# CHAPTER 113

## LONG WAVE RUNUP ON COASTAL STRUCTURES

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### Abstract

We present a general method for determining the runup and the amplification explicitly for nonbreaking long waves propagating over piecewise linear topography, using the linear shallow water wave equations. We associate each constant–depth segment and each linearly–varying depth segment with $(2 \times 2)$ matrices and we calculate the transmitted wave amplitude after propagating over any number of segments explicitly. We then extend our methodology to the three dimensional topography of a conical island. Our method is applicable in the design of dikes, sea–walls and other coastal structures.

### INTRODUCTION

The September 2, 1992 Nicaragua tsunami, the December 12, 1992 Flores island tsunami, the July 12, 1993 Hokkaido tsunami, the June 2, 1994 East Java tsunami, on October 2, 1994 Kuril Islands, Shikotan tsunami, the November 11, 1994 Mindoro Island tsunami, the February 17, 1996 Biak, Irian Jaya tsunami, and the February 27, 1996 Peru tsunami were eight major devastating geophysical events that caused severe property damage and killed an estimated 2000 people. Field observations raised new questions about the suitability of the standard paradigm of a tsunami model, i.e. a solitary wave attacking a plane beach. For one, most of the eyewitness reported waves which caused the shorelines to recede first before advancing; these waves were studied by Tadepalli and Synolakis and they are now called leading–elevation N–waves (Tadepalli and Synolakis, 1994). Then, practically all field–measured

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runup distribution showed three–dimensional effects which had been counter–intuitive. For example, during the 12/12/92 Flores Island tsunami, 263 people were killed and two fishing villages completely annihilated on Babi Island, a small volcanic islet off Flores; all damage concentrated on the tsunami wise and wind–wave wise lee side of the island. During the 9/2/92 Nicaraguan tsunami, the runup varied by a factor of 4 between locations distant less than a mile form each other, suggesting that even though the tsunamis are long waves, local bathymetric features of much smaller length scale did influence the wave runup. A recent increase in tsunami incidence and an abundance of new observations of coastal effects added urgency to the resolution of the question to determine a priori how a given topography will affect the wave evolution.

Besides tsunamis, the long wave theory presented here can be used for the calculation of the runup and evolution of certain wind generated waves particularly in the infragravity spectral band. The numerical solution of the nonlinear shallow–water wave equation were used to calculate the runup of wind swell (Raubenheimer *et al.*, 1995). Also, it has long been known that the runup predicted by the nonlinear shallow–water wave equation is mathematically the same of that predicted by the linear shallow–water wave equation for the one slope beach case (Carrier, 1966; Synolakis, 1987).

## FORMULATION OF THE PROBLEM

We will describe a new general method that will provide a closed form analytical solution to determine the relationship between the amplification factor and the incident wave amplitude for the two–dimensional topographies. Given that any physically realistic topography is unlikely to consist entirely of a single–sloping beach or a single constant–depth segment, and to allow the use of analytical (non–numerical) solutions, it is incumbent to be able to break down the physical propagation problem into a series of linear problems which can then be solved with standard method analytically. The proposed method of solution consists of representing a given topography by a series of constant–depth and linearly–varying depth segments.

We will use the linear shallow–water wave equation to solve the propagation problem over linear topographies; this equation is

$$\eta_{tt} - (\eta_x \, h)_x = 0. \tag{1}$$

We introduce dimensionless variables using a reference undisturbed offshore water depth $d$ as the characteristic length scale, and $\sqrt{d/g}$ as the characteristic time scale. With time harmonic dependence of the form $\eta(x,t) = \zeta(x)\, e^{-i\omega t}$, for the wave evolution over the dimensionless constant–depth $h(x) = h_c$, equation

(1) becomes,

$$h_c \frac{d^2\zeta(x)}{dx^2} + \omega^2 \zeta(x) = 0. \tag{2}$$

The general solution is

$$\eta(x,t) = \{A_1 e^{-\frac{i\omega x}{\sqrt{h_c}}} + B_1 e^{\frac{i\omega x}{\sqrt{h_c}}}\} e^{-i\omega t}, \tag{3}$$

where $A_1$ and $B_1$ are arbitrary constants. If the depth is linearly–varying and defined by $h(x) = mx + n$ with $m \neq 0$ and $n$ constant, the field equation is

$$(mx+n) \frac{d^2\zeta(x)}{dx^2} + m \frac{d\zeta(x)}{dx} + \omega^2 \zeta(x) = 0. \tag{4}$$

