A GENERALIZED KDV TYPE EQUATION
FOR UNEVEN DEPTHS

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A Korteweg & de Vries type equation with improved dispersion characteristics for uneven water depths is presented. The new KdV type equation contains mixed dispersion and shoaling terms, which extend its applicable range of relative depths to virtually deep waters. The wave equation also satisfies an important consistency condition that there is an exact agreement between the shoaling rate of the equation itself and the rate obtained from the constancy of energy flux. A finite-difference scheme is devised for simulating several linear and nonlinear cases over varying bathymetry. The performance the new KdV type equation is observed to be quite satisfactory.

Keywords: KdV type equation; improved linear dispersion; improved linear shoaling

INTRODUCTION

Korteweg and de Vries (1895) derived a nonlinear wave equation which might be interpreted as the one-directional form of Boussinesq's (1872) one-dimensional wave model. The KdV equation admits solitary waves and periodic "cnoidal" waves as solutions. The original KdV equation contains the third spatial derivative of the surface elevation as dispersive term and has relatively poor dispersion characteristics. Benjamin *et al.* (1972) replaced the dispersive term by an equivalent term that resulted in a regularized form which had better dispersion characteristics. This particular KdV equation has become known as the BBM equation after its authors while it was Peregrine (1966) who first used the BBM type KdV equation for undular bore simulations.

For varying depth the KdV type equations are typically augmented by a single term originating from the depth-gradient term of the depth-integrated continuity equation. Therefore, unlike Peregrine's (1967) Boussinesq equations of varying depth, which contain shoaling terms associated with dispersion terms besides the term in the continuity equation, the available KdV type equations for varying depth do not contain any dispersion related shoaling terms. This deficiency should be amended in appropriate manner since KdV type equations are weakly dispersive wave equations just like Boussinesq type equations. The present work considers such an improved KdV type equation with appropriate shoaling terms (Beji, 2016) and demonstrates its advantages for practical applications.

A robust finite-difference scheme is given for the solution of the KdV type equation with mixed dispersion and linear shoaling terms. Accuracy of the linear shoaling characteristics of the new wave model over uneven topography is demonstrated by checking numerical results against theoretical predictions based on energy flux concept. A solitary wave simulation is also included as a standard test of the numerical scheme. Nonlinear wave evolutions over a submerged bar for regular and random waves are then simulated for different relative depths and compared with the experimental measurements (Beji and Battjes, 1994). Overall, the performance of the new KdV type equation is quite well, especially for cases involving dispersive nonlinear wave evolutions over uneven depths.

GENERALIZED KDV OVER VARYING BATHMETRY

Generalized KDV Type Equation for Uneven Depths

The KdV type equation with mixed dispersion terms for better dispersion characteristics and linear shoaling terms is given by (Beji, 2016)

$$\eta_t + C\eta_x + \frac{3C}{2h}\eta\eta_x - \frac{(1+2\beta)}{6}Ch^2\eta_{xxx} - \frac{(1+\beta)}{3}h^2\eta_{xxt}$$
$$+ \frac{Ch_x}{4h}\eta - \frac{(15+32\beta)}{24}Chh_x\eta_{xx} - \frac{5(1+\beta)}{6}hh_x\eta_{xt} = 0 \tag{1}$$

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where η is the surface displacement, h the spatially varying water depth, $C = \sqrt{gh}$ the non-dispersive wave celerity, β the dispersion parameter controlling the form of the dispersion relationship of the wave equation. Subscripts stand for partial differentiation with respect to the indicated variables.

