CHAPTER 240

MATHEMATICAL MODELS FOR WAVES AND BEACH PROFILES IN SURF AND SWASH ZONES

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Abstract

The paper presents mathematical models for wave deformation and for beach profile change in the surf and swash zones. Boussinesq-type equations including a breaker-induced energy dissipation term (Watanabe *et al.*, 1994) is extended to the swash zone by introducing a periodically moving shoreward boundary. Computation is made on the wave deformation on constant slope beaches, particularly on the runup and run-down heights as well as the wave heights at the still water shoreline, which are compared with existing formulas. Beach profile change is computed with the sediment transport rate formula proposed by Dibajnia and Watanabe (1992) after a further generalization, together with Lagrangian treatment for the sediment motion in the swash zone, using the near-bottom velocity obtained from the wave computation. The validity of the models is examined through comparisons with measurement data obtained in large wave flume experiments.

Introduction

The authors have presented a numerical model for profile change of sheet-flow dominated beaches based on Boussinesq-type wave equations and a sediment transport rate formula proposed by themselves (Watanabe *et al.*, 1994). It has been demonstrated that the model can predict well the cross-shore distributions of the wave height and the transport rate as well as the beach profile change in and around the surf zone. However, the model has a crucial limitation of the incapability of computing the wave behavior and the beach evolution in the surf zone, both of which are generally very important in practical applications and fundamental studies regarding

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the coastal processes. A major effort in the present study is thus focused on the treatment of the swash zone dynamics both for waves and for sediment transport. Mathematical models are proposed for wave deformation and beach profile change and their validity is examined by comparing computations with measurements.

Mathematical Model for Wave Deformation

Watanabe *et al.* (1994) have presented a set of one-dimensional Boussinesq-type equations including a breaker-induced energy dissipation term as follows:

$$\frac{\partial \varsigma}{\partial t} + \frac{\partial Q}{\partial x} = 0, \quad \varsigma = \varsigma(x,t), \quad Q(x,t) = \int_{-h}^{\zeta} u(x,z,t)\,dz \tag{1}$$

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{Q^2}{D}\right) + gD\frac{\partial \varsigma}{\partial x} - \frac{1}{3}\overline{D}^2\frac{\partial^3 Q}{\partial t \partial x^2} - M_D = 0 \tag{2}$$

where

$$D(x,t) = h + \varsigma, \quad \overline{D}(x) = h + \overline{\zeta},$$

u is the horizontal velocity, and M_D corresponds to the momentum diffusion in the surf zone and has been expressed by Sato and Suzuki (1990) (See also Sato and Kabiling, 1994a, b) as

$$M_D = \frac{g\overline{D}}{\sigma^2} f_D \frac{\partial^2 Q}{\partial x^2} \tag{3}$$

in which σ is the angular frequency, and f_D is the following energy dissipation coefficient proposed by Watanabe and Dibajnia (1988).

$$f_D = \alpha_D \tan\beta \sqrt{\frac{g}{\overline{D}}} \cdot \sqrt{\frac{\hat{Q} - Q_r}{Q_s - Q_r}} \tag{4}$$

where $\alpha_D = 2.5$, $\tan\beta$ is the bottom slope around the breaking point, $\hat{Q}$ is the amplitude of the flow rate Q, and Q_s and Q_r correspond to $\hat{Q}$ in the dissipation zone on a uniform slope and in the recovery zone of constant depth, respectively, defined in this study by

$$Q_s = 0.4\left(0.57 + 5.3\tan\beta\right)C\overline{D}, \quad Q_r = \gamma_B \cdot C\overline{D} \tag{5}$$

where C is the wave celerity, and γ_B is a coefficient proportional to the ratio of the wave amplitude to the total mean depth. In the present model, we don't use any breaking criterion that determines the location of the breaking point *a priori* as employed in most previous models. Instead, by setting an appropriate value of γ_B and by equating f_D to zero in regions where $\hat{Q} \le Q_r$, the model can automatically handle the problem of determining the breaking point and the dissipation zone as well as the recovery zone. This also makes it unnecessary to use the factitious increase of the value of α_D from 0 to 2.5 around the breaking point that is otherwise needed to avoid numerical wave reflection from there.