The eigenfunctions of the field equation are two zeroth–order linearly independent solutions of Bessel's equation, e.g. $J_0$ and $Y_0$. Then the solution for evolution over variable depth is

$$\eta(x,t) = \{A_2 J_0(2\omega\sqrt{\xi}) + B_2 Y_0(2\omega\sqrt{\xi})\} e^{-i\omega t}, \tag{5}$$

with $\xi = \sqrt{(x + n/m)/m}$, where $A_2$ and $B_2$ are arbitrary constants.

From the solutions (3) and (5), the appropriate form of $\eta$ is chosen for each segment. Continuity of the surface elevation and of the mass flux provides two equations at each transition point between the adjacent segments and the unknowns such as $A_1$, $B_1$, $A_2$, $B_2$ can be calculated from these interface matching conditions.

In general, an $m$–segment topography involves $m$–sets of (3) and (5) type eigenfunction expansions, each with 2 unknown coefficients. Given that the incoming wave height is known, $(2m-1)$ coefficients have to be determined, requiring $(2m-1)$ equations for closure. To this end, continuity of the surface elevation and of the mass flux boundary conditions must be considered at each transition point between the adjacent segments. These boundary conditions provide $(2m-2)$ equations. The conditions of transmission at the segment, or the condition of bounded solution at the coastline or perfect reflection off a wall provide one additional equation. Adding a new segment introduces two more unknown coefficients, but at the same time also allows to write two new boundary conditions. This extends the order of the system of equations by two. Thus the wave evolution over any topography composed of any number of piecewise linear segments can be calculated.

Our method consists of establishing basic topography segments, i.e. a constant–depth segment, a linearly–varying depth segment. Each segment has a $(2 \times 2)$ matrix that incorporates its topographic feature. We associate each constant–depth segment of depth $h_r$ with a segment matrix,

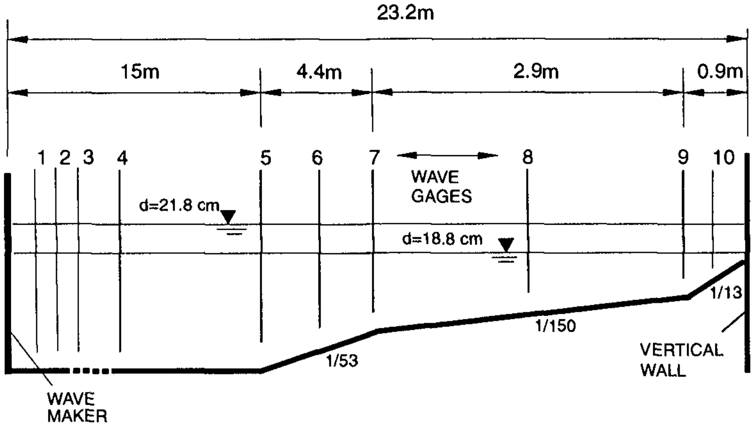


**Figure 1.** Definition sketch of the Revere Beach topography (not to scale).

$$K_{pr} = \begin{pmatrix} e^{-\frac{i\omega x_p}{\sqrt{h_r}}} & e^{\frac{i\omega x_p}{\sqrt{h_r}}} \\ i\, e^{-\frac{i\omega x_p}{\sqrt{h_r}}} & -i\, e^{\frac{i\omega x_p}{\sqrt{h_r}}} \end{pmatrix}. \tag{6}$$

We associate each linearly–varying depth segment with $h_r(x) = m_r x + n$ with $m_r \neq 0$ and $n$ constant with the segment matrix,

$$S_{pr} = \begin{pmatrix} J_0(2\omega\sqrt{\xi_{pr}}) & Y_0(2\omega\sqrt{\xi_{pr}}) \\ J_1(2\omega\sqrt{\xi_{pr}}) & Y_1(2\omega\sqrt{\xi_{pr}}) \end{pmatrix}, \tag{7}$$

with $\xi_{pr} = \sqrt{(x_p + n/m_r)/m_r}$. The subscripts $p$ and $r$ identify the transition point and the segment number increasing from the seaward to the shoreward.