Dispersion Relationship

The dispersion relationship is obtained by substituting $\eta = a_0 \exp\left[ik\left(x - C_K t\right)\right]$ into the linearized constant depth form of Eq. 1.

$$C_K = C\left[\frac{1 + \frac{1}{6}\left(1 + 2\beta\right)k^2 h^2}{1 + \frac{1}{3}\left(1 + \beta\right)k^2 h^2}\right] \qquad (2)$$

where C_K indicates the phase celerity of the generalized KdV type equation. The exact phase velocity according to linear theory is

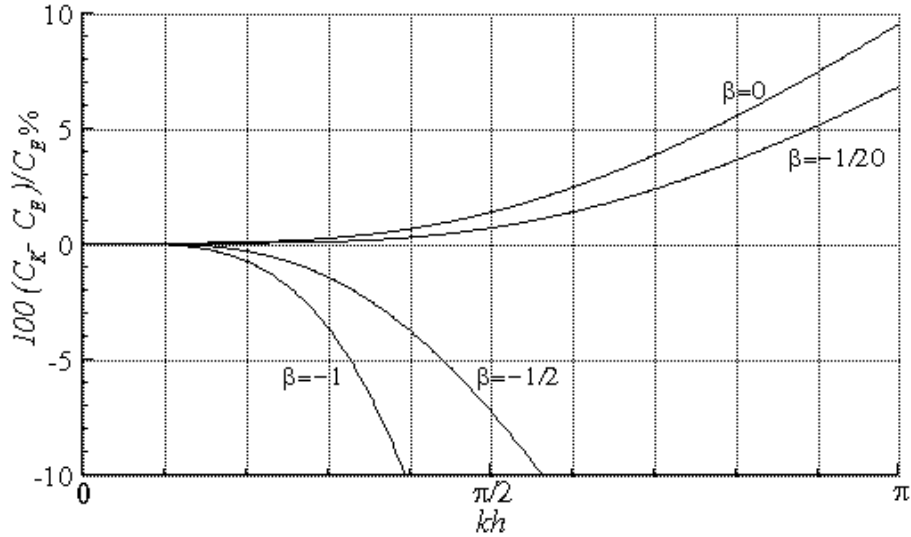$$C_E = C\sqrt{\frac{\tanh kh}{kh}} \qquad (3)$$

Figure 1. Relative error percentages of the phase celerity of the generalized KdV equation in comparison with the linear dispersion relationship for different β values.

Fig. 1 shows the relative error percentages of the KdV type equation with mixed dispersion terms for different β values. $\beta = -1$ gives the original or classical KdV equation while $\beta = -1/2$ gives the BBM type KdV equation. $\beta = -1/20$ corresponds to [2/2] Padé approximant of the exact dispersion relationship and has the lowest error percentage among others with errors $O\left(k^6 h^6\right)$. Finally, $\beta = 0$ produces a KdV equation with shoaling characteristics which are in exact agreement with those obtained from the concept of energy flux (Beji, 2016).

FINITE-DIFFERENCE SCHEME

Finite-Difference Discretization of KDV Type Equation
The KdV type equation with mixed dispersion terms given in Eq. 1 is discretized by finite-difference approximations centered in time $t + \tfrac{1}{2}\Delta t$ and in space $x + \tfrac{1}{2}\Delta x$ as follows.

$$\eta_t \cong \frac{\eta_{i+1}^{k+1} - \eta_{i+1}^{k} + \eta_{i}^{k+1} - \eta_{i}^{k}}{2\Delta t} \qquad \eta_x \cong \frac{\eta_{i+1}^{k+1} - \eta_{i}^{k+1} + \eta_{i+1}^{k} - \eta_{i}^{k}}{2\Delta x}$$

$$\eta_{xxx} \cong \frac{\eta_{i+2}^{k+1} - 3\eta_{i+1}^{k+1} + 3\eta_{i}^{k+1} - \eta_{i-1}^{k+1} + \eta_{i+2}^{k} - 3\eta_{i+1}^{k} + 3\eta_{i}^{k} - \eta_{i-1}^{k}}{2\Delta x^3} \qquad (4)$$

$$\eta_{xxt} \cong \frac{\eta_{i+2}^{k+1} - \eta_{i+1}^{k+1} - \eta_{i}^{k+1} + \eta_{i-1}^{k+1} - \eta_{i+2}^{k} + \eta_{i+1}^{k} + \eta_{i}^{k} - \eta_{i-1}^{k}}{2\Delta x^2 \Delta t} \quad \& \; etc.$$

where the subscript i and superscript k denote indices multiplying respectively the spacing Δx between the grid points and the time step Δt. The final discretized equation results in a quadro-diagonal matrix, which can be reduced to a tri-diagonal matrix by performing a single sweep in the computational domain. Afterwards use of the standard tri-diagonal solver gives the surface elevation values for the grid points.

