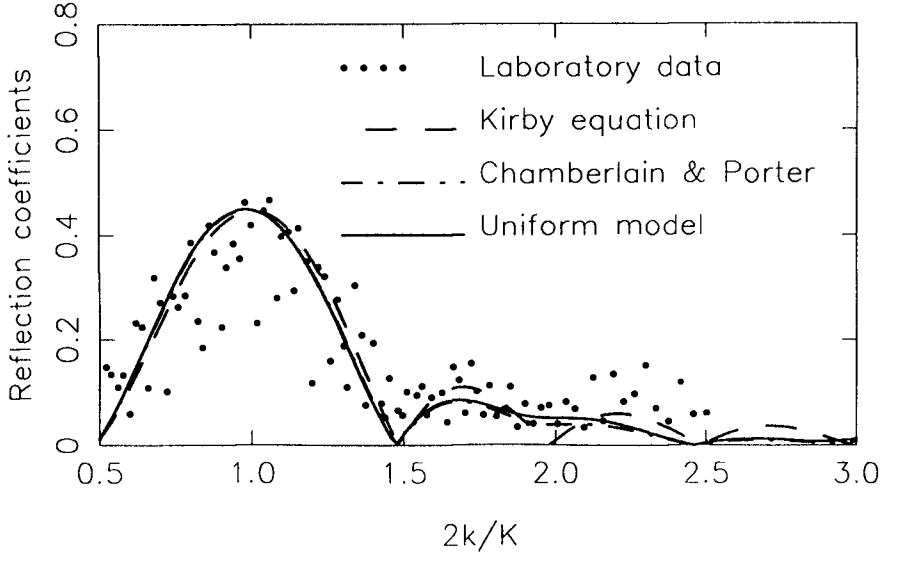

Fig.2 Singly-sinusoidal bed, n=2, b/h0=0.32

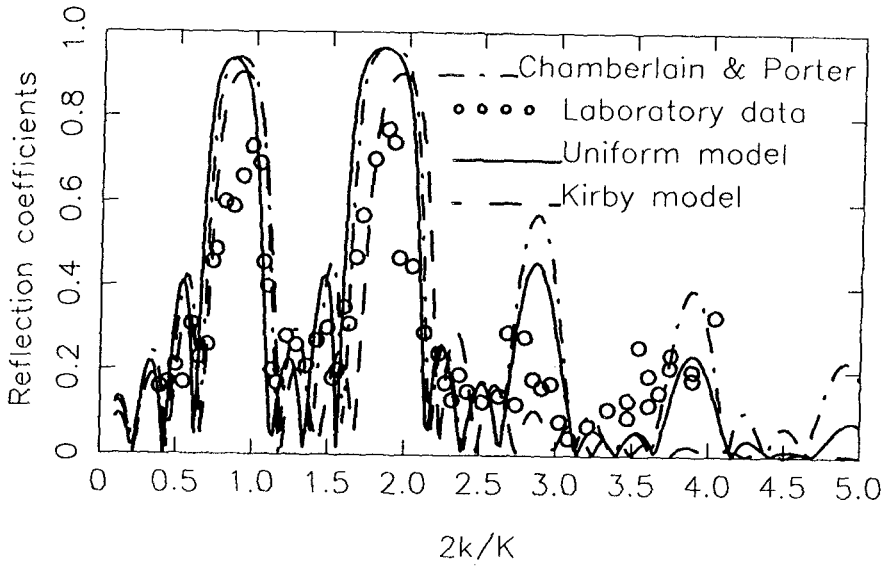

Fig.3 Doubly-sinusoidal bed, n=4,m=2,b/h0=0.4

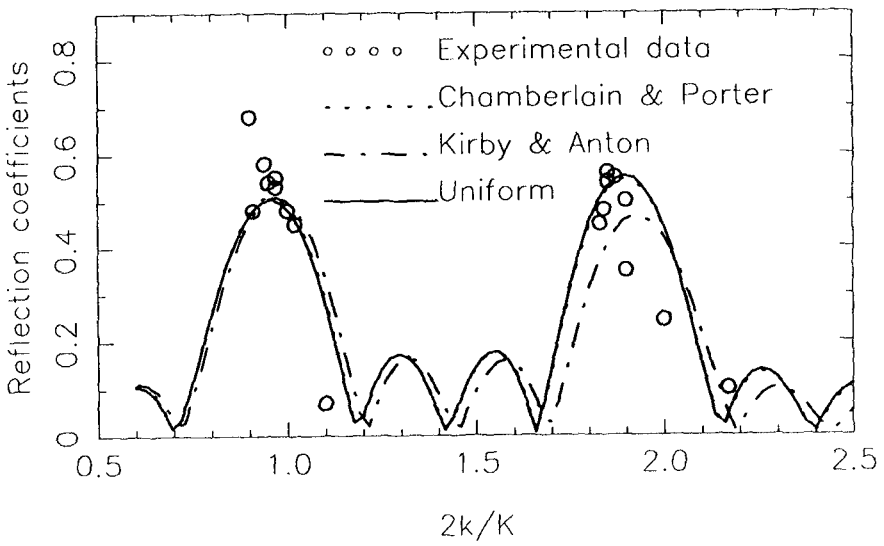

Fig.4 Four man-made bars (bar spacing=120cm)