Using these topographic–feature matrices, it is possible to write a matrix equation between the unknowns of the eigenfunction expansion. We will provide a specific example for application of this general method.

## REVERE BEACH

A physical model of the Revere Beach –located approximately 6 miles northeast of Boston City of Revere, Massachusetts– was constructed at Waterways Experiment Station (WES), US Army Corps of Engineers Coastal Engineering Research Center (CERC), by Ward who investigated the beach erosion and flooding problems (Ward, 1995). Moreover it presented a unique opportunity to evaluate the predictions of the general method presented here. The model of the Revere Beach consisted of three piecewise linear, 1 : 13, 1 : 150 and 1 : 53 slopes from seaward to shoreward respectively, with a vertical wall at the landward end of the compound slope and it is shown in figure 1.

## Analytical Solution

Alternative to the conventional method of solution –try to solve $(2m-1)$ equation numerically– using the general methodology presented here boundary conditions for each transition point can be written as a matrix equation;

$$S_w A_1 = V_1, \tag{8}$$

$$S_{11} V_1 = S_{12} V_2, \tag{9}$$

$$S_{22} V_2 = S_{23} V_3, \tag{10}$$

$$S_{33} V_3 = K_{34} V_4, \tag{11}$$

and finally

$$S_w A_1 = S_{11}^{-1} S_{12} S_{22}^{-1} S_{23} S_{33}^{-1} K_{34} V_4. \tag{12}$$

Here $V_1$, $V_2$, $V_3$ and $V_4$ are column unknown vectors for each segment. Using the naming style presented for the topographic–feature matrices,

$$\begin{pmatrix} 1 \\ -J_1(\frac{2\omega\sqrt{h_w}}{m_1})/Y_1(\frac{2\omega\sqrt{h_w}}{m_1}) \end{pmatrix} A_1 =$$
$$\begin{pmatrix} J_0(\frac{2\omega\sqrt{h_1}}{m_1}) & Y_0(\frac{2\omega\sqrt{h_1}}{m_1}) \\ J_1(\frac{2\omega\sqrt{h_1}}{m_1}) & Y_1(\frac{2\omega\sqrt{h_1}}{m_1}) \end{pmatrix}^{-1} \begin{pmatrix} J_0(\frac{2\omega\sqrt{h_1}}{m_2}) & Y_0(\frac{2\omega\sqrt{h_1}}{m_2}) \\ J_1(\frac{2\omega\sqrt{h_1}}{m_2}) & Y_1(\frac{2\omega\sqrt{h_1}}{m_2}) \end{pmatrix}$$
$$\begin{pmatrix} J_0(\frac{2\omega\sqrt{h_2}}{m_2}) & Y_0(\frac{2\omega\sqrt{h_2}}{m_2}) \\ J_1(\frac{2\omega\sqrt{h_2}}{m_2}) & Y_1(\frac{2\omega\sqrt{h_2}}{m_2}) \end{pmatrix}^{-1} \begin{pmatrix} J_0(\frac{2\omega\sqrt{h_2}}{m_3}) & Y_0(\frac{2\omega\sqrt{h_2}}{m_3}) \\ J_1(\frac{2\omega\sqrt{h_2}}{m_3}) & Y_1(\frac{2\omega\sqrt{h_2}}{m_3}) \end{pmatrix}$$
$$\begin{pmatrix} J_0(\frac{2\omega\sqrt{h_3}}{m_3}) & Y_0(\frac{2\omega\sqrt{h_3}}{m_3}) \\ J_1(\frac{2\omega\sqrt{h_3}}{m_3}) & Y_1(\frac{2\omega\sqrt{h_3}}{m_3}) \end{pmatrix}^{-1} \begin{pmatrix} e^{-\frac{i\omega x_3}{\sqrt{h_4}}} & e^{\frac{i\omega x_3}{\sqrt{h_4}}} \\ i\, e^{-\frac{i\omega x_3}{\sqrt{h_4}}} & -i\, e^{\frac{i\omega x_3}{\sqrt{h_4}}} \end{pmatrix} \begin{pmatrix} A_i \\ A_r \end{pmatrix}. \tag{13}$$