Numerical Treatment of Incoming and Outgoing Boundaries
At the first node of the incoming boundary the incident wave form is simply introduced as a function of time. However, at the second node and the last two nodes the presence of terms outside the domain due to η_{xxx} and η_{xxt} requires special care. Either they should be approximated by one-sided discretization or some simplifications to the KdV equation must be introduced. One-sided discretization does not yield a robust scheme therefore using the linear wave approximation $\eta_{xx} = -k^2\eta$ the present KdV equation in the absence of linear shoaling terms is reduced to

$$\left[1 + \frac{(1+\beta)}{3}k^2h^2\right]\eta_t + C\left[1 + \frac{(1+2\beta)}{6}k^2h^2\right]\eta_x + \frac{3C}{2h}\eta\eta_x = 0 \qquad (5)$$

in which the wave number k is computed from the dispersion relationship Eq.2 for the grid point considered. The discretization of the above equation then terminates the problems arising at the end points of the domain.

NUMERICAL SIMULATIONS
Using the finite-difference discretization of the new KdV type equation several numerical simulations are performed to demonstrate its extended range of applicability. The first group of simulations compares linear shoaling properties of the present model with those of the BBM type KdV model for sinusoidally varying bottom. Then a solitary wave simulation is given as a standard test for the scheme. Finally, experimental data of Beji and Battjes (1994) for random nonlinear wave propagation over a trapezoidal bar is used for testing the performance of the new model.

Linear Shoaling over Sinusoidal Bathymetry
Besides improved dispersion characteristics quite an important aspect of the present KdV type model is its improved linear shoaling properties. In particular, accurate estimation of wave heights in the nearshore zone depends on good shoaling characteristics of the wave model. Linear waves propagating over a sinusoidally varying bathymetry are simulated for three different wave periods by using the new KdV type equation for $\beta = -1/20$ with all the linear shoaling terms present and the BBM type KdV equation (corresponding to $\beta = -1/2$) with only the shallow water shoaling term present. At the incoming boundary the water depth is 10 m and is sinusoidally reduced to 5 m in the mid-channel at 500 m and then increased to 10 m again at the end of the channel. Wave periods used are $T = 20, 8, 6$ s which respectively correspond to $h_0/L_0 \cong 1/20, 1/7, 1/5$ values which are shallow and intermediate relative depths.