The Sommerfeld radiation condition is imposed on the offshore boundary of a computation domain in the same way as in the previous model (Watanabe *et al.*, 1994). On the other hand, the treatment of the shoreward boundary in the present model is completely different from the previous one that assumes the presence of a fictitious shoreward zone of constant depth and neglects the swash wave behavior. Here we adopt a periodically moving shoreward boundary (swash wave front) on which the condition $Q = 0$ is imposed all through the period of uprush and backwash.

Numerical computation is conducted by a finite difference method with a staggered grid scheme of the central difference except for the convection term that is treated with the upwind difference for the sake of stability.

Application of the Wave Model

Figure 1 shows one example of a time-series of wave profiles for one wave period in and near the swash zone computed by using the present model. The bottom consists of a uniform slope of 1/20 and a horizontal bed with a depth of 2 m. The time history of incident waves (H_I = 0.45 m, T = 8 s) has been calculated by the second-order cnoidal wave theory and given on the seaward boundary (x = 0 m). The swash wave behavior or runup/rundown process seems to be simulated reasonably.

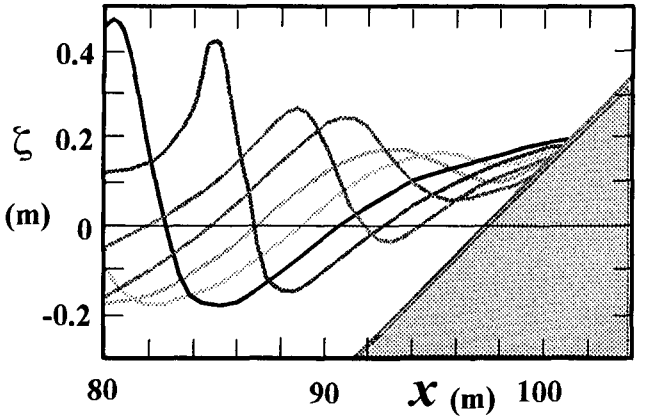

Fig. 1 Wave behavior near the shoreline; Runup and run-down.
(H_I = 0.45 m, T = 8 s, tan β = 1/20)

In order to examine the validity of the present wave model more quantitatively, swash-related parameters are computed and compared with existing formulas. Regarding the runup height R_u, Hunt (1959) proposed the following formula:

$$R_u / H_0 = \tan\beta / \sqrt{H_0 / L_0} \qquad (6)$$

On the other hand, Ogawa and Shuto (1984) formulated the runup height *Ru*, on mild

slope solid beds as the summation of the rundown height R_d and the swash zone height S_h as follows (See their paper for details):

$$R_u / H_0 = R_d / H_0 + S_h / H_0 \quad (7)$$

Comparisons of the runup height R_u are shown in **Fig. 2** between the computations by the present model and the above formulas. The agreement of the computations with Ogawa and Shuto's formula is remarkably good.