Notice that first the matrix equation (8) represents perfect reflection boundary condition, $\partial\eta/\partial x = 0$, since the topography has finite depth $(h_w)$ at the shoreline $(x = 0)$. We evaluate the following integral to find time histories of surface elevation at any given location;

$$\eta_j(x,t) = \int_{-\infty}^{+\infty} \Phi(\omega)\ \{A_j\, J_0(\tfrac{2\omega\sqrt{h_j(x)}}{m_j}) + B_j\, Y_0(\tfrac{2\omega\sqrt{h_j(x)}}{m_j})\}\ e^{-i\omega t} d\omega, \tag{14}$$

where $j = 1, 3$ represents the segment number. The unknowns of the eigenfunction expansion can be determined equations (8) through (12). Here $\Phi(\omega)$

is the Fourier transform of the initial solitary wave profile located at $x = x_s$ and given by $(2/3)\,\omega\,\mathrm{cosech}(\alpha\omega)\,e^{i\omega x_s}$ with $\alpha = \pi/2\gamma$ and $\gamma = \sqrt{3H/4}$ (Synolakis, 1986). $H$ is the dimensionless incoming waveheight.

We evaluated the integral (14) at the shoreline ($x = 0$) using the contour integration technique to find an analytical expression for the maximum runup (Kânoğlu, 1997). With asymptotic analysis, we found that the maximum runup for the Revere Beach can be given by

$$\mathbf{R} = 2\,h_w^{-1/4}\,H; \tag{15}$$

the maximum runup depends only on the incoming waveheight $H$ and on the depth at the wall $h_w$.

## Experimental Results

We performed the experiments 23.2$m$–long, 45$cm$–wide glass–walled flume. The wave maker was located 23.22$m$ away from the wall. Ten wave gages were located to record time histories of free surface displacements as shown in figure 1. Gage 4 was moved to a half–wavelength away from the toe of the 1 : 53 slope. This gage was used to define the waveheight of the incoming wave and its location, ensured that all waves propagated the same relative distance of a half wavelength ($L/2 = (1/\gamma)\mathrm{arccosh}\sqrt{20}$) between the reference location and the toe of the beach. This localization is the standard method for referencing the heights of solitary waves climbing up a sloping beach (Synolakis, 1986 and 1987). Experiments were carried out at two different water depths, 18.8$cm$ and 21.8$cm$. The experiments are described elsewhere in detail (Briggs *et al.*, 1997; Kânoğlu, 1997).

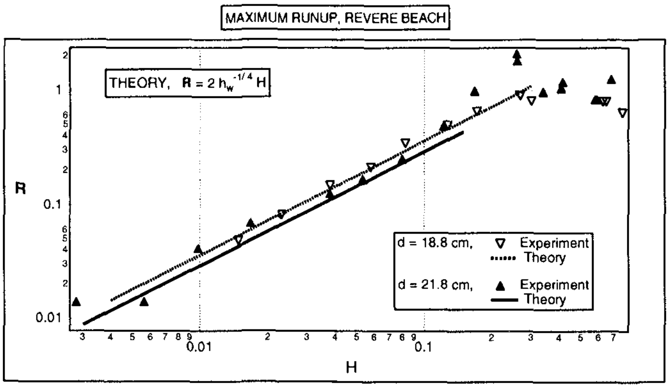


**Figure 2.** Comparison of the maximum runup between the analytical solution and the laboratory data for two different depth $d = 18.8cm$ and $d = 21.8cm$.

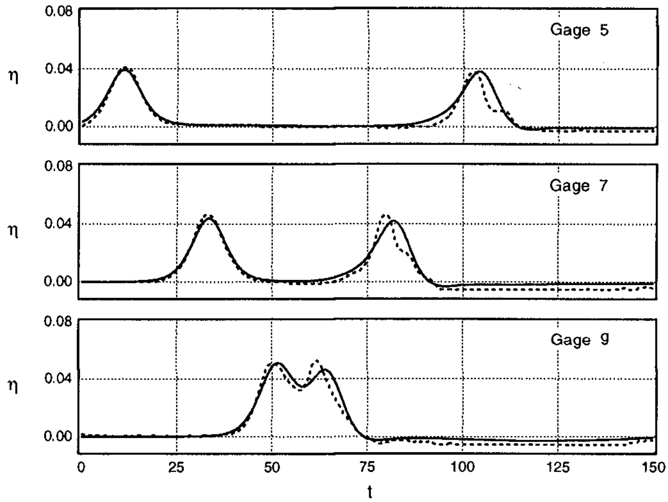


**Figure 3.** Comparison of the time histories of surface elevations between the analytical solution (evaluation of the integral 14) and the laboratory data for $d = 21.8cm$ and $H = 0.0378$ case at three gages. Dotted lines represent the laboratory results.