Fig. 2 shows the comparisons of the performances of two models against the energy flux concept $a^2 C_g = Const.$ drawn as the envelope. C_g is taken according to linear theory.
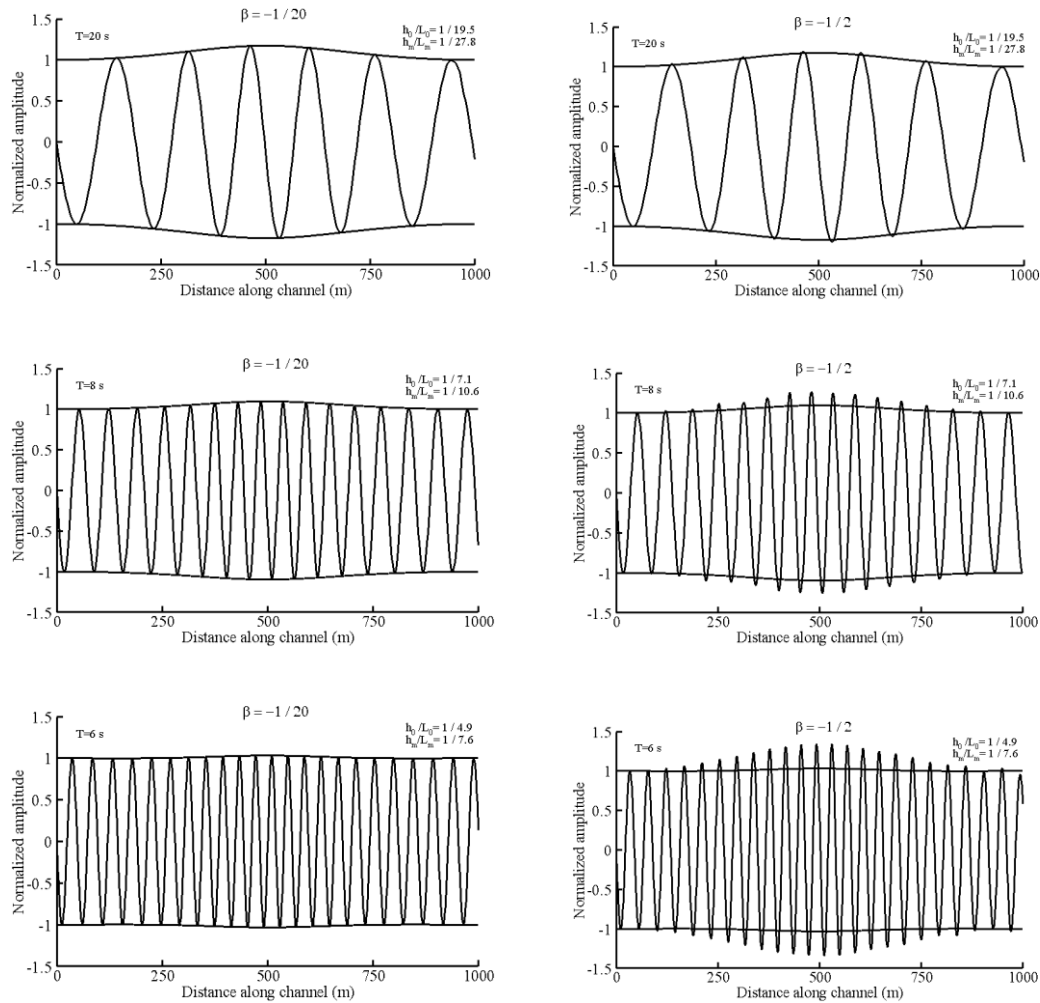

Figure 2. Linear shoaling over a sinusoidal bathymetry for T=20 s, 8 s, and 6 s waves (from top to bottom). Left column is the new wave model right column is the BBM type KdV equation with shallow water shoaling term only. Envelopes are drawn according to energy flux concept with C_g taken from exact linear theory.

Virtually perfect agreement of the amplitude variations as predicted by the new KdV type model with the amplitude envelopes is quite impressive. On the other hand, the BBM type KdV model with only the shallow water shoaling term present $(Ch_x / 4h)\eta$ predicts the shoaling rate considerably higher with increasing relative depth, as shown on the right column. The envelopes are drawn according to the energy flux concept $a^2 C_g = Const.$ with C_g taken from exact linear theory, as seen on the left column.

Solitary Waves

Cnoidal waves and solitary waves are the permanent wave solutions of any KdV type equation. The present generalized KdV equation is also capable of simulating solitary waves. As a simple demonstration a solitary wave with steepness a/h=0.25 is simulated and shown in Figure 3.
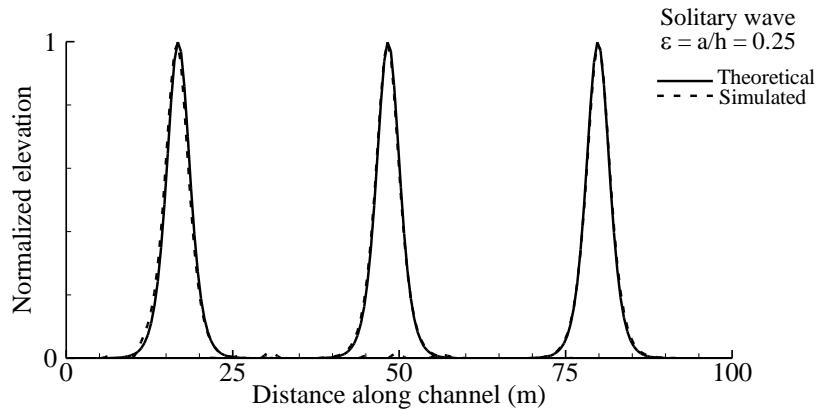