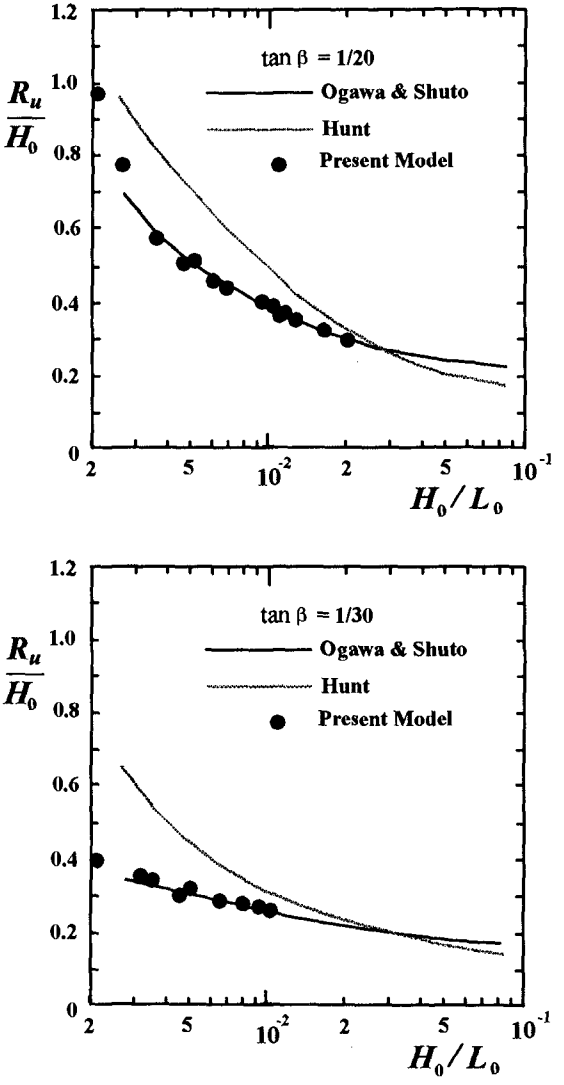

Fig. 2 Runup height R_u.

Similar comparisons are made in **Fig. 3** for the rundown height R_d and the vertical length of the swash zone S_h. The agreement with Ogawa and Shuto's formula is worse for R_d and thus for S_h than for R_u in particular for the small deepwater wave steepness, but still fairly good.

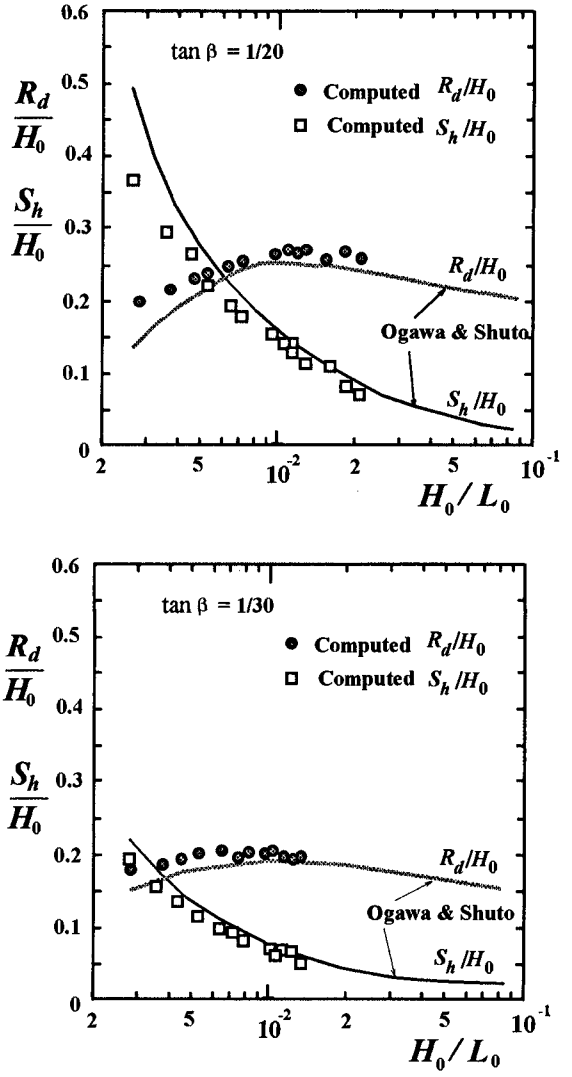

Fig. 3 Rundown height R_d and swash zone height S_h.

Now we will discuss the accuracy of the present model for evaluating the wave height H_S at the still water shoreline. Sunamura (1984) has reported the following

empirical formula for H_S based on wide-range experimental data.

$$H_S / H_B = 2.5 \tan\beta \tag{8}$$

where H_B is the breaker height, which, according to Sunamura and Horikawa (1974), is related to the deepwater wave height H_0, wavelength L_0 and the bottom slope as

$$H_B / H_0 = (\tan\beta)^{1/5} (H_0 / L_0)^{-1/4} \tag{9}$$

Substituting Eq. (9) into Eq. (8), we obtain

$$H_S / H_0 = 2.5 (\tan\beta)^{6/5} (H_0 / L_0)^{-1/4} \tag{10}$$

Figure 4 compares H_S between the computations and Eq. (10), also demonstrating the applicability of the present model to simulating the swash wave dynamics.