A comparison of theory predictions with laboratory data is presented here. We compared the maximum runup heights and the time series of surface elevations predicted by the general method with the laboratory data in figure 2 and figure 3 respectively.

## THE CONTINENTAL SHELF AND SLOPE

Our objective is to obtain the amplification factor $\mathcal{A}$ in the first segment and the reflection coefficient $A_r$ in the third segment in terms of the incident wave amplitude $A_i$. We use general methodology to write a matrix equation to obtain analytical solution;

$$S_{11}\,\mathcal{A} = S_{12}\,V_2, \tag{16}$$

$$S_{22}\,V_2 = K_{23}\,V_3, \tag{17}$$

Combining the two matrix equations, it is possible to write the following matrix equation;

$$S_{11}\,\mathcal{A} = S_{12}\,S_{22}^{-1}\,K_{23}\,V_3. \tag{18}$$

Using the naming style for the topographic–feature matrices,

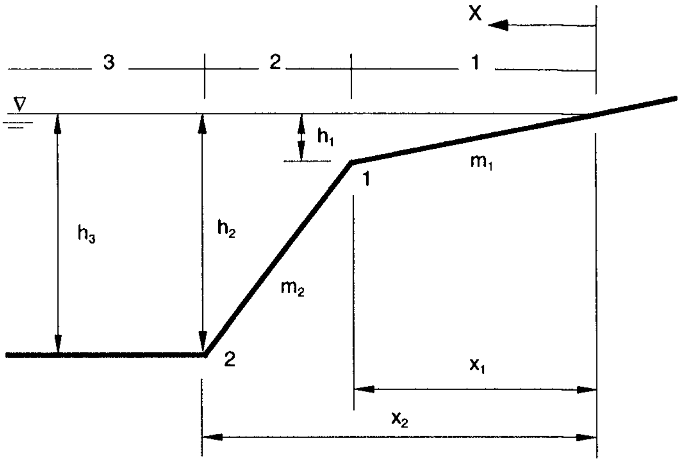


**Figure 4.** Definition sketch for the continental shelf and slope topography.

$$\begin{pmatrix} J_0(\frac{2\omega\sqrt{h_1}}{m_1}) \\ J_1(\frac{2\omega\sqrt{h_1}}{m_1}) \end{pmatrix} \mathcal{A} =$$
$$\begin{pmatrix} J_0(\frac{2\omega\sqrt{h_1}}{m_2}) & Y_0(\frac{2\omega\sqrt{h_1}}{m_2}) \\ J_1(\frac{2\omega\sqrt{h_1}}{m_2}) & Y_1(\frac{2\omega\sqrt{h_1}}{m_2}) \end{pmatrix} \begin{pmatrix} J_0(\frac{2\omega\sqrt{h_2}}{m_2}) & Y_0(\frac{2\omega\sqrt{h_2}}{m_2}) \\ J_1(\frac{2\omega\sqrt{h_2}}{m_2}) & Y_1(\frac{2\omega\sqrt{h_2}}{m_2}) \end{pmatrix}^{-1}$$
$$\begin{pmatrix} e^{-\frac{i\omega x_2}{\sqrt{h_3}}} & e^{\frac{i\omega x_2}{\sqrt{h_3}}} \\ i\,e^{-\frac{i\omega x_2}{\sqrt{h_3}}} & -i\,e^{\frac{i\omega x_2}{\sqrt{h_3}}} \end{pmatrix} \begin{pmatrix} A_i \\ A_r \end{pmatrix}. \tag{19}$$

Here $\mathcal{A}$ is the unknown scalar –To ensure a bounded solution at the coastline, as in the single beach case, the unknown coefficient in the eigenfunction equation for $Y_0$ must be set equal to zero. This gives a scalar unknown for this segment.– for the first segment and $V_2$ and $V_3$ are the column unknown vectors for the second and third segments respectively.