Figure 3. Theoretical and simulated solitary wave with steepness a/h=0.25.

Nonlinear Waves over Trapezoidal Bar

Beji and Battjes (1994) made measurements of sinusoidal and random nonlinear waves propagating over a submerged bar. Water depth is constant at 0.4 m water depth for the first 0.3 m, then an upslope of 1:20 follows for 6 meters reducing the water depth to 0.1 m. For 2 m the depth is constant again at 0.1 m then a downslope of 1:10 increases the water depth to 0.4 m in 3 m. Two different groups of waves were considered: relatively long waves with $T = 2$ s period and short waves with $T = 1.25$ s period. For each group regular sinusoidal waves and random waves with initially JONSWAP type spectrum were generated and evolutions of wave forms and spectra were recorded at seven different stations along the wave flume. For random waves periods correspond to peak periods of the spectra.
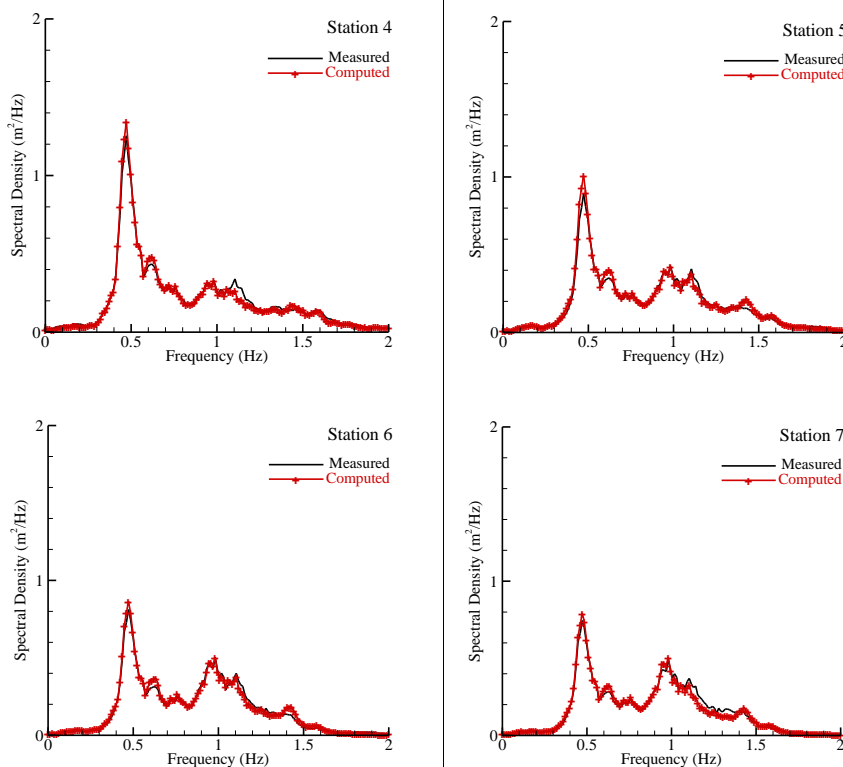

Figure 4. Comparisons of measured and computed spectra for long waves with peak frequency f_P=0.5 Hz.