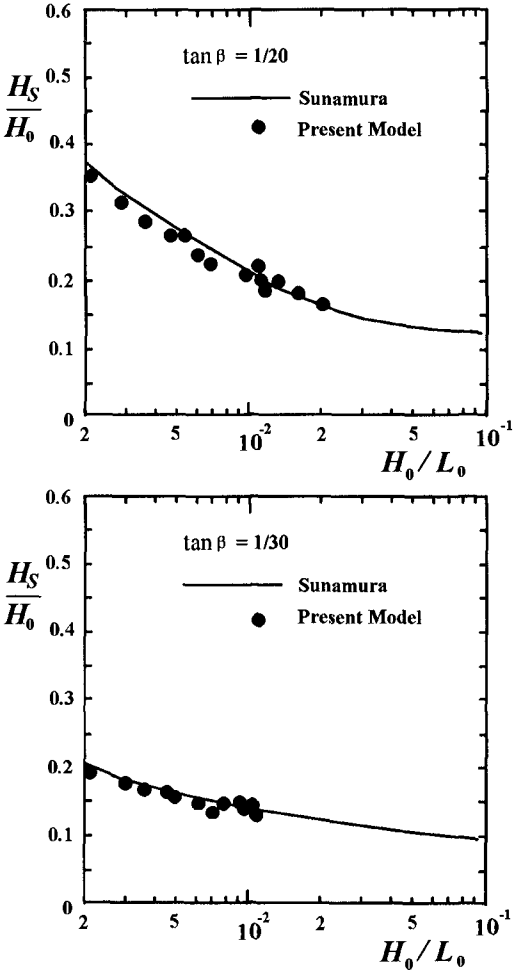

Fig. 4 Wave height H_S at the still water shoreline.

Mathematical Model for Beach Profile Change

The sediment transport rate formula proposed by Dibajnia and Watanabe (1992) is for the sheet flow in asymmetric oscillatory flow with superimposed steady flow, and has already been generalized by them for the bedload as well as for the suspended load over ripples (Dibajnia *et al.*, 1994). After a further slight modification, Dibajnia and Watanabe's sediment transport rate formula reads as follows:

$$\Phi \equiv \frac{q_{\text{net}}(1-\lambda_v)}{w_0 d} = 0.001 \cdot \text{sign}(\Gamma) \cdot |\Gamma|^{0.5} \tag{11}$$

where q_{net} is the net transport rate, and w_0, d, and λ_v are the settling velocity, grain diameter, and porosity of the sediment, respectively. The quantity Γ is defined by

$$\Gamma = \frac{u_c T_c (\Omega_c^3 + \Omega'^3_t) - u_t T_t (\Omega_t^3 + \Omega'^3_c)}{(u_c + u_t) T} \tag{12}$$

in which u_c and u_t are the equivalent root-mean-square amplitudes and T_c and T_t are the periods of the onshore and offshore velocity, respectively, and T (= T_c + T_t) is the wave period, namely,

$$u_c^2 = \frac{2}{T_c}\int_0^{T_c} (u_w + U)^2 dt, \quad u_t^2 = \frac{2}{T_t}\int_{T_c}^{T} (u_w + U)^2 dt \tag{13}$$

where u_w is the near-bottom orbital velocity and U is the steady flow velocity. Values of Ω_j are determined as follows:

$$\begin{cases} \text{if } \omega_j \le \omega_{\text{cr}} & \begin{cases} \Omega_j = \omega_j \cdot \dfrac{2 w_0 T_j}{d} \\ \Omega'_j = 0 \end{cases} \\ \text{if } \omega_j > \omega_{\text{cr}} & \begin{cases} \Omega_j = \omega_{\text{cr}} \cdot \dfrac{2 w_0 T_j}{d} \\ \Omega'_j = (\omega_j - \omega_{\text{cr}}) \cdot \dfrac{2 w_0 T_j}{d} \end{cases} \end{cases} \tag{14}$$

where the subscript j is to be replaced by either c or t, and

$$\omega_j = \frac{1}{2} \cdot \frac{u_j^2}{s g w_0 T_j}, \quad s = \frac{\rho_s - \rho}{\rho} \tag{15}$$

$$\omega_{\text{cr}} = 1 - 0.97 \cdot \sqrt{\Lambda}, \quad \Lambda = \{1 - [(\Psi_{\text{rms}} - 0.2)/0.4]^2\} \cdot \min(1, 2\lambda / d_0) \tag{16}$$

in which ρ and ρ_s are the densities of the water and sediment, respectively, Ψ_{rms} is the Shields number estimated in terms u_c and d, λ is the pitch length of ripples if any, and d_0 is the near-bottom orbital diameter.