The transmitted wave to the beach is given by

$$\eta_1(x,t) = \int_{-\infty}^{+\infty} \Phi(\omega)\,\mathcal{A}\,J_0(\tfrac{2\omega\sqrt{h_1(x)}}{m_1})\,e^{-i\omega t}d\omega. \tag{20}$$

Again here $\Phi(\omega)$ associated with initial solitary wave profile. Amplification factor $\mathcal{A}$ can be determined from equation (19). Using the asymptotic analysis, the maximum runup can be given by

$$\mathrm{R} = 2.831\sqrt{1/m_1}\,H^{5/4}. \tag{21}$$

This is the same analytical expression that Synolakis (1986) found for the single sloping beach case, implying that at least for long waves the runup only depends on the beach slope closest to the shoreline. To better understand these results, different hypothetical topographies of the continental shelf and slope will be investigated next, by changing the transition point and the wave height $H$ the effects of the slopes on the maximum runup are analyzed. Maximum runup calculations are based on the evaluation of the maximum of the integral (20). Maximum runup calculations are shown in figure 5 for the different ranges of parameters; i.e. $H$, $h_1$, $m_1$ and $m_2$. It is clear from the figure that the most dominant parameter on the maximum runup is the slope of beach closest to the shoreline; i.e. $m_1$. More parametric analysis can be found elsewhere (Kânoğlu, 1997).

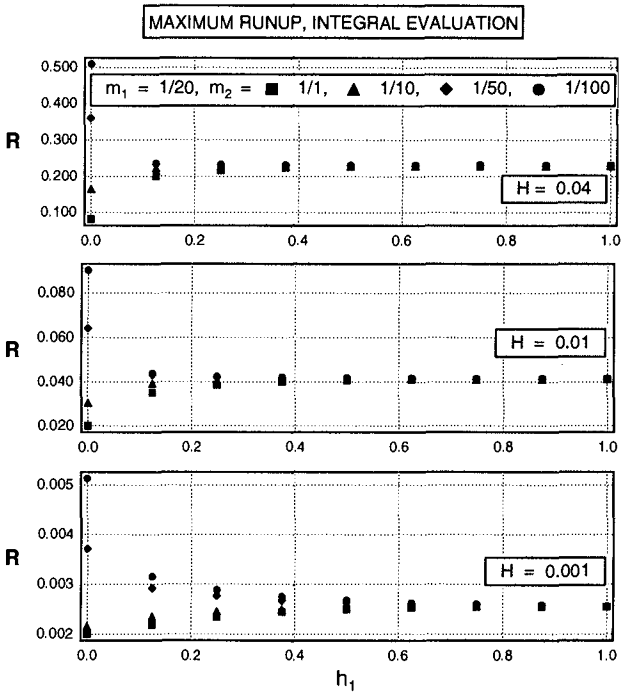


**Figure 5.** The effect of the parameters $m_2$, $h_1$ and $H$ on the maximum runup for the continental shelf and slope with $m_1 = 1/20$ –$h_1 = 0$ means topography only with $m_2$ slope and $h_1 = 1$ means topography only with $m_1$slope–.

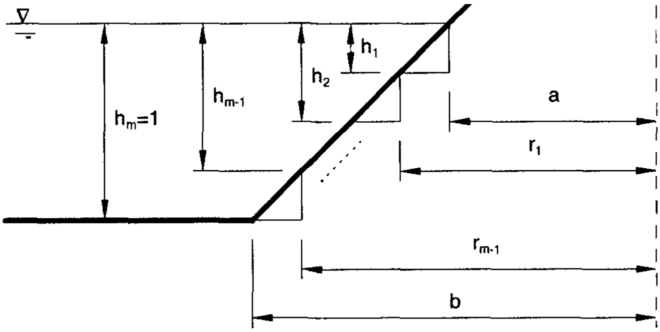


**Figure 6.** Island slope in a stepwise fashion.

## CONICAL ISLAND

The overall good agreement between the analytic results and the laboratory data for the two–dimensional topography of the Revere Beach suggested the implementation of a similar methodology, i.e. usage of piecewise linear topographies for the three–dimensional topographies.