Here, taking $\beta = -1/20$ for all the numerical simulations long and short random waves were simulated in time domain for 20480 time steps (nearly 800 seconds of actual laboratory measurement time). These time series in turn were used in producing wave spectra by ensemble averaging 10 segments of the FFT of 2048 measured/computed points. Short random waves were especially included to demonstrate the extended applicability range of the new KdV type equation. For each computation the measured time series data at Station 1 was fed into the numerical model as incident boundary condition and the computations were carried out for 20480 time steps without interruption or restart. The end of the computational domain was specified as radiation boundary as formulated in Eq. 5, which proved to be quite effective. Figure 4 shows the comparisons of measurements with the measured experimental data for Stations 4-7 for the long wave case. Stations 1-3 are not shown as the agreements in these stations are nearly perfect they are excluded to save space. Figure 5 shows the same comparisons for short or high frequency waves.
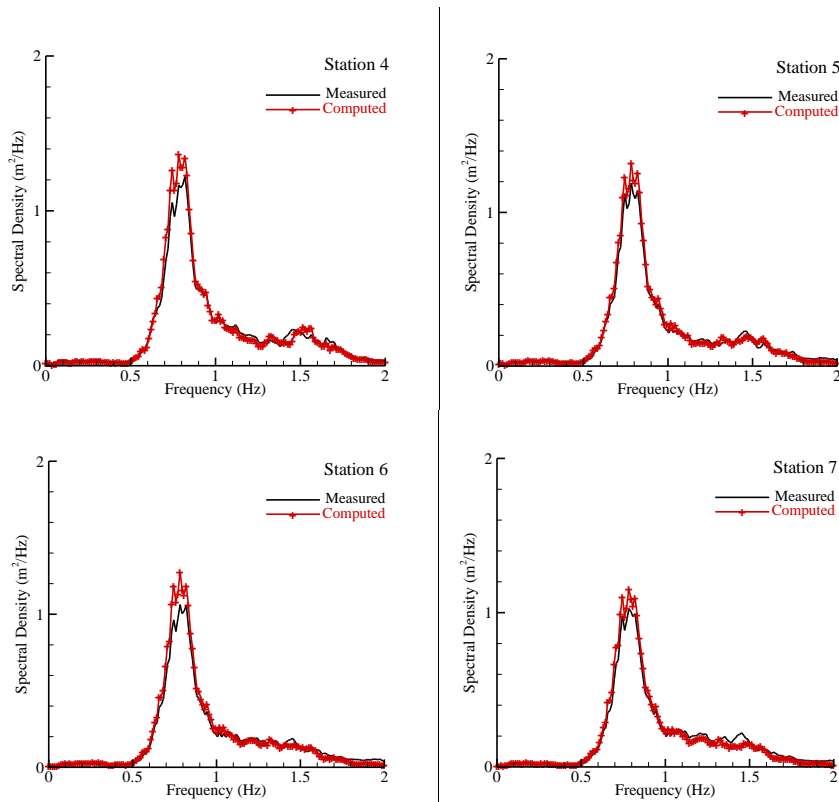

Figure 5. Comparisons of measured and computed spectra for short waves with peak frequency f_p=0.8 Hz.

CONCLUDING REMARKS

A generalized KdV type equation with improved linear dispersion and linear shoaling characteristics is presented. The new equation has a wider applicability range, extending virtually to relatively deep waters. Finite-difference algorithm described for the numerical solution of the equation is quite robust and accurate as tests on linear shoaling and solitary waves show clearly.

The linear shoaling terms accounting for the accurate and consistent shoaling properties are the most original and important part of the equation; the classical shoaling term of the KdV models alone predicts amplitude changes too high as demonstrated for relatively shorter waves. Therefore, the corrective role of the new linear shoaling terms is vital in simulations over varying water depths.

Nonlinear waves over wide range of relative depths require both good dispersion and shoaling characteristics for realistic simulations. These crucial aspects of the model are also tested for the case of nonlinear wave propagation over a trapezoidal bar, where a high rate of energy transfer takes place among harmonic wave components.

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