Now one very important issue is how to apply this generalized formula to computing the sediment transport rate in the swash zone reasonably, where the wave-induced orbital velocity drastically varies over a distance of the periodical movement of sediment mass. In the present model for beach profile change, we will adopt a Lagrangian method, namely, assuming that the velocity of every sediment mass in

motion is given by the concurrent flow velocity, we trace the motion of a sediment mass over every wave period, memorize the velocity variation experienced by it, determine its mean position by taking the time-average, and evaluate its net transport rate at this position from the above formula with the velocity variation.

The undertow velocity, which is particularly large near the breaking point in the surf zone, is also incorporated in the computation of the transport rate (For details, readers are referred to Watanabe *et al.*, 1994) . Change in beach profiles is obtained from the following conservation equation of sediment mass including the effect of local bottom slope (Watanabe *et al.*, 1986) through alternate computation of the wave deformation, the net transport rate, and the beach transformation itself.

$$\frac{\partial z_b}{\partial t} = -\frac{\partial h}{\partial t} = -\frac{\partial}{\partial x}\left(q_{\text{net}} - \varepsilon_s |q_{\text{net}}| \frac{\partial z_b}{\partial x}\right) \tag{17}$$

where z_b is the bottom elevation, and a value of the coefficient ε_s is set equal to 2.0 in consideration of the repose angle of sand.

Application of Beach Profile Model

Now we will examine the validity of the model for the beach profile change. **Figures 5** and **6** show examples of comparisons of the wave height distribution and of the beach profile change between the computations and measurements. The measurement data are those obtained in large-wave-flume experiments by Shimizu *et al.* (1985). Figure 5 corresponds to an erosional condition, whereas Fig. 6 to a depositional case. For both the cases, the computed wave height distributions agree very well with the measurements except for the under-estimation near breaking points, which is attributable to the weak nonlinearity of Boussinesq-type equations. It is seen in Fig. 5 that the present model properly reproduces not only the formation of a bar but also the recession of the shoreline. Figure 6 also indicates high capability of the model in reproducing both the accretion near the shoreline and the berm formation.

Concluding Remarks

Mathematical models have been proposed in this paper both for the nearshore wave deformation and for the beach profile change in the surf and swash zones. The wave model is based on a Boussinesq-type equations including the breaker-induced energy dissipation and adopts a periodically moving shoreline boundary for the treatment of the swash zone dynamics. In the beach profile model, a Lagrangian method has been employed to evaluate the net rate of sediment transport in and near the swash zone. The validity and applicability of the proposed models have been verified through comparisons with the existing formulas and the experimental data both for the wave deformation and the beach profile change. Improvement of the computational efficiency and application to longer-term and H-2D conditions are left for future study.