### Analytical Solution

Here we will use linear shallow–water wave equation in polar coordinates. Given the singularity of the equation of motion in the case of the conical island $h(r) = m(r-a)$ –$m$ and $a$ are the slope and the waterline radius of the conical island, respectively– one method of removing it is to approximate the surface of cone with cylindrical boxes –Henceforth a cylindrical box will be referred to as a sill– in a stepwise fashion as in figure 6. For example in the segments, the basic solutions are

$$\eta_j = \sum_{n=-\infty}^{+\infty} e^{i(n\theta-\omega t)} \begin{cases} \{A_{n,i}\, e^{-\frac{in\pi}{2}} J_n(kr) + A_{n,r}\, H_n^{(1)}(kr)\} & r \geq b, \\ \{A_n\, J_n(kr) + B_n\, Y_n(kr)\} & r_j \leq r \leq r_{j+1}, \end{cases} \tag{22}$$

where $b$ is the radius of the conical island at the toe, $k = \omega/\sqrt{h_j}$ and $j$ represents the segment number. Given that the solution is known for evolution over a sill, the dividing the solution over the conical surface involves matching solutions at the interface of steps on segments. At the edge of each sill, in other words at a discontinuity in $h$, it will be required that the surface elevation and the normal component of the mass flux are continuous. At the shoreline ($r = a$); the later matching condition requires that $\partial\eta/\partial r|_{r=a} = 0$. Again here instead of trying to solve the system of equations which can be set up from the matching conditions, we will use $(2 \times 2)$ matrices to get solutions as described two–dimensional case. Details of the analytical study are described by Kânoğlu (Kânoğlu, 1997).

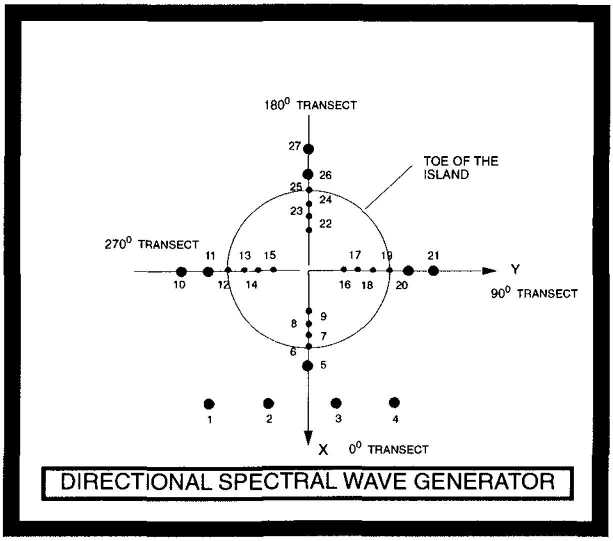


**Figure 7.** Definition sketch of the basin (not to scale).

## Experimental Study

We performed a series of large scale laboratory experiment –in a $30m$ wide and $25m$ long basin– at CERC for a conical island. Conical island –$60cm$ high, $7.2m$ toe diameter and $2.2m$ crest diameter and $1:4$ slope– was located in the middle of the basin as shown in figure 7. We varied the water depth, the solitary wave height, the horizontal length of the source and the eccentricities. We recorded the time histories of surface elevation at 27 locations and we measured maximum runup heights at 24 locations around the island. The experiments are described elsewhere in detail (Liu *et al.*, 1995; Briggs *et al.*, 1994; Kânoğlu, 1997). In most cases, the maximum runup heights are the largest at the front of the island and it decreases gradually toward the lee side of the island. Because of the collision of the two trapped waves, there is a drastic increase in the maximum runup height at the lee side of the island. Furthermore, in some cases as in figure 8, the maximum runup heights at the lee side of the island are larger than that of the front side.

We compared the maximum runup heights and the time series of surface elevations predicted by the general method with the laboratory data. Comparisons for the maximum runup and time histories of surface elevation are shown in figure 8 and 9 respectively.

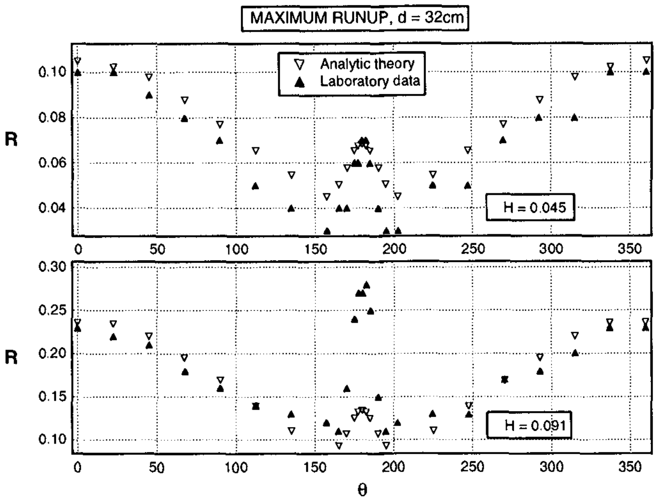


**Figure 8.** Comparison between analytical solution and experimental results for maximum runup for two different waveheights $H$.

## CONCLUSIONS

Comparisons between the analytical and experimental results are in good agreement for the maximum runup and the time series of surface elevation in both cases, i.e. the Revere Beach and the conical island experiments. For the conical island, we observe that the maximum runup height is largest in the front of the island and it decreases gradually toward the lee side of the island. Because of the collision of the two trapped waves, there is a drastic increase in the maximum runup height at the lee side. As suggested by Yeh (Yeh *et al.*, 1994), our results also confirm the mechanism of catastrophe around Babi island.

## ACKNOWLEDGEMENT

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## REFERENCES

Briggs, M. J., Synolakis, C. E., Harkins, G. S. and Green, D. 1994. Laboratory experiments of tsunami runup on a circular island. *PAGEOPH*, **144**, 569–593.

Briggs, M. J., Synolakis, C. E., Kanoglu, U., Green, D. R. 1997. Benchmark problem 3, Runup of solitary waves on a vertical wall. *Proceedings of the Second Workshop on Long Wave Runup Models.* In press.

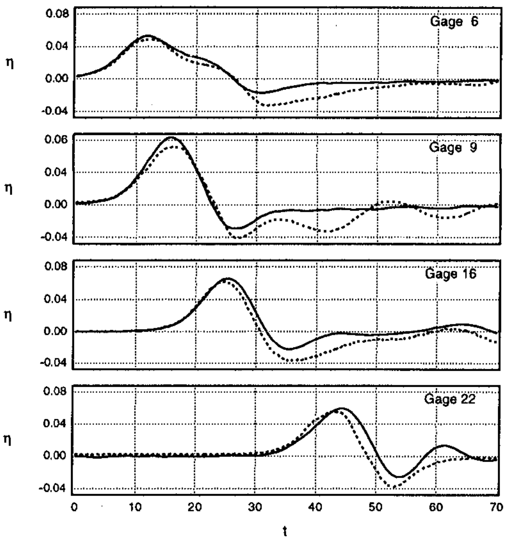


**Figure 9.** Comparison between the laboratory data and the analytical solution for the time histories of the surface elevation for a $H = 0.045$ solitary wave, at four different locations. Dotted lines represent the laboratory results.


Carrier, G. F. 1966. Gravity waves on water of variable depth. *J. Fluid Mech.*, **24**, 641-659.

Kânoğlu, U. 1997. *The runup of long waves around piecewise linear bathymetries.* Ph.D. Thesis. University of Southern California, Los Angeles, California, 90089.

Liu, P-L, Cho, Y-S, Briggs, M. J., Kanoglu, U., and Synolakis, C. E. 1995. Runup of solitary waves on circular islands. *J. Fluid Mech.*, **320**, 259–285.

Raubenheimer, B., Guza, R. T., Elgar, S. 1995. Wave transformation in the surf zone. *EOS*, AGU 1995 Fall Meeting, November 7, p. 282.

Synolakis, C. E. 1986. *The Runup of Long Waves.* Ph.D. Thesis. California Institute of Technology, Pasadena, California, 91125. 228 pp.

Synolakis, C. E. 1987. The runup of solitary waves. *J. Fluid Mech.*, **185**, 523-545.

Tadepalli, S. and Synolakis, C. E. 1994. The run-up of N-waves on sloping beaches. *Proc. of R. Soc. Lond. A*, **445**, 99-112.

Ward, D. 1995. Physical model study of Revere Beach, Massachusetts. US Army Corps of Engineers, Waterways Experiment Station Technical Report CERC–95–2, March 1995.

Yeh, H., Liu, P-L., Briggs, M., Synolakis C. E. 1994. Tsunami catastrophe in Babi island. *Nature*, **372**, 353-358.