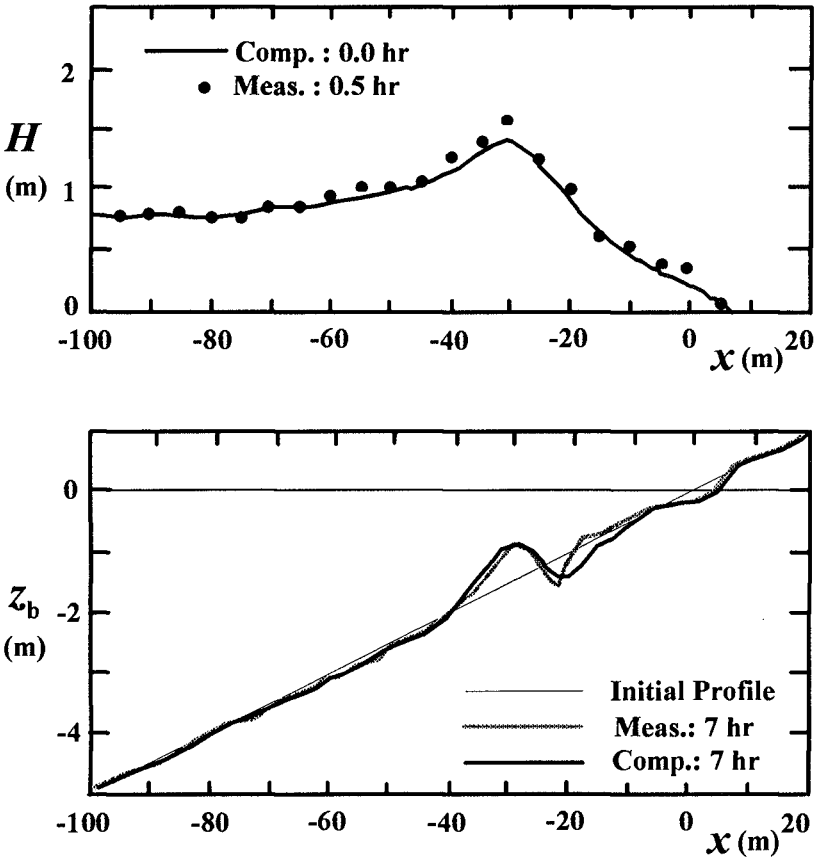

Fig. 5 Wave height distribution and beach profile change.
($H_0 = 0.81$ m, $T = 12$ s, $d = 0.27$ mm)

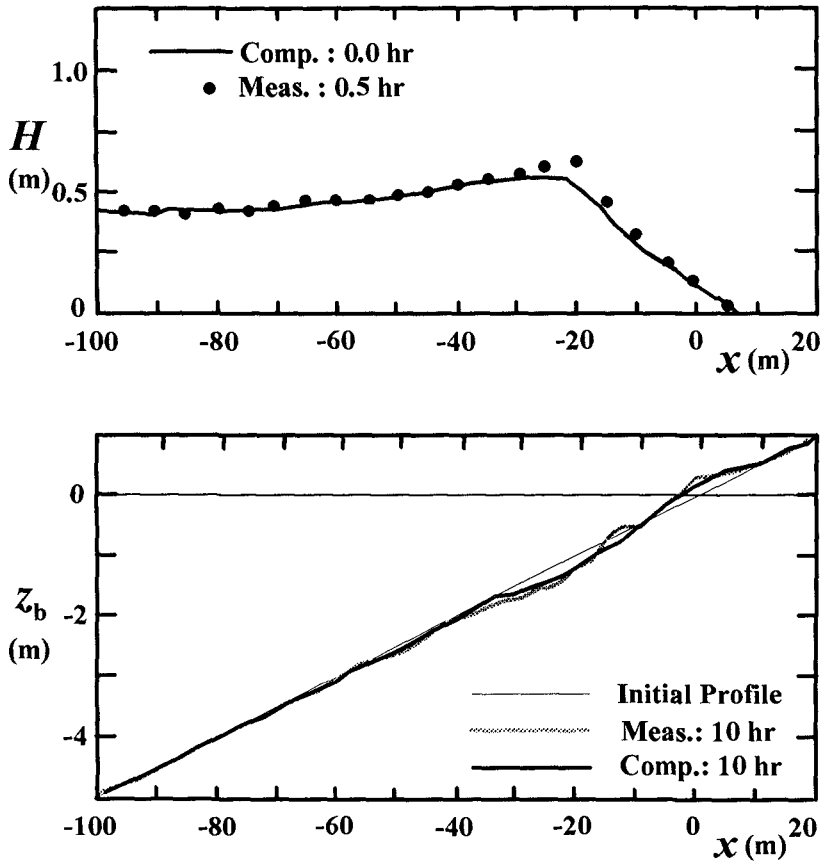

Fig. 6 Wave height distribution and beach profile change.
($H_0 = 0.46$ m, $T = 6$ s, $d = 0.27$ mm)

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